HAAR MEASURE ON $SL_2(\mathbb{R})$, DISCRETE SUBGROUPS AND $SL_2(\mathbb{Z})$

ODED ELISHA

1. Lie Groups

1.1. **Definition.** G is called Lie group if G has both group structure and smooth manifold structure such that

$$(g_1, g_2) \mapsto g_1 \cdot g_2$$
$$g \mapsto g^{-1}$$

are smooth.

1.2. **Definition.** Let G be a Lie group then we define its Lie algebra $\mathfrak{g} = \mathrm{Lie}(g) \coloneqq T_1G$. One naturally identifies it with the left-invariant vectors fields

$$\phi : \mathfrak{g} \to X(G)$$
$$\phi(h)(g) = dL_g(h)$$

where $L_g(g') \coloneqq gg'$.

A left-invariant vector field is a vector field $v \in \mathfrak{X}(G)$ such that

$$dL_g(h) \cdot v(h) = v(L_g h)\,.$$

1.3. *Exercise.* Show ϕ is a linear isomorphism between T_1G and left-invariant vector fields.

1.4. **Definition.** We define the exponent function $\exp : \mathfrak{g} \to G$ by

$$\frac{d}{dt}\exp(tg) = g$$

which is a solution of ODE and this is a well-defined map. Moreover

$$d_0 \exp = \mathrm{Id}$$

so by the inverse function theorem we have a small neighborhood of the identity element where $\log : G \to \mathfrak{g}$ is defined and is exp's inverse.

1.5. **Definition.** Given $h_1, h_2 \in \mathfrak{g}$ we define the commutator

$$[h_1, h_2] = \frac{d}{dt}_{|t=0}\frac{d}{ds}_{|s=0} \exp(th_1) \cdot \exp(sh_2) \cdot \exp(th_1)^{-1}$$

which has the following properties:

(1) $[\cdot, \cdot]$ is bi-linear

(2) $[h_1, h_2] = -[h_2, h_1]$

(3) $[h_1, [h_2, h_3]] + [h_3, [h_1, h_2]] + [h_2, [h_3, h_1]] = 0$

making $\mathfrak{g}$ a *lie algebra.*

1.6. **Proposition.** *Define* $GL_n(\mathbb{R}) := \{A \in Mat_{n\times n}(\mathbb{R}) \mid \det A \neq 0\}$. *Then this is a Lie group with*

(1) $Lie(GL_n(\mathbb{R})) = Mat_{n\times n}(\mathbb{R})$
(2) $[A, B] = AB - BA$
(3) $\exp(tA) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$
(4) $\det(\exp A) = e^{trA}$

1.7. *Exercise.* Prove the proposition.

1.8. **Definition.** For a subgroup $H \leq G$ of Lie group G we say that H is a Lie subgroup if it has a structure of a Lie group. Moreover it is closed Lie subgroup if it is closed in G's topology.

1.9. *Remark.* It is a known non-trivial theorem that closed subgroups of a given Lie group are closed Lie subgroups.

1.10. **Definition.** The adjoint representation of a Lie group G is defined by

$$Ad : G \to \text{Aut}(\text{Lie}(G)) \subseteq GL_n(\mathbb{R})$$
$$Ad(A) = Ad_A$$
$$Ad_A : \text{Lie}(G) \to \text{Lie}(G)$$
$$Ad_A(h) := \frac{d}{dt}_{|t=0} A\exp(th)A^{-1}$$

This is a $G-$representation.

1.11. *Exercise.* Prove this is a linear representation and calculate its kernel for $SL_2(\mathbb{R})$.

1.12. **Corollary.** *Adjoint representation is a homomorphism of Lie groups so its differential at identity element gives us homomorphism*

$$ad = dAd$$
$$ad : g \to Lie(Aut(Lie(G))) \subseteq Mat_{n\times n}(\mathbb{R})$$
$$ad(g)(h) := \frac{d}{dt}_{|t=0} Ad_{\exp(tg)}(h) = \frac{d}{dt}_{|t=0} \frac{d}{ds}_{|s=0} \exp(tg) \cdot \exp(sh) \cdot \exp(tg)^{-1} = [g, h]$$

Moreover a common notation is $ad_g(h) := [g, h]$.

