MONDAY'S TALK

I – ψ-MA

<u>Def:</u> A is ψ-MA $\iff$ $\exists \infty$ many $(p,q) \in \mathbb{Z}^m \times \mathbb{Z}^n$ st $\Pi(Aq-p)\,\Pi(q) \neq 0$ and:

$$\Pi(Aq-p) \leq \psi(\Pi(q)).$$

<u>Not:</u>

- $\mathfrak{a}^+ = \{(t_1, \dots, t_{m+n}) \text{ st } \sum_{i \leq m} t_i = \sum_{i > m} t_i\}$.
- $\forall t \in \mathbb{R}$ $\mathfrak{a}_t^+ = \{\mathbf{t} \in \mathfrak{a}^+ \text{ st } \sum_{i \leq m} t_i = t\}$.
- for $\mathbf{t} \in \mathfrak{a}^+$, $g_{\mathbf{t}} = \mathrm{diag}(e^{t_1}, \dots, e^{t_m}, e^{-t_{m+1}}, \dots, e^{-t_{m+n}})$

  and $$g_{\mathbf{t}} = \begin{pmatrix} {}^{(m)}g_{\mathbf{t}} & 0 \\ 0 & {}_{(n)}g_{\mathbf{t}} \end{pmatrix}$$

<u>Lemma:</u> If $(U,V) \in \mathbb{R}^m \times \mathbb{R}^n$ with $\Pi(U)\Pi(V) \neq 0$ and $t \in \mathbb{R}$ then $\exists \mathbf{t} \in \mathfrak{a}_t^+$ st:

$$1)\ e^t\,\Pi(U) = \|{}^{(m)}g_{\mathbf{t}} \cdot U\|^m$$

$$2)\ e^{-t}\,\Pi(V) = \|{}_{(n)}g_{\mathbf{t}} \cdot V\|^n.$$

Proposition: A is ψ-MA $\iff$ $\forall t_0 > 0$ $\exists t \geq t_0$ $\exists \mathbf{t} \in \mathfrak{a}_t^+$

$$g_{\mathbf{t}} \cdot [L_A] \notin K_{\varepsilon_\psi(t)}.$$

<u>Proof:</u> Suppose A is ψ-MA for (p,q) as in the def; define

t_q by $(*)$ $e^{t_q}\varepsilon_\psi(t_q) = \pi(q)$. (recall that $t \to \varepsilon(t)e^{t}$ ↗)

and chose $\mathbf{t} \in \mathfrak{a}^+_{t_q}$ as in the previous Lemma for $(U,V)=(Aq-p,q)$.

thus $\|^{(m)}g_{\mathbf{t}} \cdot (Aq-p)\|^m \overset{(1)}{=} e^{t_q}\,\pi(Aq-p) \overset{(\psi\text{-MA})}{\leqslant} e^{t_q}\,\psi(\pi(q))$

$\overset{(*)}{\leqslant} e^{t_q}\,\psi\left(\varepsilon_\psi^n(t_q)e^{t_q}\right) = e^{t_q}\,\varepsilon_\psi^m(t_q)e^{-t_q} = \varepsilon_\psi^m(t_q)$

$\Rightarrow \|^{(m)}g_{\mathbf{t}} \cdot (Aq-p)\| \leqslant \varepsilon(t_q)$

and $\|_{(n)}g_{\mathbf{t}} \cdot q\|^n \overset{(2)}{=} e^{-t_q}\,\pi(q) \overset{(*)}{=} \varepsilon_\psi^n(t_q)$

$\Rightarrow \|_{(n)}g_{\mathbf{t}} \cdot q\| < \varepsilon(t_q)$. Hence $\left\| g_{\mathbf{t}} \cdot L_A\begin{pmatrix}-p\\ q\end{pmatrix}\right\| < \varepsilon(t)$

and thus $g_{\mathbf{t}}[L_A] \notin K_{\varepsilon_\psi(t)}$; $(t_q)_q \to \infty$.

