Low Entropy Method

Homin Lee.

Notation

- $G = SL_3\mathbb{R}$, $\Gamma < G$ lattice, $X = G/\Gamma$.

  $A = \left\{ \begin{pmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{t_3} \end{pmatrix} \right\}$ μ : A-inv. prob msr on X.

- $U_{(13)} = \langle U_{12}, U_{13}, U_{23} \rangle = G^-$ $= \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$, $A'_{13} = \left\{ a'(\lambda) = \begin{pmatrix} e^{\lambda} & & \\ & e^{-2\lambda} & \\ & & e^{\lambda} \end{pmatrix} \right\} = A'$

  where

  $U_{ij} = \{ I + sE_{ij} : s \in \mathbb{R} \}$ $(E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$.

- $H_{13} = \langle U_{13}, U_{31} \rangle$, $L = C(U_{13}) = \left\{ \begin{pmatrix} x & * & * \\ & x^{-2} & * \\ & & x \end{pmatrix} \right\}$.

- $C(H_{13}) = \left\{ \begin{pmatrix} s & & \\ & s^{-2} & \\ & & s \end{pmatrix} \right\}$, $u(s) := \begin{pmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{pmatrix}$.

- $\alpha(t) := \begin{pmatrix} e^t & & \\ & 1 & \\ & & e^{-t} \end{pmatrix}$.

- For msble $J \subset \mathbb{R}$, $|J|$ = Leb msr of I.

- μ_x^{ij} : Leaf wise msr at $x \in X$ w.r.t. U_{ij}.

- As $U_{ij} \cong \mathbb{R}$, we will write $\mu_x^{ij}(J)$ instead of $\mu_x^{ij}(\{ I + sE_{ij} : s \in J \})$.

Thm (Low Entropy : $G = SL_3\mathbb{R}$ case)

Assume that μ_x^{13} is nontrivial and μ_x^{12}, μ_x^{23} are trivial a.e. x.

Then one of the following properties holds

① μ is U_{13} inv.

② Almost every A'_{13}-ergodic component of μ is supported on a single $C(H_{13})$-orbit.

(Recall that $C(H_{13}) \supset A'_{13} = \left\{ \begin{pmatrix} e^{\lambda} & & \\ & e^{-2\lambda} & \\ & & e^{\lambda} \end{pmatrix} : \lambda \in \mathbb{R} \right\}$.)

$C(H_{13}) = \left\{ \begin{pmatrix} s & & \\ & s^{-2} & \\ & & s \end{pmatrix} : s \in \mathbb{R} \right\}$.

Rmk 1) If ① holds then μ is Haar.
(by Ratner's thm)

2) When $\Gamma = SL_3\mathbb{Z}$, ② can not happen.

3) ② $\iff$ $\forall \varepsilon > 0$ $\exists K$ w $\mu(K) > 1 - \varepsilon$

"A'-returns to K is strong exceptional"

exceptional if $g \in L = C(U_{13})$.

s.t. $\exists \delta > 0$ so that

$\forall x, x' \in K$, $a'(s) \in A'_{13}$ w $x' = a'(s)x \in B_\delta(x) \cap K$

every $g \in B_\delta^G$ $x' = gx$ satisfies $g \in C(H_{13})$.

3') ② $\Longleftrightarrow$ $\exists\, a'(r) \in A'$ and $\exists\, x_0 \in X$

only for $Sl_3 \mathbb{R}$

w/ $a'(r)x_0 = x_0$ s.t.

μ is supp. by the orbit of x under $C(A')$.

(Especially μ a.e. x, $a'(r)x = x$).

Lemma 1. Under the same assumption,

$\exists\ \mu$-null set $N \subset X$ s.t.

a) $\forall x \in X \setminus N$,

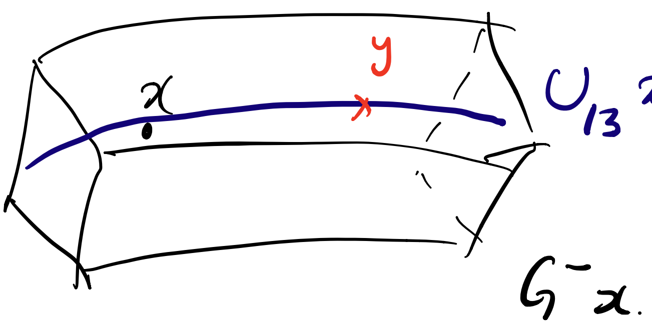
$$U_{(13)}x \cap (X \setminus N) \subset U_{13}x$$

b) Unless μ is U_{13}-inv, it can be arranged that

$$\mu_x^{13} \neq \mu_y^{13} \quad \forall x \in X \setminus N \text{ and}$$

$$\forall y \in U_{(13)}x \setminus N, \ x \neq y.$$

* In our case, $U_{(13)} = G^- = \left\{\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}\right\}$.



pf of Lemma 1.) This is basically due to product structure.

a) $\mu_x^{G^-} = \mu_x^{13} \times \mu_x^{12} \times \mu_x^{23}$. $\underbrace{\text{a.e. } x}_{\text{say } x \in X \setminus N}$ for some $\mu(N) = 0$.

By our assumption in low entropy,

$\mu_x^{G^-}$ is supported on U_{13}.

Enlarge N, if necessary, we can make that $\forall\, x, y \in N$, $y = ux \in G^- x$ satisfies

$$\mu_x^{G^-} \propto \mu_x^{G^-} u .$$

Since $\mu_x^{G^-}, \mu_y^{G^-}$ supp. on U_{13},

this implies that $u \in U_{13}$.

b) If $y \in U_{13} x$. (due to a).),

$\mu_x^{13} = \mu_y^{13}$ then μ_x^{13} is inv. under right translation by some $u \in U_{13}$.