2. Properties of $SL_2(\mathbb{R})$

2.1. **Definition.**

$$SL_2(\mathbb{R}) := \{A \in GL_2(\mathbb{R}) \mid \det A = 1\}$$

2.2. **Proposition.** $SL_2(\mathbb{R})$ *is a Lie group.*

Proof. $SL_2(\mathbb{R})$ is a subgroup and closed because det is continuous and $SL_2(\mathbb{R}) = \det^{-1}(1)$ hence by the non-trivial theorem we mentioned it is a Lie subgroup of $GL_2(\mathbb{R})$. □

2.3. *Exercise.* Show that lie group homomorphisms (a smooth map and group homomorphism between Lie groups) always have constant rank (of the differential). Use that fact to prove this proposition using the implicit function theorem on the map $A \mapsto \det A - 1$.

2.4. **Proposition.** *The Lie group $SL_2(\mathbb{R})$ is*

(1) Non-compact

(2) Connected

Proof. The matrices $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ belong to $SL_2(\mathbb{R})$ for every $n \in \mathbb{N}$. Hence $SL_2(\mathbb{R})$ is not bounded hence not compact. For connectedness [1] first define the subgroup

$$A \coloneqq \{\begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \mid r > 0\} = \{\exp(t\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \mid t \in \mathbb{R}\}$$

And the map

$$\psi : SO(2) \times A \times SO(2) \to SL_2(\mathbb{R})$$
$$\psi(z_1, r, z_2) = z_1 r z_2$$

We have that ψ is continuous and we identified $SO_2(\mathbb{R})$ (rotation matrices) with S^1. Let $B \in SL_2(\mathbb{R})$ then we have polar decomposition $B = OP$ where $O \in O_2(\mathbb{R})$ and P is positive semi-definite symmetric matrix. But $B \in SL_2(\mathbb{R})$ so $O \in SO_2(\mathbb{R})$ and P is positive symmetric matrix with determinant 1. So by the spectral theorem we have $V \in SO_2(\mathbb{R})$ such that $V^{-1}PV = \mathrm{diag}(r, \frac{1}{r})$ hence $B = (OV^{-1})\mathrm{diag}(r, \frac{1}{r})V$ giving that ψ is surjective. But S^1 is connected and $A \cong (0, \infty)$ so the domain of ψ is connected hence so is $SL_2(\mathbb{R})$. □

2.5. **Proposition.** $sl_2(\mathbb{R}) \coloneqq Lie(SL_2(\mathbb{R})) = \{A \in M_{n\times n}(\mathbb{R}) \mid trA = 0\}$

Proof. Let $\gamma : (-\epsilon, \epsilon) \to SL_2(\mathbb{R})$ be a smooth path with $\gamma(0) = \mathrm{Id}$ then $\det \gamma(t) = 1$ thus

$$0 = \frac{d}{dt}_{t=0} \det \gamma(t) = \mathrm{tr}\, \gamma'(0)$$

. In the other direction for $A \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ with $\mathrm{tr}(A) = 0$ we have

$$\det \exp(tA) = e^{\mathrm{tr(tA)}} = 0$$
$$\Rightarrow A \in sl_2(\mathbb{R})$$

□

2.6. **Definition.** The adjoint representation of a $SL_2(\mathbb{R})$ is

$$Ad_A : sl_2(\mathbb{R}) \to sl_2(\mathbb{R})$$
$$Ad_A(h) \coloneqq \frac{d}{dt}_{|t=0} A \exp(th) A^{-1} = \frac{d}{dt}_{|t=0} \exp(tAhA^{-1}) = AhA^{-1}$$

3. Haar measure on $SL_2(\mathbb{R})$

3.1. **Theorem.** *(Haar) Let G be a Lie group. Then there exists locally finite regular left-invariant Borel measure m_G, which is unique up to positive scaling. Such a measure is called a (left-invariant) Haar measure on G.*

(1) $m_G(A) = m_G(L_g A)$ *for all* $g \in G$ *and* $A \in \sigma_g$ *(the Borel sigma algebra of* G*)*

(2) Every $g \in G$ *has an open neighborhood* $g \in U$ *such that* $m_G(U) < \infty$ *(Locally finite)*

(3) (Regular) $m_G(A) = \sup_{K_{compact} \subseteq A, K \in \sigma_g} m_G(K) = \inf_{A \subseteq U \in \sigma_g, U \ open} m_G(U)$

1. *Note.* A similar definition is given for right-invariant Haar measure. A group which has a measure m that is both left and right Haar measure is called *unimodular*.