Reciprocally: let $t>0$ and $\mathbf{t} \in \mathfrak{a}^+_t$ such that $g_{\mathbf{t}}[L_A] \notin K_{\varepsilon_\psi(t)}$.
let $(p_t, q_t) \in \mathbb{Z}^{m+n}$ st.

$$\left\| g_{\mathbf{t}}\, L_A\begin{pmatrix}-p_t\\ q_t\end{pmatrix}\right\| < \varepsilon_\psi(t).$$

$$\Leftrightarrow \quad \left\| g_{\mathbf{t}} \cdot \begin{pmatrix}Aq_t - p_t\\ q_t\end{pmatrix}\right\| < \varepsilon_\psi(t).$$

$$\Leftrightarrow \begin{cases} a)\ \|^{(m)}g_{\mathbf{t}} \cdot (Aq_t - p_t)\| < \varepsilon_\psi(t_q) \text{ and} \\ b)\ \|_{(n)}g_{\mathbf{t}}\, Aq_t - p_t\| < \varepsilon_\psi(t_q) \end{cases}$$

$b \Rightarrow e^{-t}\,\pi(q_t) \leqslant \|_n g_{\mathbf{t}} \cdot q_t\|^n \leqslant \varepsilon_\psi^n(t)$

$\Rightarrow \pi(q_t) \leqslant e^{t}\,\varepsilon_\psi^n(t)$.

$\Rightarrow \psi(\pi(q_t)) \underset{\text{monotonicity}}{\geqslant} \psi(e^t\,\varepsilon_\psi^n(t)) = e^{-t}\,\varepsilon_\psi^m(t)$.

$a) \Rightarrow \pi(Aq_t - p_t) \leqslant e^{-t}\,\|^{(m)}g_{\mathbf{t}}(Aq_t - p_t)\|^m \leqslant e^{-t}\,\varepsilon_\psi^m(t)$
$\leqslant \psi(\pi(q_t))$

thus the $(-p_t, q_t)$ satisfy the equation of ψMA.

Notice that the q_t are unbounded. Suppose to a contradiction they

a) $\Rightarrow$ $\forall i \leq m$

$$\left|\left[Aq_t - p_t\right]_i\right| \leq \varepsilon_{\psi(t)} e^{-t_i} \leq \varepsilon_{\psi(t)} e^{-\inf t_i} \leq \varepsilon_{\psi(t)} e^{-t/m}$$

(i^{th} coef of $Aq_t - p_t$). Let $i(t) = \inf_{i \leq m} (t_i)$

then: one of the coef of $Aq_t - p_t$ goes to zero as $t \to \infty$. this implies that when $\|q_t\|$ is bounded above, there is a compact set of $\mathbb{R}^{m+n}$ that contains ∞ many points of $L_A \cdot \mathbb{Z}^{n+m}$ which in contradiction with the discreteness of the latter.

Drawings for $\psi\, x \to \frac{c}{x}$ ($\varepsilon_\psi = c^{\frac{1}{m+n}}$) and $\psi\, x \to \frac{1}{x^{1+\beta}}$ ($\varepsilon_\psi = e^{-\gamma t}$) ■

II – Approximations with weights.

If $V \in \mathbb{R}^{m+n}$. $V^{(m)}$ and $V_{(n)}$ are define st:

$$V = \begin{pmatrix} V^{(m)} \\ V_{(n)} \end{pmatrix}.$$

let $\underline{r} = (r_1, \ldots, r_m)$ $\quad$ $\underline{s} = (s_1, \ldots, s_n)$

$\sum r_i = 1$ $\quad$ $\sum s_i = 1$.

- $$g_t^{r,s} = \operatorname{diag}\left(e^{tr_1}, \ldots, e^{tr_m}, e^{-ts_1}, \ldots, e^{-ts_n}\right)$$

- $\forall v \in \mathbb{R}^m$ $\|v\|_{\underline{r}} = \max_{i \leq m} |v_i|^{\frac{1}{r_i}}$
- $\forall u \in \mathbb{R}^n$ $\|u\|_{\underline{s}} = \max_{i \leq n} |u_i|^{\frac{1}{s_i}}$
- $\forall V \in \mathbb{R}^{m+n}$ $\|V\|_{\underline{r},\underline{s}} = \max\left(\|V^{(m)}\|_{\underline{r}}, \|V_{(n)}\|_{\underline{s}}\right)$.

