Spectral Gap, Property (T), and Effective Mixing for $\mathrm{SL}_3(\mathbb{R})$

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December 2025

1 Spectral Gap

In the study of dynamical systems and representation theory, we are often interested in how "efficiently" a group action mixes vectors in a Hilbert space. The concept of a spectral gap quantifies this efficiency.

Let G be a locally compact group (e.g., a Lie group like $\mathrm{SL}_n(\mathbb{R})$) and let $(\pi, \mathcal{H})$ be a unitary representation of G. We denote the subspace of G-invariant vectors by:

$$\mathcal{H}^G = \{v \in \mathcal{H} \mid \pi(g)v = v \text{ for all } g \in G\}.$$

1.1 Definitions

We begin by quantifying how close a vector is to being invariant over a specific subset of the group.

Definition 1.1 ((Q, ε)-invariant vectors)**.** Let $Q \subseteq G$ be a compact subset and let $\varepsilon > 0$. A unit vector $v \in \mathcal{H}$ is called **(Q, ε)-invariant** if

$$\sup_{g \in Q} \|\pi(g)v - v\| < \varepsilon.$$

Using this local definition, we can characterize representations that behave like the trivial representation asymptotically.

Definition 1.2 (Almost Invariant Vectors)**.** The representation π is said to have **almost invariant vectors** if for every compact subset $Q \subseteq G$ and every $\varepsilon > 0$, the representation admits a (Q, ε)-invariant vector.

Definition 1.3 (Spectral Gap)**.** A unitary representation $(\pi, \mathcal{H})$ has a **spectral gap** if the restriction of π to the orthogonal complement of the invariant vectors, denoted $\pi|_{\mathcal{H}_0}$ where $\mathcal{H}_0 = (\mathcal{H}^G)^\perp$, does **not** have almost invariant vectors.

Definition 1.4 (Uniform Spectral Gap)**.** A collection of representations (or a single representation) has a **uniform spectral gap** if there exist a specific compact set $Q \subset G$ and a constant $\varepsilon > 0$ such that no representation in the collection possesses a (Q, ε)-invariant vector in $\mathcal{H}_0$.

1.2 Equivalence of Characterizations

It is often useful to characterize the spectral gap not just by vectors, but by the norm of averaging operators. To do this, we first define the action of $L^1(G)$ on the Hilbert space.

Definition 1.5 (Convolution / Smeared Representation)**.** Let $f \in L^1(G)$. We define the operator $\pi_*(f) : \mathcal{H} \to \mathcal{H}$ by the Bochner integral:

$$\pi_*(f) = \int_G f(g)\pi(g)\, dg.$$

With dg being the left Haar measure. For a vector $v \in \mathcal{H}$, this means $\pi_*(f)v = \int_G f(g)(\pi(g)v)\, dg$. The operator norm satisfies $\|\pi_*(f)\|_{op} \le \|f\|_{L^1}$.

Proposition 1.6. *Let $(\pi, \mathcal{H}_0)$ be a unitary representation without non-zero invariant vectors. The following are equivalent:*

1. *π has a spectral gap (i.e., does not admit almost invariant vectors).*
2. ***(Indicator Mean)** There exists a compact set $Q \subset G$ with positive Haar measure such that the averaging operator $\pi_*(\mathbf{1}_Q) = \int_Q \pi(g)\, dg$ satisfies*

$$\|\pi_*(\mathbf{1}_Q)\|_{op} < \mathrm{vol}(Q).$$

3. ***(Continuous Mean)** There exists a continuous probability density $f \in C_c(G)$ (non-negative, compactly supported, $\int f = 1$) such that*

$$\|\pi_*(f)\|_{op} < 1.$$

Sketch of Proof. ($1 \implies 2$): If π has no almost invariant vectors, then by contradiction, assume for all Q, $\|\pi_*(\mathbf{1}_Q)\| = \mathrm{vol}(Q)$. This would imply the existence of a sequence of vectors v_n such that $\langle \pi_*(\mathbf{1}_Q)v_n, v_n\rangle \to \mathrm{vol}(Q)$, which forces $\pi(g)v_n \approx v_n$ for $g \in Q$. This constructs almost invariant vectors.