3.2. *Exercise.* Show that any Lie group has a left or right invariant Riemannian metric, and conclude that the induced measure from the corresponding riemannian volume gives a Haar measure.

3.3. **Proposition.** *A Haar measure on* $SL_2(\mathbb{R})$ *is given by*

$$m_{SL_2(\mathbb{R})}(S) = m_{\mathbb{R}^4}([0,1] \cdot S)$$

where $m_{\mathbb{R}^4}$ *is the standard Lebesgue measure on* $\mathbb{R}^4$ *and*

$$[a,b] \cdot S := \{\lambda A \mid A \in S, \lambda \in [a,b]\}$$

(we use this notation for general intervals too).

Proof. Define the function $g : \det^{-1}((0,1]) \to SL_2(\mathbb{R})$ by

$$g(A) = \frac{1}{\sqrt{\det A}} A$$

Let $m(S) = m_{\mathbb{R}^4}([0,1] \cdot S) = m_{\mathbb{R}^4}((0,1] \cdot S)$. This is a Borel measure because g is measurable (it is continuous) and $m = g_*(\mu_{\mathbb{R}^4})$ by definition of push-forward measure. It is left invariant because if $B \in SL_2(\mathbb{R})$ then

$$m_{SL_2\mathbb{R}}(BS) = m_{\mathbb{R}^4}([0,1] \cdot BS) = m_{\mathbb{R}^4}(B([0,1] \cdot S)) \overset{\det B = 1}{=} m_{\mathbb{R}^4}([0,1] \cdot S) = m_{SL_2(\mathbb{R})}(S)$$

. For locally-finite if $B_\epsilon(A) \cap SL_2(\mathbb{R})$ is a neighborhood of $A \in SL_2(\mathbb{R})$ then $m_{SL_2\mathbb{R}}(B_\epsilon(A) \cap SL_2(\mathbb{R})) = m_{\mathbb{R}^4}([0,1] \cdot (B_\epsilon \cap SL_2(\mathbb{R}))) \leq m_{\mathbb{R}^4}([0,1] \cdot B_\epsilon(A)) < \infty$. Pushforward of outer-regular is outer-regular. $\square$

3.4. *Exercise.* Show this measure is an inner-regular measure, finishing the proof that this is a Haar measure. Moreover, show that this measure is right-invariant, so that $SL_2(\mathbb{R})$ is unimodular.

3.5. *Exercise.* In any Haar measure $m_{SL_2(\mathbb{R})}$ on $SL_2(\mathbb{R})$ we have

$$m_{SL_2(\mathbb{R})}(SL_2(\mathbb{R})) = \infty$$

.

4. Discrete subgroups

4.1. **Definition.** A subgroup $H \leq G$ of a Lie group is called discrete if it is discrete as topological subspace. That is for every $h \in H$ there's an open neighborhood $h \in U$ such that $U \cap G = \{h\}$.

4.2. **Proposition.** *Let H be a discrete subgroup. We define the right-coset space by*

$$H\backslash G := \{H \cdot g \mid g \in G\}$$

with the quotient topology. Given an invariant metric d on G (which is a metric such that $d(g_1, g_2) = d(g'g_1, g'g_2)\,\forall g' \in G$) we can define a metric on the right-coset space by

$$\bar{d}(Hg_1, Hg_2) := \inf_{h_1,h_2 \in H} d(h_1g_1, h_2g_2) = \inf_{h \in H} d(g_1, hg_2)$$

4.3. *Exercise.* Show this is a well-defined metric, use the fact that H is discrete and d is invariant.