$\leadsto$ let $B := \{ x \in X : \mu_x^{13}$ is inv. under some right translation $u \in U_{13} \}$

if $\mu(B) > 0$ then μ is inv. under U_{13}.

– Contradiction to our assumption

$\square$

Thm' Assume that $\exists N$: μ-null set $\exists \varepsilon > 0$

s.t.

$\forall K$: cpt, $K \subset X \setminus N$ w/ $\mu(K) > 1 - \varepsilon$

$\exists z \in K$, $\exists s \in \mathbb{R}$ s.t. $z' = a'(s) \in B_\delta(z) \setminus B_\delta^L(z)$

$(L = C(U_{13}))$.

Then μ is U_{13}-inv.

Rmk We can find null set N so that

$\forall K$: cpt $K \subset X \setminus N$

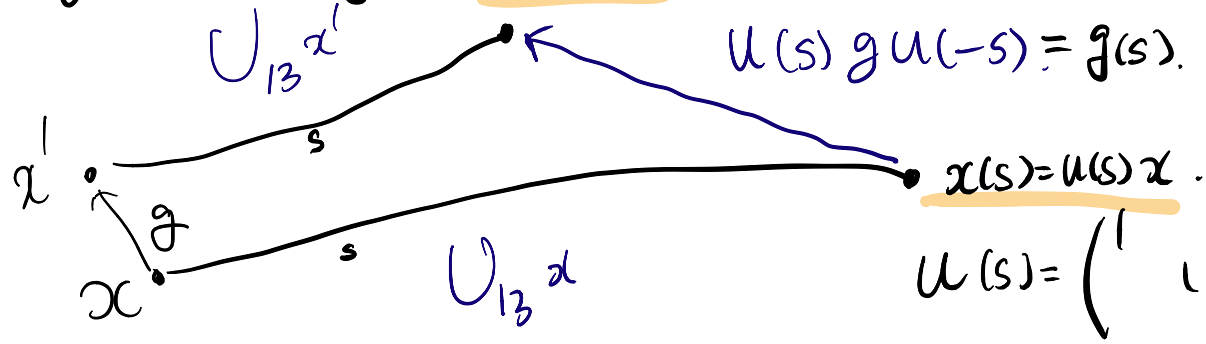
A'-returns to K is exceptional

$\Rightarrow$ strong exceptional.

Therefore, the assumption (after enlarge N) implies failure of ② in the theorem.

Sketch of pf : Need to find $p \in X \setminus N$, $q \in G^- p \setminus N$ s.t. $\mu_p^{13} = \mu_q^{13}$

Let $x' = a'(r) x \in B_\delta(x)$ $\quad x'(s) = u(s) x'$



$$u(s) = \begin{pmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{pmatrix}.$$

Let

$$g = \begin{pmatrix} a_1 & g_{12} & g_{13} \\ g_{21} & a_2 & g_{23} \\ g_{31} & g_{32} & a_3 \end{pmatrix} \notin C(H_{13}) \iff \text{Either } a_1 - a_3 \neq 0,\ g_{21} \neq 0,\ g_{31} \neq 0, \text{ or } g_{32} \neq 0.$$

$u(s)\, g\, u(-s) =$

$$\begin{pmatrix} a_1 + s g_{31} & g_{12} + s g_{32} & g_{13} + s(a_3 - a_1) - s^2 g_{31} \\ g_{21} & a_2 & g_{23} - s g_{21} \\ g_{31} & g_{32} & a_3 - s g_{31} \end{pmatrix}$$

As $g \to I_3$, $g(s)$ converges to an element in G^-.

* Here μ_x^{13} is nontrivial $\Rightarrow$ There are many s so that $x(s)$, $x'(s)$: generic.

If we can make $x, x' \to Z$ $\boxed{d(x(s), x'(s)) \approx C}$ >0

Main difficulty.

for some $s = s(x, x')$

$\boxed{x(s) \text{ and } x'(s) \text{ are in the domain of conti.}}$

for $\bullet \longmapsto \mu^{13}_{\bullet}$ then.

- As $x' = a'(s)\, x$, $a'(s) \in C(U_{13}) \Rightarrow \mu^{13}_{x} = \mu^{13}_{x'}$.
- As $x'(s) = u(s) x'$; $x(s) = u(s) x \Rightarrow \mu^{13}_{x(s)}\, u(s) \propto \mu^{13}_{x}$

$$\mu^{13}_{x'(s)}\, u(s) \propto \mu^{13}_{x'}.$$

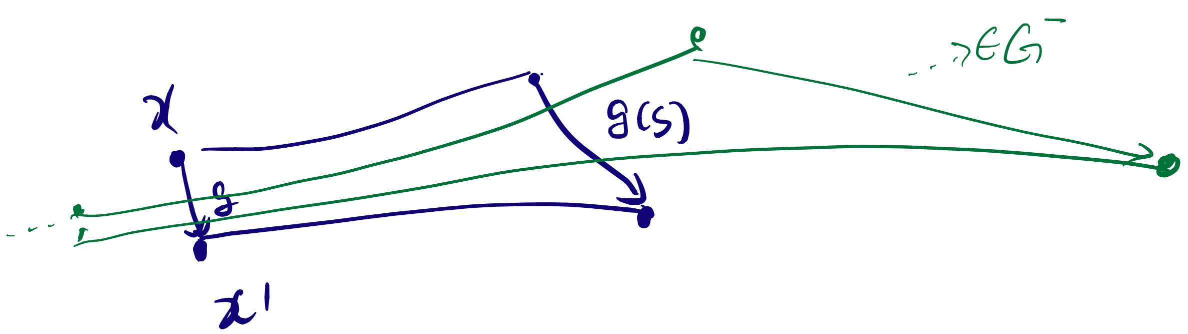
By our normalization, $(\mu^{13}_{\bullet}([-1,1]) = 1)$,

$$\mu^{13}_{x(s)}\, u(s) = \mu^{13}_{x'(s)}\, u(s) \Rightarrow \mu^{13}_{x(s)} = \mu^{13}_{x'(s)}.$$

Now up to subseq, $\mu^{13}_{x(s)} \to \mu^{13}_{y}$, $\mu^{13}_{x'(s)} \to \mu^{13}_{y'}$ for some y, y'.