($2 \implies 3$): Take Q from (2). By Urysohn's lemma and approximation, we can find a continuous function f sufficiently close to $\frac{1}{\mathrm{vol}(Q)}\mathbf{1}_Q$ in the L^1 norm such that $\|\pi_*(f)\|$ remains strictly less than 1.

($3 \implies 1$): If $\|\pi_*(f)\| = 1 - \delta$ for some $\delta > 0$, then for any unit vector v, $\|\pi_*(f)v\| = \|\int f(g)\pi(g)v\| \le 1 - \delta$. This creates a strict geometric barrier preventing vectors from staying arbitrarily close to v for all g in the support of f. □

Remark 1.7 (Relation of Constants). The constants in the definitions above are quantitatively related.

- If $\|\pi_*(\mathbf{1}_Q)\|_{op} \le \mathrm{vol}(Q)(1-\delta)$, then for any unit vector v, there exists some $g \in Q$ such that $\|\pi(g)v - v\| \ge \sqrt{2\delta}$.
- Conversely, if a representation has a uniform spectral gap such that $\sup_{g\in Q} \|\pi(g)v - v\| \ge \varepsilon$ for all unit vectors v, then the averaging operator satisfies:

$$\left\| \frac{1}{\mathrm{vol}(Q)} \pi_*(\mathbf{1}_Q) \right\|_{op} \le 1 - \frac{\varepsilon^2}{2}.$$

This shows that a "small" spectral gap ε leads to a "slow" operator decay $\varepsilon^2/2$, but the existence of one implies the other.

2 Property (T)

Property (T), introduced by Kazhdan, is a rigidity property of the group itself.

Definition 2.1 (Kazhdan's Property (T))**.** A locally compact group G has **Property (T)** if every unitary representation $(\pi, \mathcal{H})$ that possesses almost invariant vectors also possesses a non-zero invariant vector.

In the language of Section 1: G has Property (T) if every representation without invariant vectors has a spectral gap. In fact, Property (T) implies uniform spectral gap.

2.1 Implications of Property (T)

Proposition 2.2.

1. ***Amenability:*** *If G is amenable and has Property (T), then G is compact.*
2. ***Compact Generation:*** *If G has Property (T), then G is compactly generated.*
3. ***Homomorphisms:*** *If G has Property (T) and $\phi : G \to H$ is a continuous surjective homomorphism, then H has Property (T).*
4. ***Lattices:*** *Let $\Gamma \leq G$ be a lattice. Then G has Property (T) $\iff$ Γ has property (T).*

Corollary 2.3. *Let G be a group with Property (T).*

1. *The abelianization $G/[G, G]$ is compact.*
2. *G is unimodular (i.e., the modular function $\Delta_G \equiv 1$).*

Proof. **1.** The abelianization $A = G/[G, G]$ is an abelian group. As a quotient of a Property (T) group, A has Property (T). Abelian groups are amenable. By Prop 2.3(1), an amenable Property (T) group is compact.

2. The modular function $\Delta_G : G \to \mathbb{R}^+$ is a homomorphism into the abelian group $\mathbb{R}^+$. The image must have Property (T). The only Property (T) subgroup of $\mathbb{R}^+$ is the trivial group $\{1\}$. Thus $\Delta_G \equiv 1$. □

Remark 2.4. The notation (T) comes the isolation of the trivial representation $\pi(\cdot) = \mathrm{Id}$ in Fell's topology.

3 Effective Proof that $\mathrm{SL}_3(\mathbb{R})$ has Property (T)

We now seek a concrete, "effective" proof that relates these abstract properties to specific estimates on matrix coefficients for $G = \mathrm{SL}_3(\mathbb{R})$.

3.1 Smooth Vectors in $\mathrm{SL}_3(\mathbb{R})$

To discuss effective decay rates, we need to handle the issue that vectors in $\mathcal{H}$ can be arbitrarily "rough." We introduce smooth vectors to control this.