4.4. **Proposition.** *The quotient topology coincides with the topology induced by the metric. Moreover, the quotient is locally isometric (via the action) to G. That means that for every Hg there is $r > 0$ such that*

$$B_G(1, r) \tilde{\to} B_{H\backslash G}(Hg, r)$$
$$h \mapsto H(gh)$$

Motivation: the quotient space looks locally like G but can have nicer properties.

Proof. [2][Proposition 9.14]. □

4.5. **Definition.** A discrete subgroup $L \leq G$ is said to have a fundamental domain if there exists $F \subseteq G$ and

(1) F is a measurable (i.e a Borel set)
(2) For every $g \in G$ we have $|Lg \cap F| = 1$.

If the quotient admits finite-measure fundamental domain with respect to a Haar measure we say that L is a *lattice*.

If the quotient is compact we say that L is a *uniform lattice*.

4.6. *Exercise.* Prove that a uniform lattice is a lattice.

4.7. **Proposition.** *If the fundamental domain of F has finite measure then every two fundamental domains of $L \leq G$ have the same (Haar) measure and G is unimodular.*

Proof. [2][Proposition 9.20]. □

4.1. **Unimodular Lattices and $SL_2(Z)$.**

4.8. **Proposition.** *$L \subseteq (\mathbb{R}^n, +)$ is a lattice iff there exists linearly independent $v_1, \dots, v_n \in L$ such that $L = span_{\mathbb{Z}}\{v_1, \dots, v_n\}$*

4.9. **Definition.** For a lattice $L \subseteq \mathbb{R}^n$, taking a set of $\mathbb{Z}-$generators $v_1, \dots, v_n \in L$ we can write $L = A\mathbb{Z}^n \subseteq \mathbb{R}^n$ where

$$A := \begin{pmatrix} | & \dots & | \\ v_1 & \dots & v_n \\ | & \dots & | \end{pmatrix}$$

. A lattice is called *unimodular* if $\text{covol}(A) := |\det A| = 1$.

4.10. *Exercise.* Show that the *covol* is an invariant of the lattice.

4.11. **Proposition.** *We have an identification*

$$\{L \subseteq \mathbb{R}^n \mid L \text{ is a full dimensional lattice}\} \xrightarrow{\sim} GL_n(\mathbb{R}) \big/ SL_n(\mathbb{Z})$$
$$\{L \subseteq \mathbb{R}^n \mid L \text{ is a full dimensional lattice and } covol(A) = 1\} \xrightarrow{\sim} SL_n(\mathbb{R}) \big/ SL_n(\mathbb{Z})$$

4.12. *Exercise.* Prove this proposition.

5. Group actions

5.1. **Definition.** A group action of a group G on a (Hausdorff) topological space X is group homomorphism

$$\rho : G \to \text{Homeo}(X)$$

often denoted $g \cdot x \coloneqq \rho(g)(x)$. When G is a topological space, it is called continuous group action when the action map

$$f : G \times X \to X$$
$$f(g, x) = \rho(g)(x) = g \cdot x$$

is continuous. When G is a Lie group then a smooth group action is when f is smooth.

5.2. **Definition.** Let $x \in X$ where G acts on X. We define the stabilizer of x or isotropy group of x by

$$G_x \coloneqq \{g \in G \mid g \cdot x = x\}$$

.

5.3. *Exercise.* Prove G_x is always a (closed) subgroup of G and give example for when it is not a normal subgroup.

5.4. **Definition.** An action is called *faithful* if $\ker \rho = \{1\}$.

An action is called *transitive* if for every $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$.

An action is called *free* if $G_x = 1$ for every $x \in X$.

5.5. **Definition.** A pair (X, G) is called a *homogeneous space* when G is a Lie group acting smoothly and transitively on X.

5.6. **Proposition.** *Let (X, G) be a homogeneous space then there is a bijection*

$$G \big/ G_x \xrightarrow{\sim} X$$

given by $gG_x \mapsto g \cdot x$. When G is LCSC topological group and the action is continuous then this is an homeomorphism. In that case G_x is a closed Lie subgroup and there is a known theorem that says that $G \big/ G_x$ has a smooth manifold structure, thus any subgroup $N \leq G$ defines an homogeneous space $(G \big/ N, G)$.

5.7. *Exercise.* Prove this proposition when G or $G \big/ G_x$ is compact.