$\Rightarrow$ $y' = uy$ for some $u \in G^-$

& $\mu^{13}_{y} = \mu^{13}_{y'}$. $\Rightarrow$ Done!!



More precisely, we need to prove that.

$\forall \delta>0 \quad \exists x, x'$ in $x' = a'(r)x = gx \in B_\delta(x)$

$g \notin L$ for some $r \in \mathbb{R}$

& $\exists s \in \mathbb{R}$ s.t.

$u(s)x,\ u(s)x'$ is in the domain of conti. $\cdot \mapsto \mu_\cdot^{13}$,

(*) $\quad 1/C \leq \max(|(a_3-a_1)s - s^2 g_{31}|, |sg_{32}|, |sg_{21}|)$

$\leq C$, and

(**) $\quad |sg_{31}| \leq C\delta^{3/8}$

for some constant C

then we are done. $\square$

Fix $\varepsilon>0$ as in the assumption. (may assume $\varepsilon < 1/100$)

Fix $\delta>0$. Let's define

$X_1 \subset X \setminus N$: compact s.t.

$\mu(X_1) > 1-\varepsilon^4$ and μ_x^{13} depends conti.

on $x \in X_1$

As we want to find s, so that $x(s) \in X_1$, it is natural to find

$$\boxed{\begin{array}{l} X_2 \subset X \setminus N, \quad \mu(X_2) > 1 - C_1\varepsilon^2 \\ \text{s.t. } \forall R>0, \ \forall x \in X_2 \qquad \text{(for some univ. const. } C_1) \\ \mu_x^{1/2}(\{s \in [-R,R] : u(s)x \in X_1\}) \geq (1-\varepsilon^2)\mu_x^{1/2}([-R,R]). \end{array}}$$

We can find such a X_2 due to Maximal Ergodic theorem.

<u>Let's try to prove the theorem</u>: Let $\delta > 0$ suff. small so that $\forall x, x' \in X_2$, $d(x,x') < \delta$ $\exists! \, g \in B_\delta^G$ s.t. $x' = gx$.

(Assume that)
We can find $x \in X_2$, $r \in \mathbb{R}$ st
$x' = a'(r)x \in B_\delta(x) \cap X_2$, due to our non-exceptional returns

$$K(x,x') := \max(|a_3 - a_1|, |g_{31}|^{1/2}, |g_{32}|, |g_{21}|)$$

$\in (0, c\delta^{1/2})$ for some univ. const. c.

$S_0 = K(x,x')^{-1}$

$$\Rightarrow \max(|(a_3 - a_1)S_0|, |g_{31}|^{1/2}S_0, |g_{32}S_0|, |g_{21}S_0|)$$

$$= 1$$

if $|g_{32}s_0|$ or $|g_{21}s_0| = 1$ then,

we can just take $C = 1$ w/ $s = K(x,x')^{-1}$ in $\circledast$

otherwise $|(a_3 - a_1)s_0|$ or $|g_{31}|^{1/2}s_0 = 1$

possibly $|(a_3 - a_1)s_0 - g_{31}s_0^2|$ can be small.

then; $|2(a_3 - a_1)s| \approx 2$, $|4s^2 g_{31}| \approx 4$.

$\rightsquigarrow$ take $s = 2K(x,x')^{-1}$ w/ $C = 1/100$ in $\circledast$.

<u>What's wrong!:</u> As we don't know what μ_x^{13} looks like, we can not say where $x(s)$ and $x'(s)$ are in...

possibly μ_x^{13} and $\mu_{x'}^{13}$ have very large msr on two small intervals containing each root of quad. $(a_3 - a_1)s - s^2 g_{31}$.

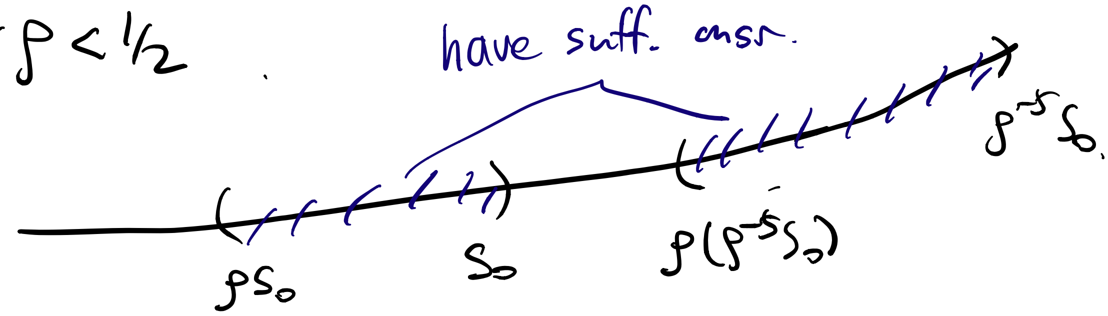
So that we can not ensure $x(s), x'(s) \in X_1$..

Actually, we will see

$$2\mu_x^{13}[-\rho s_0, \rho s_0] < \mu_x^{13}[-s_0, s_0]$$

& $$2\mu_x^{13}[-\rho(\rho^{-5}s_0), \rho(\rho^{-5}s_0)] < \mu_x^{13}[-\rho^{-5}s_0, \rho^{-5}s_0]$$

for some $0<\rho<1/2$.



If μ_x^{13} is regular enough (e.g. doubling) then this kind of estimates can become easier.

However, we don't know regularity of μ_x^{13}.

