Topological Entropy

Goal: ① Define top. entropy and give its basic properties.

② Make the connection with metric entropy (so called Variational principle).

- **Setting:** • (X,d) compact metric space.

• $T : X \to X$ continuous map.

Def: • A cover is a familly of open subsets of X $\mathcal{U} = (\mathcal{U}_i)_{i\in I}$ st $X = \bigcup_{i\in I} \mathcal{U}_i$.

• the complexity of $\mathcal{U}$ is the smallest number of subset in $\mathcal{U}$ required to cover $\mathcal{U}$:

$$N(\mathcal{U}) = \inf\{|I_0| \text{ st } I_0 \subseteq I \ \& \bigcup_{i\in I_0} \mathcal{U}_i = X\}$$

- **Notations:** let $\mathcal{U} = (\mathcal{U}_i)_{i\in I}$ & $\mathcal{V} = (\mathcal{V}_j)_{j\in J}$ covers.

- $\mathcal{U} < \mathcal{V}$ when $\mathcal{V}$ is a refinement of $\mathcal{U}$.
- $\mathcal{U} \vee \mathcal{V} = \{ \mathcal{U}_i \cap \mathcal{V}_j \}_{(i,j) \in I \times J}$
- $T^{-1}(\mathcal{U}) = \{ T^{-1}(\mathcal{U}_i) \}_{i \in I}$.

Proposition (*)

① $\mathcal{U} < \mathcal{V} \Rightarrow N(\mathcal{V}) \geqslant N(\mathcal{U})$.

② $N(\mathcal{U} \vee \mathcal{V}) \leqslant N(\mathcal{U}) N(\mathcal{V})$.

③ $N(T^{-1}(\mathcal{U})) \leqslant N(\mathcal{U})$ (= if T onto)

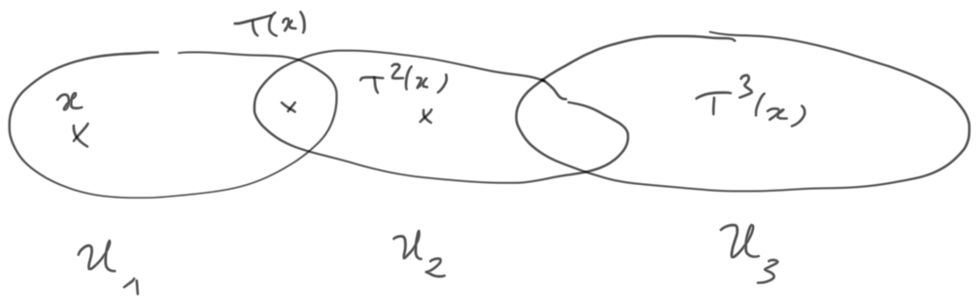
Def (Entropy of T w.r.t a cover): let $\mathcal{U} = (\mathcal{U}_i)_{i \in I}$ cover.

$$h_{Top}(T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log N(\mathcal{U}_n^T)$$

where $$\mathcal{U}_n^T = \bigvee_{k=0}^{n-1} T^{-k} \mathcal{U} = \{ T^{-(n-1)}(\mathcal{U}_{i_{n-1}}) \cap \dots \cap T^{-1}(\mathcal{U}_{i_1}) \cap \mathcal{U}_{i_0} \}_{i_{n-1}, \dots, i_0}$$

Remark: well defined because of proposition * and Fekete's lemma as in Yiftach's talk.

Interpretation: Sample X with $\mathcal{U}$: cannot tell part points in the same $U_i \in \mathcal{U}$. $\rightarrow \mathcal{U}_n^T$ roughly records the trajectory one can then observe from $t=0$ to $t=n-1$ with this "limited resolution".



$$x \in \mathcal{U}_1 \cap T^{-1}(\mathcal{U}_2) \cap T^{-2}(\mathcal{U}_2) \cap T^{-3}(\mathcal{U}_3).$$

Notice also $x \in \mathcal{U}_1 \cap T^{-1}(\mathcal{U}_1) \cap T^{-2}(\mathcal{U}_2) \cap T^{-3}(\mathcal{U}_3)$

$\rightarrow$ there is some redundance if one code trajectory by elements of $\mathcal{U}_n^T$
$\hookrightarrow$ we consider minimal subcover to avoid this the most we can.