6. $SL_2(\mathbb{R})$ and Hyperbolic half-plane

6.1. **Proposition.** *A Riemannian metric g on a smooth manifold M is defined by giving for every $x \in M$ a smooth inner-product $\langle \cdot, \cdot \rangle_g$ on its tangent space $T_x M$. In a sense that choosing any local chart near x gives smooth coordinate functions for the metric.*

6.2. **Definition.** Given a Riemannian metric we can define length of a curve $\gamma : [a, b] \to M$ by

$$\mathrm{Len}(\gamma) = \int_a^b \sqrt{\langle \gamma'(t), \gamma'(t) \rangle}\, dt$$

and then define the induced distance function between points $x, y \in M$

$$d(x, y) = \inf_{\gamma(0)=a, \gamma(1)=b} \mathrm{Len}(\gamma)$$

which is a metric.

6.3. *Exercise.* Show this is always a metric.

6.4. **Proposition.** *One defines the hyperbolic riemannian metric on $\mathbb{H}$ by*

$$\langle u, v \rangle_{(x,y)} = \frac{u \cdot v}{y^2}$$

We have an isometric transitive action of $SL_2(\mathbb{R})$ on $\mathbb{H}$ via Mobius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

That is each Mobius transformation preserves the Riemannian metric

$$\langle u, v \rangle_z = \langle d_z f \cdot u, d_z f \cdot v \rangle_{f(z)}$$

6.5. *Exercise.* Prove this proposition.

6.6. **Definition.** We define the unit tangent bundle of $\mathbb{H}$ by

$$T^1\mathbb{H} \coloneqq \{(x, v) \mid x \in H, v \in T_x\mathbb{H},\, |v|_g = 1\}$$

Note that the norm is taken with respect to the hyperbolic metric.

6.7. **Corollary.** *Define $PSL_2(\mathbb{R}) = {SL_2(\mathbb{R})} \big/ {\{\pm 1\}}$ then there is a natural identification*

$$PSL_2(\mathbb{R}) \xrightarrow{\sim} T^1\mathbb{H}$$

given by

$$g \mapsto (g(i), d_i g(i))$$

6.8. *Exercise.* Show that $PSL_2(\mathbb{R})$ acts freely and transitively on $T^1\mathbb{H}$ and conclude by Proposition 5.6.

6.1. **Horocycle and Geodesic flows [2].**

6.9. **Definition.** Let (X, g) be a smooth manifold with Riemannian metric. Then $\gamma : I \to M$ (where I is an interval) is called *geodesic* if it is length-minimizing between $\gamma(t_0)$ and $\gamma(t_1)$ for all $t_0 < t_1$ in I. That is $\mathrm{Len}(\gamma_{[t_0,t_1]}) = d(\gamma(t_0), \gamma(t_1))$.

6.10. **Proposition.** *In $\mathbb{H}$ with the hyperbolic metric, the following path is a geodesic*

$$\gamma : (-\infty, \infty) \to \mathbb{H}$$
$$\gamma(t) = ie^t$$

6.11. *Exercise.* Prove this.

6.12. **Proposition.** *The isometric action of $SL_2(\mathbb{R})$ maps geodesics to geodesics. Hence the free and transitive action of $PSL_2(\mathbb{R})$ on the unit tangent bundle $T^1\mathbb{H}$ implies that there is a unique geodesic through every point $z \in \mathbb{H}$ and any chosen unit direction $v \in T_z\mathbb{H}$. To find this geodesic for (z, v), we find a $g \in SL_2(\mathbb{R})$ such that $g \cdot (z, v) = (i, i)$ and then the geodesic is given by*

$$\gamma(t) = g \cdot (e^t i, e^t i) = (g \circ \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}) \cdot (i, i)$$

. Note this is generated by the one parameter group

$$\exp t \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$$

6.13. **Definition.** In similar fashion we define the subgroups

$$U^- = \{\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R}\} = \{\exp t \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\}$$
$$U^+ = \{\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \mid s \in \mathbb{R}\} = \{\exp t \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$$

and their corresponding flows

$$u^-(s)(z, v) = (g \circ \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}) \cdot (i, i)$$
$$u^+(s)(z, v) = (g \circ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}) \cdot (i, i)$$

where u^- is called the stable horocycle flow and u^+ is called the unstable horocycle flow. They define flow on the unit tangent bundle, and notice that our three generators - for geodesics, stable horocycle and unstable horocycle form a basis for the Lie algebra of $SL_2(\mathbb{R})$.