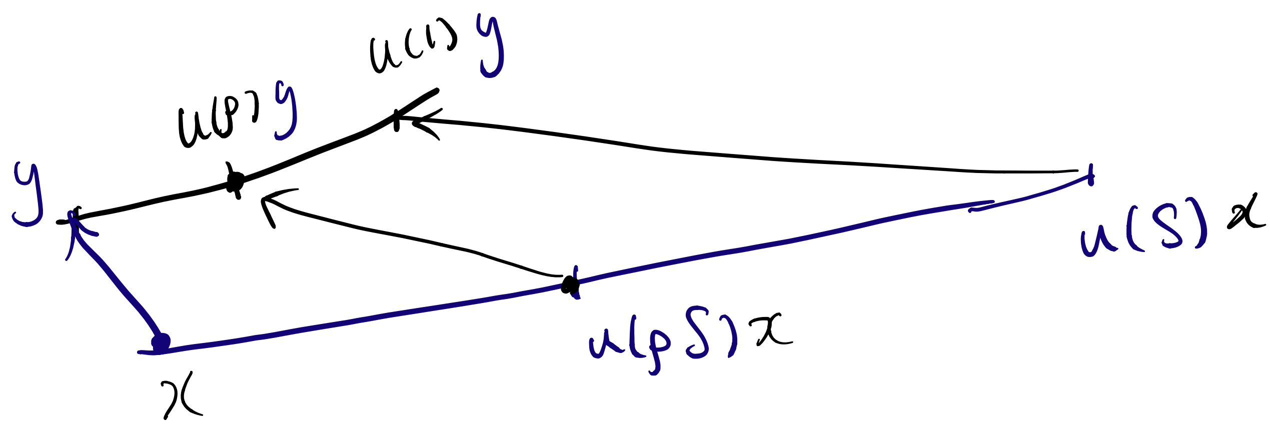
We only know that

$\exists \rho \in (0, 1/2)$ s.t.,

$$\mu(X(\rho)) > 1-\varepsilon^2 \quad \text{where}$$

$$X(\rho) := \{x \in X \setminus N : \mu_x^{13}[-\rho, \rho] < 1/2\}$$

(Recall that we assume that μ_x^{13} is non trivial.)
→ non-atomic.



However, we have $\begin{pmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{pmatrix}$ which

$y = \begin{pmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{pmatrix} x$

$$\begin{pmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{pmatrix} u(\rho s)\, x$$

$$= \begin{pmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{pmatrix} \begin{pmatrix} 1 & & \rho s \\ & 1 & \\ & & 1 \end{pmatrix} x$$

$$= \begin{pmatrix} e^{-t} & & e^{-t}\rho s \\ & 1 & \\ & & e^{t} \end{pmatrix} x$$

$$= \begin{pmatrix} e^{-t} & & e^{-t}\rho s \\ & 1 & \\ & & e^{t} \end{pmatrix} \begin{pmatrix} e^{t} & & \\ & 1 & \\ & & e^{-t} \end{pmatrix} y$$

$$= \begin{pmatrix} 1 & & e^{-2t}\rho s \\ & 1 & \\ & & 1 \end{pmatrix} y$$

Recall $\alpha(-t) = \begin{pmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{pmatrix}$.

This leads to define

$X_3 \subset X \setminus N$ cpt s.t. $\mu(X_3) > 1 - C_2\varepsilon$

& $\forall z \in X_3$ $\frac{1}{T}\int_0^T \mathbb{1}_{X_2}(\alpha(-t)z)\, dt \geq 1-\varepsilon$

Maximal ergodic thm $\to$ $\Big\{$ $\frac{1}{T}\int_0^T \mathbb{1}_{X(\rho)}(\alpha(-t)z)\, dt \geq 1-\varepsilon.$

for any $T>0$

⌜We will start from $z, z' \in X_3$

$\rightsquigarrow x, x' \in X_2$

$\rightsquigarrow x(s), x'(s) \in X_1$.

so that ⊛ & ⊛⊛ hold.⌟

As A'-returns to X_3 are not exceptional,

find $z \in X_3$, $a'(r) \in A'$ s.t.

$$z' = a'(r) z \in B_\delta(z) \cap X_3 .$$

so that $\kappa(z, z') = \max\left(|a_3 - a_1|, |g_{31}|^{1/2}, |g_{32}|, |g_{21}|\right)$

$\in (0, c\delta^{1/2})$ for some univ const. c.

where $z' = gz$ for $g \in B_\delta^G$, $g \notin L$.

$$g = \begin{pmatrix} a_1 & g_{12} & g_{13} \\ g_{21} & a_2 & g_{23} \\ g_{31} & g_{32} & a_3 \end{pmatrix}.$$

Lemma (Find x, x')

Let $z, z' \in X_3$ as above. Let T be $e^{4T} = K(z,z')^{-1}$ $=: S_0$.

then $\exists\, t \in [0,T]$ s.t.

i) $x = \alpha(-t)\, z$, $x' = \alpha(-t)\, z' \in X_2$ ← desired property

ii) $K(x,x') < C\delta^{3/8}$ ← x, x' are still close together

for some univ. const C.

iii) $\mu_x^{13}[-\rho S_0, \rho S_0] < \frac{1}{2}\mu_x^{13}[-S_0, S_0]$

$\mu_x^{13}[-\rho(\rho^{-5}S_0), \rho(\rho^{-5}S_0)] < \frac{1}{2}\mu_x^{13}[-\rho^{-5}S_0, \rho^{-5}S_0]$

Same ineq hold for x' as well. ← Intervals we want have enough msr.

proof of Lemma

$K_a(z,z') := |a_2 - a_1| \in (0, c\delta^{1/2})$

$K_u(z,z') := \max(|g_{31}|^{1/2}, |g_{32}|, |g_{21}|) \in (0, c\delta^{1/2})$.

Corresponding quantities for x, x' are defined similarly.