Then we expect to see more and more trajectory as time increases:

$h_{Top}(T, \mathcal{U})$ is the exponential growth rate of the number of traj when $n \nearrow$. At time $n \gg 1$, we observe $\sim e^{n \cdot h_{Top}(T,\mathcal{U})}$ when X is sampled by $\mathcal{U}$.

Def (Topological entropy):

$$h_{Top}(T) = \sup_{\mathcal{U}} h_{Top}(T, \mathcal{U})$$

How to compute the topological entropy in concrete cases?

Def: Assume T is invertible. A topological generator for T is a finite cover $\mathcal{U} = (\mathcal{U}_i)_{i \in I}$ st for any sequence of indices $(i_k)_{k \in \mathbb{Z}}$:

$$\bigcap_{k \in \mathbb{Z}} T^{-k}(\overline{\mathcal{U}_{i_k}}) \text{ contains at most one point.}$$

Proposition: If $\mathcal{U}$ is a topological generator for T (invertible), then:

$$h_{Top}(T) = h_{Top}(T, \mathcal{U}).$$

Sketch of proof:

① Show that if $(\mathcal{U}_n)$ is a sequence of covers whose diameters $\to 0$ then $\lim_{n \to \infty} h_{Top}(T, \mathcal{U}_n) = h_{Top}(T)$.

because $\mathcal{U}_n$ refine any $\mathcal{V}$ when $n \gg 1$ and thus $h_{Top}(T, \mathcal{V}) \leq h_{Top}(T, \mathcal{U}_n) \leq h_{Top}(T)$

Lebesgue number.

② Show that $\operatorname{diam}\left(\bigvee_{k=-n}^{n} T^{-k} \mathcal{U} \right) \to 0$.

if not contradicts the assumption.

③ Show that

(3) show that $h_{Top}\left(T, \bigvee_{k=-n}^{n} T^{-k} \mathcal{U}\right) = h_{Top}(T,\mathcal{U})$.

easy computation using $(*)$.

Conclusion: ① + ② $\Rightarrow$ $\lim_{k\to\infty} h_{Top}\left(T, \bigvee_{k=-n}^{n} T^{-k}\mathcal{U}\right) = h_{Top}(T)$.

$$\lim_{k\to\infty} h_{Top}\left(T, \bigvee_{k=-n}^{n} T^{-k}\mathcal{U}\right) \overset{③}{=} h_{Top}(T,\mathcal{U})$$

$\blacksquare$

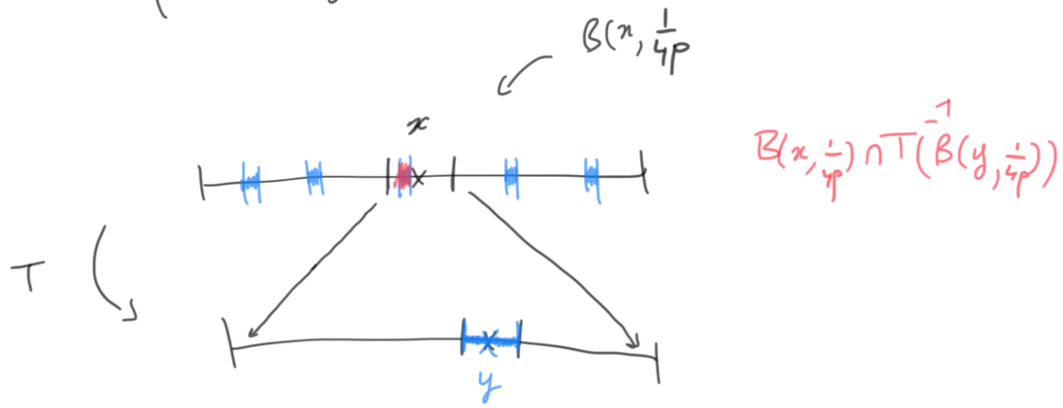
- Computation of Entropy in a concrete case using top. generator:

let $X = S^1$ and $T : \begin{array}{l} S^1 \to S^1 \\ x \mapsto p.x \mod 1 \end{array}$

Proposition: $h_{Top}(T) = \log(p)$.

proof: $\mathcal{U} = \left(B(x,\frac{1}{4p})\right)_{x\in X}$ $\quad B(x,\frac{1}{4p}) =]x-\frac{1}{4p}, x-\frac{1}{4p}[$.

