Schmidt Games

"On Badly Approximable Numbers and Certain Games" (Wolfgang Schmidt, 1965)

Let Ω and M be sets and $\varphi : \Omega \to P(M)$.

We say $F : \Omega \to P(\Omega)$ is a φ-ball-function ~~is~~ if for every $x \in \Omega$ we have $\emptyset \neq F(x) \subset \Omega$ such that for every $y \in F(x)$ we have $\varphi(y) \subset \varphi(x)$.

Let $\mathcal{F}, \mathcal{G} : \Omega \to P(M)$ be φ-ball-functions and let $S \subseteq M$. We define the following alternating two-player game between "Alice" and "Bob":

Step 1: Bob choses some $y_1 \in \Omega$ (or $\Omega' \subset \Omega$).

Step 2: Alice choses some $x_1 \in \mathcal{F}(y_1)$.

Step 3: Bob choses some $y_2 \in \mathcal{G}(x_1)$

Step 4: Alice choses some $x_2 \in \mathcal{F}(y_1)$

etc.

We now define for every $i \in \mathbb{N} \setminus \{0\}$:

$$A_i = \varphi(x_i) \subseteq M \quad \text{and} \quad B_i = \varphi(y_i) \subseteq M$$

We thus have:

$$B_1 \supseteq A_1 \supseteq B_2 \supseteq A_2 \supseteq \dots$$

Alice wins the game is

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} B_i \subset S.$$

Else, Bob wins.

We say that S is an $(\mathcal{F}, G)$-winning set is Alice can always win.

Example of a Shmidt Game

Let $d \in \mathbb{N} \setminus \{0\}$. Denote:

$$\Omega = \mathbb{R}^d \times \mathbb{R}_{>0}, \quad M = \mathbb{R}^d$$

$$\varphi : \mathbb{R}^d \times \mathbb{R}_{>0} \longrightarrow P(\mathbb{R}^d)$$

$$(x, r) \longmapsto B(x, r)$$

closed ball of radius $r>0$ and center $x \in \mathbb{R}^d$.

We define the following φ-ball-functions:
Let $\alpha, \beta \in (0,1)$. Then

$$\alpha : \mathbb{R}^d \times \mathbb{R}_{>0} \longrightarrow P(\mathbb{R}^d \times \mathbb{R}_{>0})$$

$$(x, r) \longmapsto \{(z, \alpha r) \mid B(z, \alpha r) \subset B(x, r)\}$$

$$\beta : \mathbb{R}^d \times \mathbb{R}_{>0} \longrightarrow P(\mathbb{R}^d \times \mathbb{R}_{>0})$$

$$(x, r) \longmapsto \{(z, \beta r) \mid B(z, \beta r) \subset B(x, r)\}$$

They are indeed φ-ball-functions!
By the definition.

Note that in this setting we must have

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} B_i = \{x^*\} \text{ for some } x^* \in \mathbb{R}^2.$$

If $x^* \in S$ then Alice wins.

We call this game the (α,β)-game.
We can take M to be any complete metrical space, and $\Omega = M \times \mathbb{R}_{>0}$.
If M is a Banach space with positive dimension, then $B = (x,r) \longleftrightarrow \varphi(x,r) = B(x,r)$.

(a^*b)-games

identity

$M = \mathbb{R}$, $\Omega = \{[a,b] \subset \mathbb{R} \mid a < b\}$, $\varphi([a,b]) = [a,b]$

For $a, b \in (1,\infty)$ we define the φ-ball functions:

$$\overset{*}{a} : \Omega \longrightarrow P(\Omega)$$

$$I \longmapsto \left\{[v,w] \subset \mathbb{R} \mid |w-v| = \frac{|I|}{a} \text{ and } [v,w] \cap I \neq \emptyset\right\}$$

$$\overset{*}{b} : \Omega \longrightarrow P(\Omega)$$

$$I \longmapsto \left\{[v,w] \subset \mathbb{R} \mid |w-v| = \frac{|I|}{b} \text{ and } [v,w] \cap I \neq \emptyset\right\}$$

Variant of (a^*a)-game: a-digit game.

$M = [0,1]$, $\Omega = \bigcup_{n=1}^{\infty} \{0,1,\dots,a-1\}^n$, $\Omega' = \{0,1,\dots,a-1\}$.