Definition 3.1 (Smooth Vectors)**.** A vector $v \in \mathcal{H}$ is called C^r **smooth** if the maps $g \mapsto \langle \pi(g)v, u\rangle$ from G to $\mathbb{C}$ are C^r-smooth for all $u \in \mathcal{H}$. The space of smooth vectors is denoted $\mathcal{H}^r$. Note that if v is smooth and since $\mathcal{H}^* = \mathcal{H}$ $X \mapsto d\pi(X)v$ is a well defined linear transformation from $\mathfrak{g}$ to $\mathcal{H}$.

Fact 3.2. *The space of smooth vectors $\mathcal{H}^r$ is dense in $\mathcal{H}$, and vectors may be approximated by $\pi_* (f)\, v$ for $f \in C_c^\infty (G)$ s.t. fm_G is a probability measure approximating δ_e, which is smooth by differentiating under the integral sign.*

Definition 3.3 (Sobolev Norm)**.** Fix a basis $X_1, \dots, X_d$ of the Lie algebra $\mathfrak{g}$ of G. We view elements of $\mathfrak{g}$ as differential operators on $\mathcal{H}^\infty$. For an integer $k \geq 1$, the **Sobolev norm** of order k is defined as:

$$\|v\|_{\mathcal{S}_k} = \left(\sum_{\mathrm{ord}(D)\leq k} \|d\pi(D)v\|^2 \right)^{1/2}$$

where D ranges over monomials in X_i of degree at most k.

Remark 3.4 (Lie Algebra Action). It is important to understand how $X \in \mathfrak{g}$ acts as an operator. For $v \in \mathcal{H}^\infty$, the action is the differentiation of the group action:

$$d\pi(X)v = \frac{d}{dt}\bigg|_{t=0} \pi(\exp(tX))v.$$

Since π is unitary, $d\pi(X)$ acts as a skew-symmetric operator (essentially i times a self-adjoint operator).

3.2 Unitary Representations of Abelian Groups

To prove Property (T) for $\mathrm{SL}_3(\mathbb{R})$, we utilize the representation theory of its abelian subgroups (specifically copies of $\mathbb{R}^2$). We will begin by defining the generalized Fourier transform for general abelian groups. Let A be a locally compact Hausdorff abelian group.

Definition 3.5 (Fourier Transform for measures)**.** Let μ be a probability measure on $\widehat{A}$. Define its Fourier transform $\widehat{\mu} : A \to \mathbb{C}$ as:

$$\widehat{\mu}(a) = \int_{\widehat{A}} \overline{\chi(a)}\, d\mu(\chi).$$

.

Fact 3.6.

- *The Fourier Transform can be extended to any complex-valued measure Hahn's decomposition.*
- *Also, if μ is Borel, the Fourier transform may be extended to $L^1 (\mu)$ by $\hat{f} = \widehat{fd\mu}$.*
- *In particular - the Plancherel equality holds for $f \in L^1 (\mu) \cap L^2$: (μ): $\|f\|_2 = \|\hat{f}\|_2$.*

Remark 3.7. This is a clear generalization to the famous cases of Fourier Series ($A = S^1, \widehat{A} = \mathbb{Z}$) and Fourier Transform $A = \widehat{A} = \mathbb{R}^n$.

Definition 3.8 (Positive Type Functions)**.** A function $\phi : A \to \mathbb{C}$ is of **positive type** if for any $n \in \mathbb{N}$ and any $a_1, \dots, a_n \in A$, and complex numbers $c_1, \dots, c_n$,

$$\sum_{i,j=1}^{n} c_i\overline{c_j}\phi(a_j^{-1}a_i) \geq 0.$$

For any vector $v \in \mathcal{H}$, the matrix coefficient $\phi_v(a) = \langle \pi(a)v, v\rangle$ is a function of positive type.

Fact 3.9 (Bochner's Theorem)**.** *A continuous function $\phi : A \to \mathbb{C}$ is of positive type with $\phi(e) = 1$ if and only if it is the Fourier transform of a unique finite Borel measure μ on the dual group $\widehat{A}$, That is:*

$$\phi(a) = \int_{\widehat{A}} \overline{\chi(a)}\, d\mu(\chi).$$

Definition 3.10 (Projection-Valued Measure)**.** Let $(X, \mathcal{B})$ be a measurable space, where X is a set and $\mathcal{B}$ is a σ-algebra of subsets of X. Let $\mathcal{H}$ be a separable complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$.