Indeed, $x' = \tilde{g}\, x$ where

$$\tilde{g} = \alpha(-t)\, g\, \alpha(t)$$

$$= \begin{pmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{pmatrix} \begin{pmatrix} a_1 & g_{12} & g_{13} \\ g_{21} & a_2 & g_{23} \\ g_{31} & g_{32} & a_3 \end{pmatrix} \begin{pmatrix} e^{t} & & \\ & 1 & \\ & & e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & e^{-t} g_{12} & e^{-2t} g_{13} \\ e^{t} g_{21} & a_2 & e^{-t} g_{23} \\ e^{2t} g_{31} & e^{t} g_{32} & a_3 \end{pmatrix}$$

$S_0 = K(z,z')^{-1}$

$\dashv K_a(x,x') = K_z(x,x')$

$$K_u(x,x') = e^t K_u(z,z') \leq e^T \cdot K(z,z')$$
$$= S_0^{1/4} \cdot S_0^{-1} = S_0^{-\frac{3}{4}}$$
$$= K(z,z')^{3/4} < (C\delta^{\frac{1}{2}})^{3/4}.$$

$$\Rightarrow K(x,x') = \max(K_a(x,x'), K_u(x,x'))$$
$$< C^{3/4} \delta^{3/8}.$$

This proves ii).

For i) & iii) we need following lemma:

$\ulcorner$ Lemma' $\forall \eta \in \{0,1\}$, $\forall \theta \in [4T, 6T]$

$\exists P, P' \subset [0,T]$ s.t. $|P|, |P'| > 1 - 9\varepsilon$.

and $\forall t \in P$, $\forall t' \in P'$

(i) $x = \alpha(-t) z \in X_2$, $x' = \alpha(-t') z' \in X_2$

(iii) $\mu_x^{13}[-\rho S, \rho S] < \frac{1}{2} \mu_x^{13}[-S, S]$

where $S = S(t) = e^{\theta - \eta t}$.

$\mu_{x'}^{13}[-\rho S', \rho S'] < \frac{1}{2} \mu_{x'}^{13}[-S', S']$

where $S' = S'(t') = e^{\theta - \eta t'}$. $\lrcorner$

We will prove later.

Using Lemma', we will find

$P_a, P_a', P_u, P_u' \subset [0,T]$ so that

$|P_a|, |P_a'|, |P_u|, |P_u'| > (1-9\varepsilon)T$ &

$\forall t \in P_a \cap P_a' \cap P_u \cap P_u'$ satisfies (i) & (iii) in the lemma.

Recall that

$$K(x,x') = \max(K_a(x,x'), K_u(x,x'))$$

$$K_a(x,x') = |a_3 - a_1|,$$

$$K_u(x,x') = \max(|g_{31}|^{1/2} e^t, |g_{32}| e^t, |g_{21}| e^t)$$

- P_a, P_a': if $K(x,x') > K_a(x,x')$ for all $t \in [0,T]$ we set $P_a = [0,T]$.

Otherwise, $K_a(x,x') = K(x,x')$ for some $t_0 \in [0,T]$

$$\rightsquigarrow K_a(z,z') = K_a(x,x') = K(x,x')$$

$$\geq K_u(x,x')$$

$$= e^{t_0} K_u(z,z')$$

$$\geq K_u(z,z')$$

$$\Rightarrow K(z,z') = K_a(z,z').$$

$$\begin{cases} \eta = 0 \\ \theta = -\ln K_a(z,z') = 4T. \\ S_a = e^{\theta} = e^{4T} = K(z,z')^{-1} = K_a(z,z')^{-1} \overset{\forall t}{=} K_a(x,x')^{-1}. \end{cases}$$

$\to$ Lemma' : $\exists P_a \subset [0,T]$ $\quad |P_a| > (1-9\varepsilon)T$, $\forall t \in P_a$

$x = \alpha(-t) z \in X_2$

$$\mu_x^{13}([-\rho S_a, \rho S_a]) < \frac{1}{2} \mu_x^{13} [-S_a, S_a].$$

Same process gives P_a' starting from x'. ✗

- P_u, P_u' : If $K(x,x') > K_u(x,x')$ $\forall t \in [0,T]$

  set $P_u = [0,T]$.

otherwise, $K(x,x') = K_u(x,x')$ for some t_0

$$= e^{t_0} K_u(z,z')$$

$$\geq K(z,z')$$

Since $K_a(x,x') = K_a(z,z') \leq K_u(x,x')$

$K_u(z,z') \leq e^{t_0} K_u(z,z')$

$$\therefore K(z,z') \leq e^{t_0} K_u(z,z') \leq \underbrace{e^{T}}_{K(z,z')^{-1/4}} K_u(z,z')$$

$$\Rightarrow \quad K_u(z,z') \in [K(z,z')^{5/4}, K(z,z')]$$

$\eta = 1$

$\theta = -\ln K_u(z, z') \in [4T, 5T] \; \forall t$

$S_u = S_u(t) = e^{\theta - t} = K_u(x, x')^{-1}$.

$\Rightarrow$ Lemma'

Find $P_u \subset [0, T]$, $|P_u| > (1 - 9\varepsilon)T$

$\forall t \in P_u$, $x = \alpha(-t) z \in X_2$ &

$\mu_x^{13}[-\rho S_u, \rho S_u] < \frac{1}{2} \rho_x^{13}[-S_u, S_u]$.

Same process gives P_u' starting from x' $\times$

As $K(x, x') = K_a(x, x')$ or $K_u(x, x')$,

$\forall t \in P_a \cap P_u \Rightarrow \alpha(-t) z = x \in X_2$ &

$\mu_x^{13}[-\rho S_a, \rho S_a] < \frac{1}{2} \rho_x^{13}[-S_a, S_a]$ &

$\mu_x^{13}[-\rho S_u, \rho S_u] < \frac{1}{2} \rho_x^{13}[-S_u, S_u]$

Recall that $S_a = K_a(x, x')^{-1}$

$S_u = K_u(x, x')^{-1}$

$\therefore \mu_x^{13}[-\rho S_0, \rho S_0] < \frac{1}{2}\mu_x^{13}[-S_0, S_0]$

where $S_0 = \kappa(x, x')^{-1}$.