$\ldots$ V^{m+n} $\ldots$

- $\delta_{\underline{r},\underline{s}} \quad \Lambda \longrightarrow \mathbb{R}_{>0}$

$$[M] \longmapsto \inf_{v \in \mathbb{Z}^{m+n} - \{0\}} \delta_{\underline{r},\underline{s}}(M \cdot v).$$

Proposition: K is compact $\Leftrightarrow$ $\delta_{\underline{r},\underline{s}}(K) > 0$.

Def: A is $(\underline{r},\underline{s})$ BA $\Leftrightarrow$ $\exists c>0 \ \forall (p,q) \in \mathbb{Z}^{m+n}$ $\|Aq-p\|_{\underline{r}} > c\|q\|_{\underline{s}}^{-1}$

Prop: A is $(\underline{r},\underline{s})$- BA $\Leftrightarrow$ $\{ g_t^{\underline{r},\underline{s}} \cdot L_A, t \geq 0 \}$ is bounded.

"$\Rightarrow$" if A is $(\underline{r},\underline{s})$ BA, $\exists c > 0$ $\forall (p,q)$:

$$\|Aq-p\|_{\underline{r}} \ \|q\|_{\underline{s}} > c \qquad \text{let } t \geq 0.$$

$$\left\| g_t^{\underline{r},\underline{s}} \cdot \begin{pmatrix} Aq-p \\ q \end{pmatrix} \right\|_{\underline{r},\underline{s}}^2 \geq \|{}^{\underline{r}}g_t\, Aq-p\|_{\underline{r}} \ \|{}_{\underline{s}}g_t\, q\|_{\underline{s}}$$

$$\begin{pmatrix} {}^{\underline{r}}g_t \cdot (Aq-p) \\ {}_{\underline{s}}g_t \cdot q \end{pmatrix} \qquad \geq e^{t}\|Aq-p\|_{\underline{r}}\, e^{-t}\|q\| > c.$$

then $\forall t \geq 0$ $\delta_{\underline{r},\underline{s}}(g_t L_A) > 0$ and by previous prop:

$$\{ g_t^{\underline{r},\underline{s}} [L_A] \ t \geq 0 \} \text{ is bounded.}$$

Reciprocally: $\{ g_t^{\underline{r},\underline{s}} [L_A] \ t \geq 0 \}$ bounded $\Rightarrow$ $\exists c$ st:

$\forall t \qquad \delta_{\underline{r},\underline{s}}(g_t^{\underline{r},\underline{s}}[L_A]) > c$. let $(p,q) \in \mathbb{Z}^{m+n}$.

Chose t st $\qquad \|{}_{\underline{s}}g_t \cdot q\|_{\underline{s}} = \frac{c}{2} \ \left(\Leftrightarrow \|q\|_{\underline{s}} = e^{t}\frac{c}{2}. \right)$

then $\quad \delta_{\underline{r},\underline{s}}(g_t^{\underline{r},\underline{s}} [L_A]) > c \Rightarrow \|{}^{\underline{r}}g_t (Aq-p)\| > c$

$\ldots (g_t [L_A]) > \ldots \| g_t \ldots \|_{\underline{r}} >$

$$\Leftrightarrow \| Aq - p \|_{\underline{r}} > c e^{-t} = \frac{c^2}{2} \| q \|_{\underline{s}}^{-1}$$

$$\Rightarrow \| Aq - p \|_{\underline{r}} \, \| q \|_{\underline{s}} > \frac{c^2}{2}.$$

$\Rightarrow$ A is $(\underline{r}, \underline{s})$ Badly approx.

Proposition: A is $(\underline{r}, \underline{s})$ - Singular $\Leftrightarrow$ $g_t^{r,s} [L_A]_{t>0}$ divergent

proof: thm 7.4 in "Flows on homogeneous spaces and dioph. approximations", Kleinbock.

⚠ Upshot when one works with weights, all the dynamical criteria remain the same provided one replaces g_t by $g_t^{r,s}$.