A **projection-valued measure** (PVM) is a mapping $E : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ satisfying the following properties:

1. **Projection Property:** For every $A \in \mathcal{B}$, $E(A)$ is an orthogonal projection, i.e., $E(A)^2 = E(A)$ and $E(A)^* = E(A)$.
2. **Normalization:** $E(\emptyset) = 0$ (the zero operator) and $E(X) = I$ (the identity operator).
3. **Additivity:** If $A, B \in \mathcal{B}$ are disjoint, then $E(A \cup B) = E(A) + E(B)$. Furthermore, $E(A)E(B) = 0$ (orthogonality).
4. **σ-Additivity:** If $\{A_n\}_{n=1}^{\infty}$ is a countable collection of pairwise disjoint sets in $\mathcal{B}$, then

$$E\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} E(A_n),$$

where the series converges in the strong operator topology (i.e., for every $v \in \mathcal{H}$, $\lim_{N\to\infty} \left\| E\left(\cup_{n=1}^{\infty} A_n\right) v -\right.$ 0).

Theorem 3.11 (Stone-Naimark-Ambrose-Godement Theorem)**.** *Let $(\pi, \mathcal{H})$ be a unitary representation of A. There exists a unique projection-valued measure P on $\widehat{A}$ such that*

$$\pi(a) = \int_{\widehat{A}} \overline{\chi(a)}\, dE(\chi)$$

In particular, for any $v \in \mathcal{H}$, the measure $\mu_{v,v}$ associated to the positive type function $\langle \pi(a)v, v\rangle$ satisfies

$$\langle \pi(a)v, v\rangle = \int_{\widehat{A}} \overline{\chi(a)}\, d\mu_{v,v}(\chi)$$

where $\mu_{v,v}(E) = \langle P(E)v, v\rangle$.

Proof. We construct $P(E)$ by lifting the uniqueness of the Fourier transform from measures to operators.

1. Construction of scalar measures: Let $v \in \mathcal{H}$. The function $\phi_v(a) = \langle \pi(a)v, v\rangle$ is of positive type with $\phi_v(e) = \|v\|^2$. By Bochner's Theorem, there is a unique finite measure $\mu_{v,v}$ on $\widehat{A}$ such that $\langle \pi(a)v, v\rangle = \widehat{\mu_{v,v}}(a)$. By polarization, for any $v, w \in \mathcal{H}$, we define the complex measure $\mu_{v,w}$ such that $\langle \pi(a)v, w\rangle = \int \overline{\chi(a)} d\mu_{v,w}$.

2. Construction of operators $P(E)$: Fix a Borel set $E \subset \widehat{A}$. Consider the map $B_E : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ given by $B_E(v, w) = \mu_{v,w}(E)$. This is a bounded sesquilinear form. By the Riesz Representation Theorem for operators, there exists a unique bounded operator $P(E)$ such that $\langle P(E)v, w\rangle = \mu_{v,w}(E)$.

3. Verifying $P(E)$ is a Projection: We must show $P(E)^2 = P(E)$ and $P(E)^* = P(E)$.

1. **Self-adjointness:** $\mu_{w,v} = \overline{\mu_{v,w}}$ (uniqueness of Fourier transform of $\overline{\langle \pi(a)v, w\rangle} = \langle \pi(a^{-1})w, v\rangle$). Thus $\langle P(E)w, v\rangle = \overline{\langle P(E)v, w\rangle}$, implying $P(E)^* = P(E)$.

2. **Multiplicativity:** We show $P(E \cap F) = P(E)P(F)$ using the following exercise:

   Exercise 3.12. the trigonometric polynomials on $\widehat{A}$ defined as $p(\chi) = \sum_{i=0}^{n} c_i \chi(a_i)$ for $a_i \in A, c_i \in \mathbb{C}$ are dense in $C(A)$. *hint: Stone-Weirstrass.*

   Note that for all $a, b \in A$:

   $$\int\limits_{\widehat{A}} \overline{\chi(a)\chi(b)} \mathrm{d}P(\chi) = \pi(ab) = \pi(a)\pi(b) = \int\limits_{\widehat{A}} \overline{\chi(a)} \mathrm{d}P(\chi) \int\limits_{\widehat{A}} \overline{\chi(b)} \mathrm{d}P(\chi)$$

   and by linearity and continuity, for any $f, g \in C\left(\widehat{A}\right)$:

   $$\int\limits_{\widehat{A}} fg dP = \int\limits_{\widehat{A}} f dP \int\limits_{\widehat{A}} g dP$$

   So, approximating $\mathbf{1}_E$ and $\mathbf{1}_F$, one obtains $P(E \cap F) = P(E) \cdot P(F)$.