Same conclusion is true for P_a', P_u'.

$\Rightarrow$ $\forall t \in P_a \cap P_u \cap P_a' \cap P_u'$ we have

$$\mu_x^{13}[-\rho S_0, \rho S_0] < \frac{1}{2}\mu_x^{13}[-S_0, S_0]$$

$$\mu_{x'}^{13}[-\rho S_0, \rho S_0] < \frac{1}{2}\mu_{x'}^{13}[-S_0, S_0]$$

& $x, x' \in X_2$ $\quad x = \alpha(-t)z, \quad x' = \alpha(-t)z'$.

Further, $P_a \cap P_u \cap P_a' \cap P_u' \subset [0, T]$

$|P_a \cap P_u \cap P_a' \cap P_u'| > (1 - 36\varepsilon)T$

$\Rightarrow$ we can find $t \in P_a \cap P_u \cap P_a' \cap P_u'$.

For the statement $\rho^{-5} S_0$, we can use different θ in the above so that find additional 4 sets.

Intersect $P_a \cap P_u \cap P_a' \cap P_u'$ in 4 additional subsets is not empty! (Recall $\varepsilon < 1/100$.)

$\boxed{//}$

Lemma' $\forall \eta \in \{0,1\}$, $\forall \theta \in [4T, 6T]$

$\exists P, P' \subset [0,T]$ s.t. $|P|, |P'| > 1 - 9\varepsilon$.

and $\forall t \in P$, $\forall t' \in P'$

(i) $x = \alpha(-t) z \in X_2$, $x' = \alpha(-t') z' \in X_2$

(ii) $$\mu_x^{13}[-\rho S, \rho S] < \frac{1}{2} \mu_x^{13}[-S, S]$$

where $S = S(t) = e^{\theta - \eta t}$.

$$\mu_{x'}^{13}[-\rho S', \rho S'] < \frac{1}{2} \mu_{x'}^{13}[-S', S']$$

where $S' = S'(t') = e^{\theta - \eta t'}$.

pf of Lemma'

$\exists Q_1 \subset [0,T]$, $|Q_1| > (1-\varepsilon)T$

s.t. $x = \alpha(-t) z \in X_2$ $\forall t \in Q_1$.

$\exists Q_2 \subset [0, 4T]$, $|Q_2| > (1-\varepsilon) \cdot 4T$

s.t. $\alpha(-v) z \in X(\rho)$ $\forall v \in Q_2$.

Define

$Q = \{ t \in [0,T] \; ; \; (\theta + (\theta - \eta) t) \in Q_2 \}$

$Q_3 = \{ t \in [0, T] : \frac{1}{2}(\theta + (2-\eta)t) \in Q_2 \}$

$\rightsquigarrow \ |Q_3| > (1-8\varepsilon)T.$ (direct calculation.)

Let $v = \frac{1}{2}(\theta + (2-\eta)t)$

$\Rightarrow \ y = \alpha(-v) z \in \mathcal{X}(\rho) \quad \forall t \in Q_3$

<u>Claim</u> $P = Q_1 \cap Q_3 \subset [0, T]$ satisfies all assertions in the lemma

1) $|P| > (1-9\varepsilon)T$

2) $\forall t \in P, \quad x = \alpha(-t) z \in X_2.$

Let $\omega = \frac{1}{2}(\theta - \eta t)$

$\rightarrow \quad y = \alpha(-\omega) x = \alpha(-\omega - t) z = \alpha(-v) z \in \mathcal{X}(\rho)$

$\left(-(\omega + t) = -\frac{1}{2}(\theta + (2-\eta)t) = -v \right)$

$\therefore \ \mu_y^{12}[-\rho, \rho] < \frac{1}{2} \mu_y^{12}[-1, 1] = \frac{1}{2}.$

$\frac{1}{2} > \frac{\mu_y^{13}[-\rho, \rho]}{\ } = \frac{\alpha(-\omega) \mu_x^{13} \alpha(\omega)([-\rho, \rho])}{\ }$

$$\mu_y^{13}[-1,1] \qquad \alpha(-\omega)\,\mu_x^{13}\,\alpha(\omega)\big([-1,1]\big)$$

$$= \frac{\mu_x^{13}\big([-\rho e^{2\omega}, \rho e^{2\omega}]\big)}{\mu_x^{13}\big([-e^{2\omega}, e^{2\omega}]\big)}$$

$$\underbrace{\begin{pmatrix} e^{\omega} & & \\ & 1 & \\ & & e^{-\omega} \end{pmatrix}}_{\alpha(\omega)} \begin{pmatrix} 1 & & s \\ & 1 & \\ & & 1 \end{pmatrix} \underbrace{\begin{pmatrix} e^{-\omega} & & \\ & 1 & \\ & & e^{\omega} \end{pmatrix}}_{\alpha(-\omega)}$$

$$= \begin{pmatrix} 1 & & e^{2\omega}s \\ & 1 & \\ & & 1 \end{pmatrix} \rightsquigarrow \quad s \in [-\rho,\rho] \Rightarrow e^{2\omega}s \in [-\rho e^{2\omega}, \rho e^{2\omega}]$$

We can start from x' and get ρ' as same as above $\square$

<u>proof of theorem:</u> $0<\epsilon<1/100$ as in the theorem. Fix $\delta>0$.

we find $z, z' \in X_3$ using Poincaré recurrence & A'-returns to X_3 are not exceptional.