$F, G : \Omega \longrightarrow P(\Omega)$ $\quad F((a_1,\dots,a_n)) = \{(a_1,\dots,a_{n+1}) \in \Omega \mid a_{n+1} \in \{0,1,\dots\}\}$

Exercise: Every (a^*a)-winning set is a a-digit-winning set.

Strategies

Denote by H_n^A all histories of length $2n-1$ of "permissible" choices by the players. Analogously, denote by H_n^B all histories of $2n$ permissible choices of the two players.

<u>A strategy for Alice</u> in the game is a function $f : \bigcup_{n=1}^{\infty} H_n^A \to \Omega$ such that

$$f(B_1, A_1, B_2, A_2, \dots, A_{k-1}, B_k) \in \mathcal{F}(B_k)$$

for every $k \in \mathbb{N} \setminus \{0\}$ and $(B_1, \dots, B_k) \in H_k^A$.

We will denote sometimes $f(B_1, B_2, \dots, B_k)$

A strategy for Bob in the game is a function $g : \bigcup_{n=1}^{\infty} H_n^B \longrightarrow \Omega$ such that

$$g(B_1, A_1, B_2, A_2, \ldots, B_k, A_k) \in G(B_k)$$

for every $k \in \mathbb{N} \setminus \{0\}$ and $(B_1, \ldots, B_k, A_k) \in H_k^B$

Every two strategies generate a unique "play":

$$\langle f, g \rangle = (B_1, A_1, B_2, A_2, B_3, A_3, \ldots)$$

We say the sequence $(B_1, B_2, \ldots, B_k, \ldots)$ is an f-chain is the play $(B_1, f(B_1), B_2, f(B_1, B_2), B_3, \ldots)$ is consistent with f. Finite chains too.

- Winning strategy!

Lemma 1 (exercise)

Let $(B_1, B_2, B_3, \ldots)$ be a sequence such that $(B_1, B_2, \ldots, B_k)$ is a finite f-chain for every $k \in \mathbb{N} \setminus \{0\}$. Then this sequence is an f-chain.

(α, β)-games

Let $x \in M$ and $B(c,r) \in \Omega$

Then
$$e(x, B(c,r)) = \frac{d(x,c)}{r}$$

$$e(x, B(c,r)) = 0 \iff x = c$$

$$e(x, B) \leq 1 \iff x \in B.$$

M is Banach space with positive dimension.

$\Omega = M \times \mathbb{R}_{>0}$

$\varphi(x,r) = B(x,r)$

$\longleftrightarrow$

Lemma 5

Suppose $\alpha, \beta \in (0,1)$ such that $2\alpha \geq 1+\alpha\beta$. Then the only (α,β)-winning set is M.

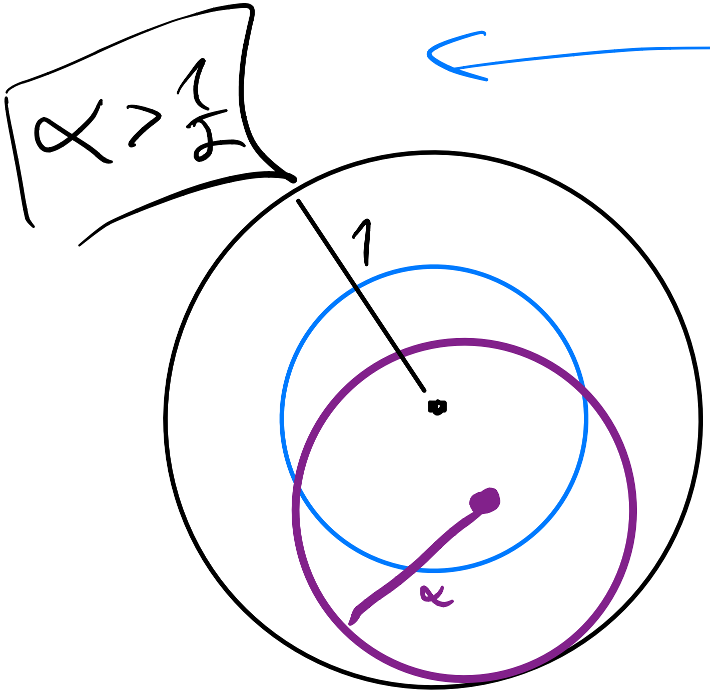
Proof

Let $x \in M$. Suppose by contradiction that $S \subset M$ is an (α,β)-winning set such that $x \notin S$. Suppose w.l.o.g. that Bob choses $B_1 = B(x,1)$.

