Dani correspondance for Diophantine approximations

Let $A \in \mathcal{M}_{m,n}(\mathbb{R})$. Any Diophantine approximation property on A can be translated in the language of divergence of flows on the space $X_{m+n} = SL_{m+n}(\mathbb{R})/SL_{m+n}(\mathbb{Z})$.

The leitmotiv is the following. Let $L_A = \begin{pmatrix} Id_m & A \\ 0 & Id_n \end{pmatrix} \in X_{n+m}$.

good rational approximations of A $\longleftrightarrow$ far excursions of the orbit of $L_A \in X_{n+m}$ under diagonal flows in the cusps of X_{n+m}.

what is a cusp?

<u>Definition</u>: • let $\Delta : X_{n+m} \to \mathbb{R}_+$
$$[g] \mapsto \inf_{v \in \mathbb{Z}^{n+m} - \{0\}} \|g \cdot v\|_2$$ où $\| \ \|_2$ euclid norm.

• $K_\varepsilon = \Delta^{-1}([\varepsilon, \infty[)$.

<u>Proposition</u>: $\forall \varepsilon > 0$, the set K_ε is compact.

we will exhibit a basis of $g_n\mathbb{Z}^2$ $\langle v_{1,n}, v_{2,n} \rangle$ with $\varepsilon \leq \|v_{1,n}\| < \|v_{2,n}\| < C(\varepsilon)$

proof in the case $n = m = 2$. (general case is the same but more technical)

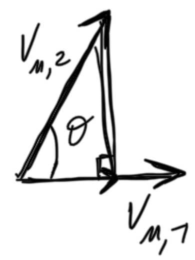
let $(g_n)_n \in SL_2(\mathbb{R})^{\mathbb{N}}$ s.t. $([g_n])_{n\in\mathbb{N}} \in K_\varepsilon^{\mathbb{N}}$.

for any $n \in \mathbb{N}$, let $v_{n,1} \in g_n \cdot \mathbb{Z}^2$ st
$$\|v_{n,1}\|_2 = \Delta([g_n]) = \inf_{v \in g_n\mathbb{Z}^2 - \{0\}} \|v\|_2$$ and let

$v_{n,2}$ tq $\|v_{n,2}\|_2 = \inf \{ \|v\|_2,\ v \in g_n \mathbb{Z}^2 - \{0\},\ v$ not proport. to $v_{n,1} \}$.

up to replacing $v_{n,2}$ by $-v_{n,2}$, one can assume that

$(v_{1,n}, v_{2,n})$ is positively oriented.



One can write $v_{n,2} = t\, v_{n,1} + u_n$ with $u_n \in \langle v_{n,1} \rangle^{\perp}$.
note that $t \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ or $v_{n,2} - v_{n,1}$ is shorter than $v_{n,2}$ and non colinear to $v_{n,1}$.

<u>Then</u>: $\det(v_{n,1}; v_{n,2}) = 1 = \|v_{n,2}\|_2 \|v_{n,1}\|_2 \sin\theta$.

but: $\cos\theta = \dfrac{|t| \|v_{n,1}\|_2}{\|v_{n,2}\|_2} \leq \dfrac{1}{2}$.

thus $\exists \delta$ tq $\sin\theta \geq \delta$. and since $\|v_{n,1}\| \geq \varepsilon$.

we have $\|v_{n,2}\|_2 = \dfrac{1}{\|v_{n,1}\| \sin\theta} \leq \dfrac{1}{\delta\varepsilon}$

let $\gamma_n \in SL_2(\mathbb{Z})$ such that

$$g_n\gamma_n (1,0) = v_{n,1} \text{ and } g_n\gamma_n (0,1) = v_{n,2}.$$

and write $g_n\gamma_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. we have $[g_n\gamma_n] = [g_n]$ and

- $\left\| g_n\gamma_n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} a_n \\ c_n \end{pmatrix} \right\|_2 = \|v_{n,1}\|_2 \leq \|v_{n,2}\|_2 \leq \dfrac{1}{\delta\varepsilon}$.
- $\left\| g_n\gamma_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} b_n \\ d_n \end{pmatrix} \right\|_2 = \|v_{n,2}\|_2 \leq \dfrac{1}{\delta\varepsilon}$

then up to taking a subsequence $g_n\gamma_n \to g$ and thus

$$[g_n] = [g_n\gamma_n] \to [g]. \qquad \blacksquare$$

Definition: A cusp is a set of the form $X_{n+m} - K_\varepsilon$.