1) $\mathcal{U}$ is a topological generator:



$$\mathrm{diam}\left| B(x,\tfrac{1}{4p}) \cap T^{-1}\left(B(y,\tfrac{1}{4p})\right)\right| \leq \frac{1}{p^2}.$$

Show by recurrence that the diameter $\bigcap^{N} T^{-k} \overline{B(x_k, \tfrac{1}{4p})}$ go to zero. thus contains at most one element.

2) Compute $N(\mathcal{U}^T_n)$.

Lemma 1: diameter of elements of $\mathcal{U}^T_n$ are $\leq \frac{1}{p^{n+1}}$

Lemma 2: $\mathcal{U}^T_n$ contains the balls of radius $\frac{1}{2p^{n+1}}$

lemma 1 $\Rightarrow N(\mathcal{U}^T_n) \geq p^{n+1} + 1$.

lemma 2 $\Rightarrow N(\mathcal{U}^T_n) \leq p^{n+1} + 1$.

thus: $h_{Top}(T,\mathcal{U}) = \lim \frac{1}{n}\log(p^{n+1}+1) \to \log p$.

$\blacksquare$

Remark: last time Yiftach showed:

$\forall \nu \in \mathcal{P}(X) \quad h_\nu(T) \leq \log(p)$ and can show
$h_{Leb}(T) = \log p$

$\Rightarrow \quad \sup h_\nu(T) = h_{Top}(T)$!

Not a coincidence:

<u>Variational principle.</u>

<u>Theorem:</u> X compact, $T: X \to X$ continuous.

$$\sup_\nu h_\nu(T) = h_{Top}(T).$$

<u>proof:</u> $\boxed{1 - \leq}$ let $\xi = \{P_1, \dots, P_k\}$ finite partition of X, and $\nu \in \mathcal{P}(X)$

- Show $\exists C > 0$, $\exists \mathcal{U}$ cover of X st $\forall T': X \to X$:

$$h_\nu(T', \xi) \leq h_{Top}(T', \mathcal{U}) + C.$$

Apply to $T' = T^k$

$$\to \quad h_\nu(T^k, \xi) = k\, h_\nu(T, \xi) \leq k\, h_{Top}(T, \mathcal{U}) + C$$

$$\to \quad h_\nu(T, \xi) \leq h_{Top}(T, \mathcal{U}) \leq h_{Top}(T)$$

$$\Rightarrow \quad \sup_\nu h_\nu(T) \leq h_{Top}(T).$$

let's construct $\mathcal{U}, C$: let $\varepsilon < \frac{1}{k\log(k)}$

$\forall i\ \exists Q_i \subseteq P_i$ compact and $\nu(P_i \setminus Q_i) \leq \varepsilon$

(ν is inner regular).

Define: $\eta = \{Q_0, Q_1, \dots, Q_k\}$ $Q_0 = X - \bigcup_{i=1}^{k} Q_i$

$$\mathcal{U} = \{Q_0 \cup Q_i\}_{i \in [1,k]}$$

η partition and $\mathcal{U}$ cover.

Compute:

$$h_\nu(T,\xi) \leqslant h_\nu(T,\eta) + \underbrace{H_\nu(\xi \mid \eta)}_{\leq 1 \text{ since } \varepsilon \leq \frac{1}{k\log k}}$$

remains to estimate $h_\nu(T,\eta) \leqslant \log N(\eta_n^T)$

Any element of $\mathcal{U}$ is union of 2 elements of η

$\Rightarrow$ any element of $\mathcal{U}_n^T$ is union of 2^n elements of η_n^T

$\Rightarrow \log N(\eta_n^T) \leqslant \log(N(\mathcal{U}_n^T) \cdot 2^n)$

$\Rightarrow h_\nu(T,\eta) \leqslant h_{Top}(T) + \log 2$

Thus: $h_\nu(T';\xi) \leq h_{Top}(T') + \underbrace{1 + \log 2}_{C}$.

$\boxed{2 - \geq}$

for any $\varepsilon > 0$, we will exhibit, denoting by $\mathcal{U}_\varepsilon$ the cover by balls of radius ε:

1) $\mu_n \to \mu \in P(X)$ T-invariant.