Thus P is a projection-valued measure. □

Corollary 3.13. *Denote by $\mu_v = \mu_{v,v}$ for all $v \in \mathcal{H}$. Then: $\mu_v(E) = \|P(E)v\|^2$, and so: If $\mu_v \perp \mu_w$, then $v \perp w$, $\mu_{P(E)v}(F) = \mu_v(E \cap F)$.*

Remark 3.14. Since A is abelian, $P(E) \cdot \pi(a) = \pi(a) \cdot P(E)$ for all $a \in A, B \in \mathcal{B}\left(\widehat{A}\right)$, meaning that $\pi = \pi\restriction_{\mathrm{ImP(E)}} \oplus \pi\restriction_{\ker P(E)}$. In particular, $P(\{\chi\}) \neq 0$ implies that there exists $\chi \in \mathcal{H}$ s.t. $\pi(g)v = \chi(g)v$ for all $g \in G$.

Example 3.15 (Concrete Spectral Decomposition)**.**

1. **Case $A = \mathbb{R}^n$:** The dual is $\widehat{A} \cong \mathbb{R}^n$ with pairing $\chi_\xi(x) = e^{2\pi i \xi \cdot x}$. For any representation π of $\mathbb{R}^n$, there is a projection-valued measure P on $\mathbb{R}^n$ such that $\pi(x) = \int_{\mathbb{R}^n} e^{-2\pi i \langle \xi, x\rangle} dP(\xi)$.

2. **Case $A = S^1 \cong \mathbb{R}/\mathbb{Z}$:** The dual is $\widehat{A} \cong \mathbb{Z}$. Any representation decomposes as a direct sum $\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}_n$ where $\pi(\theta)v = e^{2\pi i n\theta} v$ for $v \in \mathcal{H}_n$.

3.3 Effective Property (T) for $\mathrm{SL}_3(\mathbb{R})$

We now prove that $\mathrm{SL}_3(\mathbb{R})$ has Property (T) by showing effective mixing for representations without invariant vectors. This proof relies on the interaction between the abelian unipotent subgroups and the diagonal subgroup.

Theorem 3.16. *Let $G = \mathrm{SL}_3(\mathbb{R})$. There exist $C > 0$ such that for any unitary representation $(\pi, \mathcal{H})$ without G-invariant vectors, and any C^1-smooth vectors $v, w \in \mathcal{H}$,*

$$|\langle \pi(g)v, w\rangle| \leq C\|g\|_{HS}^{-\frac{3}{8}} \mathcal{S}(v)\,\mathcal{S}(w)$$

with $\mathcal{S}$ being the sobolev norm.

In the proof, we shall consider the embedded subgroup

$$\mathrm{ASL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 = \left\{ \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} \mid g \in \mathrm{SL}_2(\mathbb{R}), x \in \mathbb{R}^2 \right\}$$

- the group of affine transformations on $\mathbb{R}^2$, with subgroups $L = \mathrm{SL}_2(\mathbb{R})\,(x = 0), U = \mathbb{R}^2\,(g = I)$ accordingly. We will apply spectral decomposition for U and $K = \mathrm{SO}_2(\mathbb{R}) < \mathrm{SL}_2(\mathbb{R})$. Let P be the spectral measure of the restriction $\pi|_U$ on the dual space $\widehat{U} \cong \mathbb{R}^2$. Denote by $\mu_v = \langle P(v), v\rangle$ v's spectral measure. for $l \in L$ denote by θ_l the inner automorphism induced by l: $\theta_l(x) = (l, 0)(\mathrm{Id}, x)\left(l^{-1}, 0\right) = (\mathrm{Id}, lx) \in U$. Also:

$$\widehat{\mu_{\pi(l)v}}(x) = \langle \pi(x)\pi(l)v, \pi(l)v\rangle = \langle \pi(\theta_{l^{-1}}x)v, v\rangle = \langle \pi\left(l^{-1}\cdot x\right)v, v\rangle = \widehat{\mu_v}\left(l^{-1}\cdot x\right)$$

implying that

$$\int_{\mathbb{R}^2} e^{-2\pi i\langle \xi, x\rangle} d\mu_{\pi(l)v}(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i\langle \xi, l^{-1}x\rangle} d\mu_v(\xi) = \int_{\mathbb{R}^2} e^{-2\pi i\langle \xi, x\rangle} d\mu_v(\left(l^t\right)\xi)$$

and so overall $\mu_{\pi(l)v}(E) = \mu_v(\left(l^t\right)E)$.

Define $a_t = \begin{pmatrix} e^{\frac{t}{2}} & \\ & e^{-\frac{t}{2}} \end{pmatrix}$ for $t \in \mathbb{R}$.

Finally, we will use the following result:

Exercise 3.17 (Mautner Phenomenon for $\mathrm{SL}_3(\mathbb{R})$)**.** if v is U-invariant then it is G-invariant.

Corollary 3.18.

$$P(\{0\}) = 0$$

.

Step 1: Treating K-eigenvectors Assume that v, w are K-eigenvector. By spectral decomposition of $\widehat{K} = \mathbb{Z}$, we get that $\pi(r_\theta)v = e^{in\theta}v$ and $\pi(r_\theta)w = e^{im\theta}w$, meaning

$$\mu_v(E) = \mu_{\pi(r_\theta)v}(E) = \mu_v(r_{-\theta}E)$$

, making μ_v invariant under rotations in $\widehat{U}$, meaning

$$\mu_v(\{(r\cos\theta, r\sin\theta) \mid r > 0, \theta \in A\}) = m_{S^1}(A)$$

The same is true for w.

Lemma 3.19. *There exist $C > 0$ independent of π v, w K-eigenvectors and $t \in \mathbb{R}$:*

$$\langle \pi(a_t)v, w\rangle \le C \cdot e^{-\frac{|t|}{4}}\|v\|\|w\|$$

Proof. WLOG $t > 0$. Take $B_t = \{(x_1, x_2) \mid \left|\frac{x_2}{x_1}\right| \le e^{-\frac{t}{2}}\}$ the cone of angle $e^{-\frac{t}{2}}$. Note that

$$\mu_v(B_t) = m_{S^1}\left(\{\theta \mid |\tan(\theta)| < e^{-\frac{t}{2}}\}\right) \le c \cdot e^{-\frac{t}{2}}$$

since $\theta \le \tan(\theta)$ for small enough θ. Now take $v_1 = P(B_t)v$ and $w_1 = P\left(B_t^-\right)w$ with $B_t^- = \{(x_1, x_2) \mid \left|\frac{x_2}{x_1}\right| \ge e^{\frac{t}{2}}\}$ and $a_tB_t^- = \{(x_1, x_2) \mid \left|\frac{x_2}{x_1}\right| \ge e^{-\frac{t}{2}}\}$. So:

$$\mu_{\pi(a_t)v_1}\left(B_t^-\right) = \mu_v\left(a_tB_t^- \cap B_t\right) = 0$$

So $\mu_{\pi(a_t)v_1} \perp \mu_{w_1}$, implying $\pi(a_t)v_1 \perp w_1$. Overall:

$$|\langle \pi(a_t)v, w\rangle| \le |\langle \pi(a_t)v_1, w_1\rangle| + |\langle \pi(a_t)v, w_1\rangle| + |\langle \pi(a_t)v_1, w\rangle| \le 2\sqrt{c}\cdot e^{-\frac{t}{4}}$$

□

Step 2: Generalizing to smooth vectors and L Now let $v, w \in \mathcal{H}^1$ be C^1-smooth vectors.