I - Dirichlet improvable matrices.

Denote $g_t = \mathrm{Diag}(\underbrace{e^{t/m}, \dots, e^{t/m}}_{m}, \underbrace{e^{-t/n}, \dots, e^{-t/n}}_{n})$.

Recall: A is ε-DI $\iff$ $\exists Q_0 > 0$ $\forall Q \geq Q_0$ $\exists q \in \mathbb{Z}^n - \{0\}$ and $p \in \mathbb{Z}^m$ such that:

$$\begin{cases} \|q\|^n < \varepsilon Q \\ \|Aq + p\|^m < \varepsilon Q^{-1} \end{cases} \qquad \|v\| = \max(|v_i|)$$

Proposition: A is ε-DI $\iff$ $\exists t_0 > 0$ $\forall t \geq t_0$ $g_t \cdot [L_A] \in X_{m+n} - K_{\varepsilon^{\frac{2}{m+n}}}$

proof:

$\exists t_0 > 0$ $\forall t \geq t_0$ $g_t \cdot [L_A] \in X_{m+n} - K_{\varepsilon^{\frac{2}{m+n}}}$ $\iff$

$\exists t_0 > 0$ $\forall t \geq t_0$ $\exists q \in \mathbb{Z}^n - \{0\}$ $\exists p \in \mathbb{Z}^m$ $\begin{cases} e^{t/m} \|Aq + p\| \leq \varepsilon^{\frac{2}{m+n}} \\ e^{-t/n} \|q\| \leq \varepsilon^{\frac{2}{m+n}} \end{cases}$ $\iff$

$\exists t_0 > 0$ $\forall t \geq t_0$ $\exists q \in \mathbb{Z}^n - \{0\}$ $\exists p \in \mathbb{Z}^m$ $\begin{cases} \|Aq + p\|^m \leq \varepsilon^{\frac{2m}{m+n}} e^{-t} = \varepsilon^{\frac{m+n+m-n}{m+n}} e^{-t} = \varepsilon \overbrace{\varepsilon^{\frac{m-n}{m+n}} e^{-t}}^{Q^{-1}} \\ \|q\|^n \leq \varepsilon^{\frac{2n}{m+n}} e^{t} = \varepsilon^{\frac{m+n-(m-n)}{m+n}} e^{t} = \varepsilon \underbrace{\varepsilon^{-\frac{m-n}{m+n}} e^{t}}_{Q} \end{cases}$

$\iff$ $\forall Q > e^{t_0} \varepsilon^{-\frac{m-n}{m+n}}$ $\exists q \in \mathbb{Z}^n - \{0\}$ $\exists p \in \mathbb{Z}^m$ $\begin{cases} \|Aq + p\|^m < \varepsilon Q^{-1} \\ \|q\|^n < \varepsilon Q. \end{cases}$

$\iff$ A is ε-DI.

Recall: A matrix is said to be singular := it is ε-DI for any $\varepsilon > 0$.

Corollary: A is singular $\Leftrightarrow$ $g_t \cdot [L_A]$ is divergente.

II – Ψ-approximability

Let Ψ be a decreasing continuous function $[x_0, \infty[\to]0, \infty[$. $\lim_{x \to \infty} \Psi(x) = 0$

Recall: A is Ψ approximable if $\exists \infty$ many $q \in \mathbb{Z}^n - \{0\}$ st $\exists p \in \mathbb{Z}^m$ st

$$\| Aq + p \|^m \leq \Psi(\|q\|^n).$$

Lemma: (Change of Variables) for any Ψ as previously, there is a unique $\varepsilon : [t_0, \infty[\to]0, \infty[$ with $e^{(m+n)t_0} = x_0^m / \Psi(x_0)^n$.

st:
1) $t \mapsto e^t \varepsilon^n(t)$ increasing and unbounded.
2) $t \mapsto e^{-t} \varepsilon^m(t)$ decreasing and $\to 0$.
3) $\Psi(e^t \varepsilon^n(t)) = e^{-t} \varepsilon^m(t)$.

proof: Lemma 8.3 in Kleinbock-Margulis: "Logarithm laws ...."