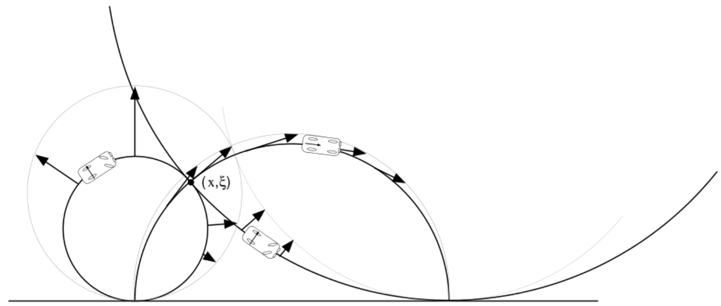

FIGURE 1. Illustration of horocycles and geodesic [3]

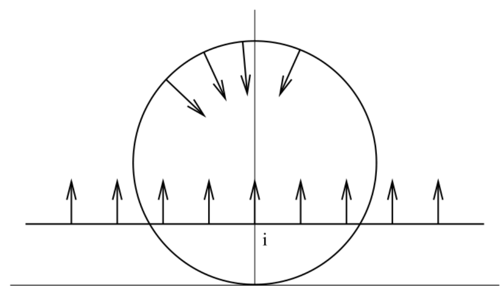

Fig. 9.3 Two particular orbits of the stable horocycle U^- on $\mathrm{T}^1\mathbb{H}$

FIGURE 2. Illustration of horocycles and geodesic at real axis [2]

7. Fundamental domain of the action of $SL_2(\mathbb{R}) \big/ SL_2(\mathbb{Z})$ on $\mathbb{H}$

7.1. **Proposition.** *[2][Proposition 9.18] We have an action of $SL_2(\mathbb{Z})$ on $\mathbb{H}$ Mobius transformations. Then a fundamental domain is*

$$F := \{z \in \mathbb{H} \mid |z| \geq 1, |\Re(z)| \leq \frac{1}{2}\}$$

. That is

(1) $\mathbb{H} = \bigcup_{g \in SL_2(\mathbb{R})} g \cdot F$

(2) $\lambda(g \cdot F \cap F) = 0$ *for all* $\pm 1 \neq g \in SL_2(\mathbb{R})$

7.2. *Exercise.* Show that our discussion gives that

$$T^1 F \cong SL_2(\mathbb{R}) \big/ SL_2(\mathbb{Z}) \,.$$

Moreover, show this is a measure preserving map, where the measure on the right is the pushforward of a Haar measure and the measure on the left is the hyperbolic measure (as defined in [2][Proposition 9.4.1]).

7.3. *Exercise.* Show that the hyperbolic measure of $T^1 F$ is finite and conclude that $SL_2(\mathbb{Z})$ is a lattice but not a uniform lattice.

References

[1] luxerhia (https://math.stackexchange.com/users/677368/luxerhia). *Prove that* $\mathrm{SL}(n, \mathbb{R})$ *is connected.* Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3241298 (version: 2023-08-07). eprint: `https://math.stackexchange.com/q/3241298`. URL: `https://math.stackexchange.com/q/3241298`.

[2] Manfred Einsiedler and Thomas Ward. *Ergodic Theory: with a view towards Number Theory*. en. London: Springer London, 2011. ISBN: 978-0-85729-020-5 978-0-85729-021-2. DOI: `10.1007/978-0-85729-021-2`. URL: `https://link.springer.com/10.1007/978-0-85729-021-2` (visited on 12/06/2025).

[3] John H. Hubbard and Robyn L. Miller. *Equidistribution of Horocyclic Flows on Complete Hyperbolic Surfaces of Finite Area*. en. arXiv:0712.1300 [math]. Mar. 2008. DOI: `10.48550/arXiv.0712.1300`. URL: `http://arxiv.org/abs/0712.1300` (visited on 12/06/2025).