Proposition 3.20. *There exists* $C' > 0$ *independent of the representation s.t. for* $l \in L$ *and* $B \in \mathcal{L}(\mathcal{H})$ *a linear operator s.t.* B *commutes with* $\pi(k)$ *for all* $k \in K$*:*

$$\langle \pi(l) Bv, w \rangle \leq C' \cdot \|B\|_{op} \|l\|_{HS}^{-\frac{1}{2}} \mathcal{S}(v) \mathcal{S}(w)$$

Proof. First, use the cartan decomposition $l = k_1 a_t k_2$ with $k_1, k_2 \in K, t \in \mathbb{R}$. Since K is compact, $c_2 \leq \|k\|_{HS} < C_2$, and so $c_2 \leq \frac{\|k_2 v\|}{\|v\|}, \frac{\|k_1^{-1} w\|}{\|w\|} \leq C_2$, and so up to a constant one may assume that $l = a_t$. Recall that since $\pi \restriction_K$ is also a unitary representation, $\mathcal{H}$ may be decomposed into $\mathcal{H} = \bigoplus_{n \in \mathcal{Z}} \mathcal{H}_n$ with

$$\pi(r_\theta) \restriction_{\mathcal{H}_n} = \exp(in\theta) \operatorname{Id}_{\mathcal{H}_n}$$

Also, v, w can both be decomposed into the sum of orthogonal vectors $v = \sum_{n \in \mathbb{Z}} v_n$ with $d\pi(\Theta) v_n = \frac{d}{dt} |_{t=0} \pi(r_\theta) v = inv_n$, where $\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \exp(t\Theta) = r_t$. So:

$$|\langle \pi(a_t) v, w \rangle| = \left| \sum_{m,n \in \mathbb{Z}} \langle \pi(a_t) v_n, v_m \rangle \right| \leq C_1 e^{-\frac{|t|}{4}} \sum_{n \in \mathbb{Z}} \|v_n\| \sum_{m \in \mathbb{Z}} \|w_m\|$$

With C_1 being the constant from Lemma 3.19. Now note that:

$$\sum_{n \in \mathbb{Z}} \|v_n\| = \|v_0\| + \sum_{n \in \mathbb{Z} \backslash \{0\}} \frac{1}{n} \|d\pi(\Theta) v_n\|$$

$$\leq \|v_0\| + \sqrt{\sum_{n \in \mathbb{Z} \backslash \{0\}} \frac{1}{n^2}} \sqrt{\sum_{n \in \mathbb{Z} \backslash \{0\}} \|d\pi(\Theta) v_n\|^2} = \|v_0\| + \frac{\pi}{\sqrt{6}} \|d\pi(\Theta) v\| \leq 3\mathcal{S}(v)$$

Also, note that since B commutes with all K, it acts on any $\mathcal{H}_n$, and so it multiplies the result by at most $\|B\|_{op}$. So overall:

$$\langle \pi(l) Bv, w \rangle \leq 9 \cdot C_2 C_1 \cdot \|B\|_{op} \|l\|_{HS}^{-\frac{1}{2}} \mathcal{S}(v) \mathcal{S}(w)$$

□

Step 3: Generalizing to all of G

Proof. Let $g \in G$ and let v, w be C^1 smooth vectors. use the Cartan decomposition of g to get $g = k_1 a k_2$, now with $k_1, k_2 \in \mathrm{SO}_3(\mathbb{R})$ and $a = \operatorname{diag}(e^{t_1}, e^{t_2}, e^{t_3})$. Since $\mathrm{SO}_3(\mathbb{R})$ is compact, one may assume that $k_1 = k_2 = \mathrm{Id}$ and enlarge the constanct by a bit. Now, note that:

$$a = b a_{t_1 + \frac{t_3}{2}}$$

with $b = \operatorname{diag}\left(e^{\frac{-t_3}{2}}, e^{\frac{-t_3}{2}}, e^{t_3}\right)$. Note that b commutes with K, meaning $\pi(b)$ commutes with $\pi(k)$ for all $k \in K$, meaning:

$$\langle \pi(g) v, w \rangle \leq C \cdot e^{-\left|\frac{t_1 - t_2}{4}\right|} \mathcal{S}(v) \mathcal{S}(w)$$