We will denote ε_Ψ to emphasize the dependance.

give examples $x \mapsto \frac{c}{x} \rightsquigarrow \varepsilon_\Psi = c^{\frac{1}{m+n}}$

$x \mapsto \frac{1}{x^{1+\beta}} \rightsquigarrow \varepsilon_\Psi(t) = e^{-\gamma t}$ $\quad \gamma = \frac{\beta}{(1+\beta)n+m}$

Proposition: A is Ψ-approximable $\Leftrightarrow$ $\exists$ arbitrarily large t with $g_t \cdot [L_A] \notin K_{\varepsilon_\Psi(t)}$.

proof: Suppose A is Ψ-approximable. For any $q \in \mathbb{Z}^n - \{0\}$ st $\exists p \in \mathbb{Z}^m$ $\|Aq + p\|^m \leq \Psi(\|q\|^n)$, chose

$t > 0$ st $\quad \|q\|^n = \varepsilon^n(t) e^t$. t exists if $\|q\|$ is large enough.

..

then $\| e^{-t/n} q \| < \varepsilon_\psi(t)$ and:

$$\| Aq + p \|^m < \psi(\| q \|^n) = \psi\left(\varepsilon_\psi^n(t) e^{t}\right) = \varepsilon_\psi^m(t) e^{-t}$$

$\Rightarrow \| e^{t/m} Aq + p \| < \varepsilon_\psi(t)$ and thus:

$$\left\| g_t \cdot \begin{pmatrix} Aq + p \\ q \end{pmatrix} \right\| < \varepsilon_\psi(t).$$

$$\Rightarrow \quad g_t[L_A] \notin K_{\varepsilon_\psi(t)}$$

Reciprocally, Suppose $\exists t$ arbitrarily large such that

$g_t.[L_A] \notin K_{\varepsilon_\psi(t)}$. then for such t:

$\exists q \in \mathbb{Z}^n - \{0\}, \exists p \in \mathbb{Z}^m$ st We can assume $\|Aq+p\|_{\mathbb{R}^m} \neq 0$. $(k \cdot q)_k$.

$$\left\| \begin{pmatrix} e^{t/m} & Aq+p \\ e^{-t/n} & q \end{pmatrix} \right\| < \varepsilon_\psi(t) \iff \begin{array}{l} \| Aq + p \|^m \leq \varepsilon^m(t) e^{-t} \\ \| q \|^n \leq \varepsilon^n(t) e^{t} \end{array}$$

$$\Rightarrow \quad \| Aq + p \|^m < \varepsilon_\psi^m(t) e^{-t} = \psi(\varepsilon_\psi^n(t) e^{t}) \leq \psi(\| q \|^n).$$

Since $\| Aq + p \| < \varepsilon^m(t) e^{-t}$ is non zero and decreases, $\| q \|$ cannot be bounded or Λ is not discrete.

Example: 1) $\psi = x \longrightarrow \frac{c}{x}$ then by 3: $\varepsilon_\psi = C^{\frac{1}{m+n}}$.

Recall: A is badly approximable $\iff$ A is not $\frac{c}{x}$ - approximable

$\Rightarrow$ Corollary: A is badly approximable $\iff$ $g_t.[L_A]$ is bounded.

2) ψ ... 1 ... $e^{-\gamma t}$ ... $\frac{\beta}{(1+\beta)m+n}$

2) $\psi = x \longmapsto \dfrac{\quad}{x^{1+\beta}}$ $\qquad \xi_\psi(t) = e \quad , \; \gamma =$

Recall: A is very well approximable $\iff$ A is $\dfrac{1}{x^{1+\beta}}$-approx for $\beta > 0$.

Corollary: A is very well approximable $\iff$ $\exists \gamma > 0$ $\exists$ arb. large t
$g_t \cdot [L_A] \notin K_{e^{-\gamma t}}$.

III – Ψ multiplicatively approximable matrices.

Denote 1) $a^+ = \{ \mathbb{t} = (t_1, \dots, t_{m+n}) \in \mathbb{R}^{m+n}_{\geq 0} \; \sum_{i \leq m} t_i = \sum_{i \geq m+1} t_i \}$

2) $a^+_t = \{ \mathbb{t} \in a^+ \mid \sum_{i=1}^{m} t_i = t \}$.

3) $\forall \mathbb{t} \in a^+ \quad g_{\mathbb{t}} = \mathrm{diag}(e^{t_1}, \dots, e^{t_m}, e^{-t_{m+1}}, \dots, e^{-t_{m+n}})$.