2) a measurable partition ξ of X $\mu(\partial\xi) = 0$

3) $\forall m$, sequences $\alpha_n^{(m)} \to 1$; $\varepsilon_n^{(m)} \to 0$

st:
$$\frac{1}{m} H_{\mu_n}(\xi_m^T) \geq \alpha_n^{(m)} \cdot \frac{1}{n} \log N(\mathcal{U}_\varepsilon{}_n^T) + \varepsilon_n^{(m)}$$

If that is so, taking the limit as $n \to \infty$. (lemma 5.7 in the book)

$$\frac{1}{m} H_\mu(\xi_m^T) \geq h_{Top}(T, \mathcal{U}_\varepsilon)$$

$$\Rightarrow \sup_\mu h_\mu(T) \geq h_\nu(T,\xi) \geq h_{Top}(T,\mathcal{U}_\varepsilon)$$

$$\Rightarrow \sup_\mu h_\mu(T) \geq h_{Top}(T, \mathcal{U}_\varepsilon).$$

$$\Rightarrow \sup_\mu h_\mu(T) \geq \lim_{\varepsilon \to 0} h_{Top}(T, \mathcal{U}_\varepsilon) = h_{Top}(T).$$

↑ prop 5.13 in the book.

let's find ν, ν_n ξ $\alpha_n^{(m)}$ $\varepsilon^{(m)}$ as before.

Let $(x_i^{(m)})$ be a maximal family of points of X for the following property:

$$\forall i,j \left(\forall k \leq m-1 \ d(T^k x_i^m, T^k(x_j^m)) \leq \varepsilon \Rightarrow x_j^{(m)} = x_i^{(m)} \right) (*)$$

We denote by N_m the cardinal of such a maximal family.

<u>Key fact</u> 1:

$$N_m \geq N((\mathcal{U}_\varepsilon)_m^T)$$

<u>proof</u>: Notice that $\left(\bigcap_{k=0}^{m-1} T^{-k}(T^k(x_i^{(m)}), \varepsilon) \right)_{i \in \{1,\dots,N\}}$ is a subcover of $(\mathcal{U}_\varepsilon)_m^T$. Indeed, otherwise, pick $x \in {}^c\left(\bigcup_{i=1}^{N_m} \bigcap_{k=0}^{m-1} T^{-k}(T^k x_i, \varepsilon) \right)$ and show $\{x, x_1^{(m)}, \dots, x_{N_m}^{(m)}\}$ verifies $(*)$ which is a contradic

(End of the proof of Key fact

<u>Define</u>:

$$\nu_m = \frac{1}{N_m} \sum_{i=1}^{N_m} \delta_{x_i^{(m)}}$$

$$\mu_m = \frac{1}{m} \sum_{k=0}^{m-1} (T^k)_* \nu_m$$

$\mu_m \to \mu$ that is invariant by T (up to extraction).

Now choose ξ a finite partition whose atoms have diameter $\leq \varepsilon$ + $\mu(\partial \xi) = 0$ (easy, see Lemma 3.13 in) and denote by a the number of atoms in

ξ.

Key fact 2: any atom of ξ_n^T contains at most one of the $(x_i^{(n)})$

proof: Suppose $x_i^{(n)}, x_j^{(n)}$ are in an atome A of ξ_n^T.

this means that $\forall k \leq T \quad d(T^k(x_i^{(n)}), T^k(x_j^{n})) \leq \varepsilon$.

and thus by $(*)$ $x_i^{(n)} = x_j^{(n)}$

End of proof of key fact 2 $\square$

As a consequence of key fact 2

$$H_{\mu_n}(\xi_n^T) = \sum_{A \in \xi_n^T} -\underset{\substack{\uparrow \\ 0 \text{ or } \frac{1}{N_n}}}{\mu(A)} \log \mu(A) = \log(N_n)$$

Finally a simple computation shows (more details page 148 of the book)

$$\exists \ \underset{\substack{\downarrow \\ 1}}{\alpha_n^{(n)}}, \quad \underset{\substack{\downarrow \\ 0}}{\varepsilon_n^{(n)}} \quad \text{st:} \quad \frac{1}{n} H_{\nu_n}(\xi_n^T) \geq \frac{\alpha_n^{(n)}}{n} H_{\mu_n}(\xi_n^T) + \varepsilon_n^{(n)}$$

$$\geq \alpha_n^{(n)} \frac{1}{n} \log N_n + \varepsilon_n^{(n)}$$

$$\geq \alpha_n^{(n)} \frac{1}{n} \log N(\mathcal{U}_\varepsilon T_n) + \varepsilon_n^{(n)}$$

$\blacksquare$