By conjugating g with permutation matrices, we get that:

$$\langle \pi(g) v, w \rangle \leq C \cdot e^{-\left|\frac{t_i - t_j}{4}\right|} \mathcal{S}(v) \mathcal{S}(w)$$

as well. WLOG $t_1 \leq t_2 \leq t_3$, so: $-|t_3 - t_1| = t_1 - t_3 \leq -\frac{3}{2}t_3$ and recall that $\|g\|_{HS} = \Theta\left(e^{t_3}\right)$, so:

$$\langle \pi(g) v, w \rangle \leq C \cdot \|g\|_{HS}^{-\frac{3}{8}} \mathcal{S}(v) \mathcal{S}(w)$$

□

Remark 3.21. In general, property (T) for semisimple Lie groups implies uniform effective mixing for smooth vectors.

1. Introduction & Section 1: Spectral Gap (15–20 Minutes)

Goal: Establish the intuition of mixing and the rigorous analytical definitions.

- **Introduction (5–7 mins):**
  - Motivation: "Efficiency" of group actions mixing vectors.
  - Setup: Locally compact groups, unitary representations.
- **Definitions (5–7 mins):**
  - (Q, ε)-invariant vectors vs. Almost Invariant Vectors.
  - **Crucial Concept:** The definition of the **Spectral Gap** (the *absence* of almost invariant vectors in the complement of the trivial rep).
- **Equivalence of Characterizations (5–7 mins):**
  - Defining the convolution operator $\pi_*(f)$, and only stating the equivalence of definitions.

2. Section 2: Property (T) (15 Minutes)

Goal: Define the rigidity property and show its immediate algebraic consequences.

- **Definition (5 mins):**
  - Kazhdan's Property (T) definition.
  - Relation to Sec 1: "Every representation without invariant vectors has a spectral gap."
- **Implications & Corollary (10 mins):**
  - List the implications: Amenability implies compactness, compact generation, lattice inheritance, compact abelization, unimodularity.

3. Section 3: Effective Proof for $\mathrm{SL}_3(\mathbb{R})$ (45–50 Minutes)

Goal: The technical core. Moving from abstract existence to concrete mixing estimates.

- **3.1 Smooth Vectors & Sobolev Norms (5–10 mins):**
  - Quick definition of C^∞ vectors and the Sobolev norm setup.
  - *This should be the end of the first academic hour.*
- **3.2 Unitary Reps of Abelian Groups (10 mins):**
  - **SNAG Theorem:** Skip proof and general case.
  - *Focus on the result and only sketch proof: projection-valued measures allow us to decompose the representation.*
- **3.3 Proof of the effective property (T) (25–30 min):**
  - **The Setup:** Introduce $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$.

- **Step 1 (The Geometric Core):** This is the most critical part of the lecture. Explain the "Cone" argument (Lemma 3.14). Draw the cone B_t and its image under the flow to show how the measures become orthogonal.
- **Step 2 & 3 (Generalization):** Briefly explain how to move from K-eigenvectors to smooth vectors (Sobolev norms appear here) and how Cartan decomposition extends this to the whole group G.

Pacing Strategy & Shortcuts

If you only have **60 Minutes**:

1. **Section 1:** Compress the definitions. Skip the proof of "Equivalence of Characterizations" and just state the result.
2. **Section 2:** State the definition of Property (T) and list the Corollaries without proof.
3. **Section 3.2: Skip the proof of the SNAG theorem entirely.** State it as a "Black Box" tool: *Abelian representations decompose into integrals over the dual group.*
4. **Focus:** Spend the majority of your time on **Section 3.3**, specifically the **Cone Argument (Lemma 3.14)**, as that is the specific mechanism that produces the decay estimate for $SL_3(\mathbb{R})$.

References

[1] Einsiedler, M., & Ward, T. (2017). *Unitary Representations and Unitary Duals.* Springer Graduate Texts in Mathematics.

[2] Bekka, B., de la Harpe, P., & Valette, A. (2008). *Kazhdan's Property (T).* Cambridge University Press.

[3] Howe, R., & Moore, C. C. (1979). "Asymptotic properties of unitary representations". *Journal of Functional Analysis.*

[4] Generated with the assistance of **NotebookLM** (Google).