4) $\forall \mathbb{t} \in a^+ \quad {}^{m}g_{\mathbb{t}} = \mathrm{diag}(e^{t_1}, \dots, e^{t_m}) \in GL_m(\mathbb{R})$

$$ {}_{n}g_{\mathbb{t}} = \mathrm{diag}(e^{-t_{m+1}}, \dots, e^{-t_{m+n}}) \in GL_n(\mathbb{R}) $$

$$ g_{\mathbb{t}} = \begin{pmatrix} {}^{m}g_{\mathbb{t}} & 0 \\ 0 & {}_{n}g_{\mathbb{t}} \end{pmatrix}. $$

Recall: A is Ψ-multiplicatively approximable $\equiv$ $\exists \infty$ many $q \in \mathbb{Z}^m - \{0\}$ st $\exists p \in \mathbb{Z}^n - \{0\}$ st:

$$ \Pi(Aq + p) \leq \Psi(\Pi_+(q)). $$

Proposition: A is ψ-multiplicatively approximable $\iff$ $\exists$ arbitrarily large t st $\exists \underline{t} \in \mathfrak{a}_t^+$ with

$$g_{\underline{t}} \cdot [L_A] \notin K_{\varepsilon_\psi(t)}.$$

proof: "$\Leftarrow$" let $\underline{t} \in \mathfrak{a}_t^+$ s.t. $g_{\underline{t}} [L_A] \notin K_{\varepsilon_\psi(t)}$

then there are $p \in \mathbb{Z}^n - \{0\}$ and $q \in \mathbb{Z}^m$ s.t.

$$\left\| g_{\underline{t}} \cdot \begin{pmatrix} Aq + p \\ q \end{pmatrix} \right\| < \varepsilon_\psi(t). \qquad *$$

$$\Rightarrow \begin{cases} \| {}^m g_{\underline{t}} \cdot Aq + p \| < \varepsilon_{\psi(t)} & (1) \\ \| {}_n g_{\underline{t}} \cdot q \| < \varepsilon_{\psi}(t) & (2) \end{cases}$$

But $e^t \pi(Aq+p) \leq \| {}^m g_{\underline{t}} \cdot Aq + p \|^m < \varepsilon_{\psi(t)}^m$

$\Rightarrow \pi(Aq+p) \leq \varepsilon^m_{\psi(t)} e^{-t} = \psi(\varepsilon^n_{(t)} e^t)$

(2) $\Rightarrow$ $e^{-t} \pi_+(q) \leq \varepsilon_{\psi(t)}^n \Rightarrow \pi_+(q) \leq e^t \varepsilon_{\psi(t)}^n$

thus: $\pi(Aq+p) \leq \psi(\varepsilon^n_{(t)} e^t) \leq \psi(\pi_+ q)$ by monotonocity of ψ

We now need to show the $\|q\|$ are unbounded. Suppose to a contradiction that they are.

$$\| Aq + p \| < \| g_{\underline{t}} \cdot Aq + p \| < \varepsilon(t)$$

then $\|Aq+p\| < \varepsilon(t)$.

$*$ If one of the coordinate of $Aq+p$ is 0 then (kq, kp)

gives the desired sequence. then

$$\| g_{\underline{t}} \cdot Aq+p \|^m < \varepsilon^m(t)$$

$$\Rightarrow \quad e^{m t_{i_0}} [Aq+p]_{i_0}^m < \varepsilon^m(t) \quad \text{with } t_{i_0} = \sup_{i \leq m} t_i$$

$$\Rightarrow \quad [Aq+p]_{i_0}^m < \varepsilon^m(t) e^{-m t_{i_0}} < \varepsilon^m(t) e^{-t} \underset{t \to \infty}{\longrightarrow} 0.$$

then $B(0, c)$ contains infinitely many vectors of $\begin{pmatrix} Aq+p \\ q \end{pmatrix}$

$C = \max (N, \varepsilon(t_0))$. which contradicts the discreteness of

$L_A \cdot \mathbb{Z}^n$

<u>Corollary</u>: 1) A is badly multiplicatively approximable iff

$$\{ g_{\underline{t}}[L_A], \ \underline{t} \in a_t^+, \ t \in \mathbb{R}_{>0} \} \text{ is bounded.}$$

2) A is very well multiplicatively approximable iff there are arbitrarily large $t \in \mathbb{R}$ with $\underline{t} \in a_t^+$ such that

$$g_{\underline{t}}[L_A] \notin K_{e^{-\gamma t}} \quad \text{for some } \gamma > 0.$$