So that $z \in X_3$, $a'(r) \in A'$ ⇝

$$z' = a'(r) z \in B_\delta(z) \cap X_3 .$$

& $\kappa(z, z') = \max\left(|a_3 - a_1|, |g_{31}|^{1/2}, |g_{32}|, |g_{21}|\right) \in (0, c\delta^{1/2})$ for some univ const. c.

where $z' = gz$ for $g \in B^G_\delta$, $g \notin L$.

⇝ $\exists t$ so that

Lemma $\quad x = \alpha(-t) z$, $x' = \alpha(-t) z' \in X_2$

& $\mu^{13}_x[-\rho S_0, \rho S_0] < \frac{1}{2}\mu^{13}_x[-S_0, S_0]$

$$\mu^{13}_x[-\rho(\rho^{-5}S_0), \rho(\rho^{-5}S_0)] < \frac{1}{2}\mu^{13}_x[-\rho^{-5}S_0, \rho^{-5}S_0].$$

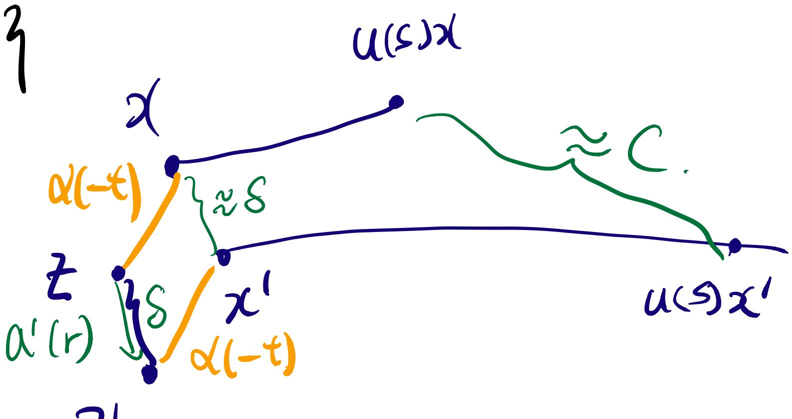
- $a'(s)$ acts on U^{13}-leaves isometrically (Recall that $[A', U^{13}] = 1$).

⇒ $\mu^{13}_z = \mu^{13}_{z'}$ and $\mu^{13}_x = \mu^{13}_{x'}$ as we discussed before

$S_0 = \kappa(x, x')^{-1}$.

$P = \{s \in [-S_0, S_0] : u(s)x \in X_1\}$

$P' = \{s \in [-S_0, S_0] : u(s)x' \in X_1\}$



then, by def of X_2,

both have density

$(1-\xi^2)$ w.r.t. $\mu_x^{13} = \mu_{x'}^{13}$.

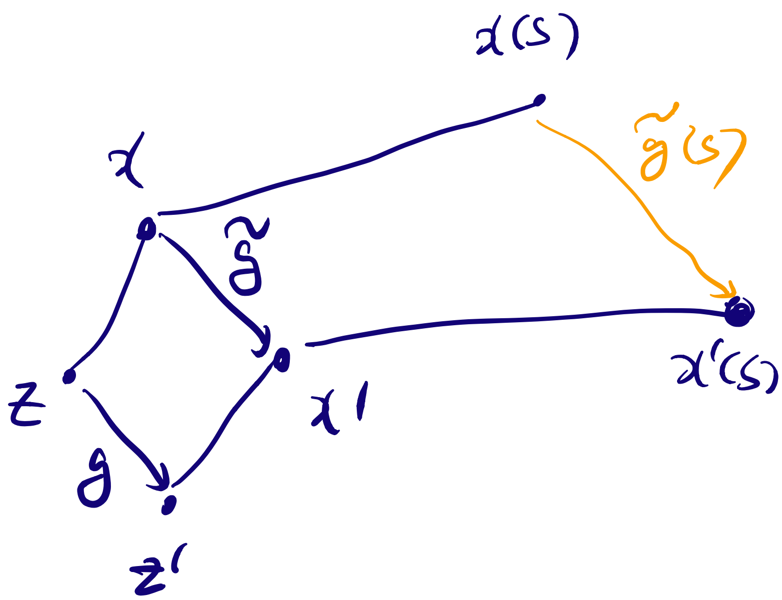
As $\mu_x^{13}([-\rho S_0, \rho S_0]) < \frac{1}{2}\mu_x^{13}[-S_0, S_0]$

we can find $s \in P \cap P_0 \setminus [-\rho S_0, \rho S_0]$

Define $x(s) = u(s)x \in X_1$

$x'(s) = u(s)x' \in X_1$

$\Rightarrow \mu_{x(s)}^{13} = \mu_{x'(s)}^{13}$ as we discussed before



$$\tilde{g} = \alpha(-t)\, g\, \alpha(t)$$

$$= \begin{pmatrix} e^{-t} & & \\ & 1 & \\ & & e^{t} \end{pmatrix} \begin{pmatrix} a_1 & g_{12} & g_{13} \\ g_{21} & a_2 & g_{23} \\ g_{31} & g_{32} & a_3 \end{pmatrix} \begin{pmatrix} e^{t} & & \\ & 1 & \\ & & e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & e^{-t} g_{12} & e^{-2t} g_{13} \\ e^{t} g_{21} & a_2 & e^{-t} g_{23} \\ e^{2t} g_{31} & e^{t} g_{32} & a_3 \end{pmatrix} = \begin{pmatrix} a_1 & \tilde{g}_{12} & \tilde{g}_{13} \\ \tilde{g}_{21} & a_2 & \tilde{g}_{23} \\ \tilde{g}_{31} & \tilde{g}_{32} & a_3 \end{pmatrix}$$

$\tilde{g}(s)$
$= u(s)\,\tilde{g}\,u(-s)$

$$= \begin{pmatrix} a_1 + s\tilde{g}_{31} & g_{12} + s\tilde{g}_{32} & \tilde{g}_{13} + s(a_3 - a_1) - s^2\tilde{g}_{31} \\ \tilde{g}_{21} & a_2 & \tilde{g}_{23} - s\tilde{g}_{21} \\ \tilde{g}_{31} & \tilde{g}_{32} & a_3 - s\tilde{g}_{31} \end{pmatrix}$$

Claim It is possible to achieve

(a) $C^{-1} \leq \max\left(|s(a_3 - a_1) - s^2 \tilde{g}_{31}|,\ |\tilde{g}_{21} s|,\ |\tilde{g}_{32} s| \right) \leq C.$

(b) $|\tilde{g}_{31} s| \leq C\delta^{3/8}$

for some univ. const. C. (Indeed if claim is true ...)

For (b): $|\tilde{g}_{21}|^{1/2} \leq K(x,x') \Rightarrow |\tilde{g}_{21}| \leq K(x,x')^2$

$$|\tilde{g}_{21} s| \leq K(x,x')^2 S_0 = K(x,x') \leq C\delta^{3/8}$$

Recall that

$$\begin{cases} K_a(x,x') = K_z(x,x') \\ K_u(x,x') = e^t K_u(z,z') \le e^T \cdot K(z,z') \\ \qquad = S_0^{1/4} \cdot S_0^{-1} = S_0^{-\frac{3}{4}} \\ \qquad = K(z,z')^{3/4} < (C\delta^{\frac{1}{2}})^{3/4} \\ \Rightarrow K(x,x') = \max(K_a(x,x'), K_u(x,x')) \\ \qquad < C^{3/4}\delta^{3/8} \end{cases}$$

//

For ⓐ; Recall that $s \in [-S_0, S_0] \setminus [-\rho S_0, \rho S_0]$.

$$\rho \le \max(|(a_3-a_1)s|, |\tilde{g}_{31}|^{1/2}|s|, |\tilde{g}_{21}s|, |\tilde{g}_{32}s|) \le 1.$$

Unless $\max(|(a_3-a_1)s|, |\tilde{g}_{31}|^{1/2}|s|) \ge \rho$

$|(a_3-a_1)s - \tilde{g}_{31}s^2| < \rho^2 < \frac{\rho}{2}$,

take $C = \rho^{-2}$. then ⓐ holds.

If $\max(|(a_3-a_1)s|, |\tilde{g}_{31}|^{1/2}|s|) \ge \rho$

$|(a_3-a_1)s - \tilde{g}_{31}s^2| < \rho^2 < \frac{1}{2}\rho$,

case 1) $|\tilde{g}_{31}|^{1/2}|s| \ge |(a_3-a_1)s|$;

$$|\tilde{g}_{31}|^{1/2}|s| \ge \rho \Rightarrow |\tilde{g}_{31}| \ge (\rho|s|^{-1})^2 \ge \rho^2 S_0^{-2}.$$

case 2) $|(a_3-a_1)s| \ge |\tilde{g}_{31}|^{1/2}|s|$

$$\overbrace{|(a_3-a_1)s|}^{\rho \le} - |\tilde{g}_{31}|s^2 < \frac{1}{2}\rho \Rightarrow |\tilde{g}_{31}|s^2 > \frac{1}{2}\rho \ge \rho^2$$

$\le$

$$\rho - |\tilde{g}_{31}|s^2 \qquad \Rightarrow |\tilde{g}_{31}| \ge (\rho|s|^{-1})^2 \ge \rho^2 S_0^{-2}$$

$\Rightarrow |\tilde{g}_{31}| \geq \rho^2 S_0^{-2}$ in any case. $(\because S_0^{-1} < c\,\delta^{1/2})$

We do same process using last assertion of Lemma.

$\rightsquigarrow$ We can find $s' \in [\rho^{-5}S_0, \rho^{-5}S_0] \setminus [\rho(\rho^{-5}S_0), \rho(\rho^{-5}S_0)]$

so that $x(s'), x'(s') \in X_1$

$\left(\text{and } \mu^{13}_{x(s')} = \mu^{13}_{x'(s')}\right)$.

Further, $\rho^{-4} \leq \max\left(|(a_3-a_1)s'|, |\tilde{g}_{31}|^{1/2}|s'|, |\tilde{g}_{21}||s'|, |\tilde{g}_{32}||s'|\right)$

$$\leq \rho^{-5}.$$

$\Rightarrow |(a_3-a_1)s'| \leq \rho^{-5}$ &

$$|\tilde{g}_{31}s'^2| \geq \rho^2 S_0^{-2}\cdot|s'|^2 \geq (\rho^2 S_0^{-2})(\rho^{-4}S_0)^2$$
$$= \rho^{-6}.$$

$\Rightarrow |(a_3-a_1)s' - \tilde{g}_{31}s'^2| \geq \rho^{-6} - \rho^{-5}$.

Take $C = \rho^{-100}$.

(For this s', (b) still holds.;

$|\tilde{g}_{31}s'| \leq K(x,x')^2\cdot(\rho^{-5}S_0) = \rho^{-5}K(x,x') \leq \rho^{-5}c^{3/4}\delta^{3/8}$) $\#$

This proves claim. $\boxed{//}$

Indeed, if the claim is true, than for each $\delta = \frac{1}{n} > 0$, we can find two seq. y_n, y_n' in X_1 w $\mu^{13}_{y_n} = \mu^{13}_{y_n'}$.

Since X_1 is cpt. $y_n \to y \in X_1$ up to subseq in X_1

$y_n' \to y' \in X_1$

Further, claim shows that $y' \in G^- y$.

and $y' \neq y$

This implies $\mu^y_{13} = \mu^{y'}_{13}$ for some $y' \in G^- y$

$y \neq y'$.

By the **Lemma 1**, this implies that μ is U^{13}-inv. and proves theorem. $\square$