Alice now need to choose a ball of radius α inside $B(x,1)$, i.e an element from $\alpha(B_1)$. Let $A_1 \in \alpha(B_1)$.

Then $\tilde{e}_1 = e(x, A_1) = \frac{d(x, c_1)}{\alpha} \leq \frac{1-\alpha}{\alpha} < 1$

$x \in A_1$



Now we have

$$\beta \leq \beta + (2\alpha - 1 - \alpha\beta) \cdot \frac{1}{\alpha} =$$

$$= \beta + 2 - \frac{1}{\alpha} - \beta = 2 - \frac{1}{\alpha} = 1 - \frac{1-\alpha}{\alpha} \leq 1 - \tilde{e}_1$$

Bob now can chose $B_2 = B(x, \alpha\beta) \in \mathcal{B}(A_1)$,

since $\beta \leq 1 - \tilde{e}$
$\Rightarrow \alpha\beta \leq \alpha - d(x, c_1)$. Thus bob

can always ensure that $\bigcap_{i=1}^{\infty} B_i = \{x\} \notin S$. Contradiction

Q.E.D.

Lemma 4 (exercise)

Let $e = e(x, B) \leq 1$ and $\gamma \in (0,1)$.
Every ball $B' \in \gamma(B)$ has:

$$\max\left\{0, \frac{e+\gamma-1}{\gamma}\right\} \leq e(x, B') \leq \frac{(e+1-\gamma)}{\gamma}$$

Lemma 6

Let $\alpha, \beta \in (0,1)$ such that $\alpha\beta \geq 1 + \alpha\beta$.
Then every dense set $S \subset M$ is (α, β)-mixing.

Proof

Let S be a dense set, and suppose Bob picks a ball of radius ρ and center $c \in M$. There is an $x \in S$ such that $d(x,c) \leq (1-\alpha)\cdot\rho$. Thus Alice can choose the ball $b(x,\alpha\rho) \in \alpha(B)$. Then we apply the method from lemma 5.

Q.E.D

Lemma 8

Let $\alpha, \beta \in (0,1)$ and $\alpha' \in (0, \alpha]$.
Then every (α, β)-winning set is also
an $(\alpha, \frac{\alpha\beta}{\alpha'})$-winning set.

Proof Let S be an (α, β)-winning set.

Let h be an $(\alpha, \beta; S)$-winning strategy.
We define a strategy f for the $(\alpha', \frac{\alpha\beta}{\alpha'})$-game
by: $\forall_{n \in \mathbb{N} \cup \{0\}} \left(f(B_1, B_2, \ldots, B_n) = A_n \in \frac{\alpha'}{\alpha}(h(B_1, \ldots, B_n)) \right)$
The strategy f is well defined since:

$$h(B_1) \in \alpha(B_1) \Rightarrow \frac{\alpha'}{\alpha}(h(B_1)) \subset \alpha'(B_1) \Rightarrow$$

$f(B_1) \in \frac{\alpha'}{\alpha}(h(B_1)) \subset \alpha'(B_1)$, and $f(B_1)$
is indeed a ball of radius $\frac{\alpha'}{\alpha} \cdot r$ $\alpha = \alpha' \cdot r$
included in $B_1 = B(x, r)$.

Suppose that $f(B_1, \dots, B_{n-1})$ is well defined,

i.e. $f(B_1, \dots, B_{n-1}) \in \alpha'(B_{n-1})$.

Let $B_n \in \frac{\alpha\beta}{\alpha'}(f(B_1, \dots, B_{n-1}))$.

Then we also have $B_n \in \beta(h(B_1, \dots, B_{n-1}))$, because
A_{n-1} is a ball inside B_{n-1} with radius $\alpha' \cdot r$.

Thus B_n is a ball of radius $\frac{\alpha\beta}{\alpha'} \cdot \alpha' \cdot r$ inside of A_{n-1}.

So B_n has radius $\alpha \cdot \beta \cdot r$ inside $A_{n-1} \subset B_{n-1}$.

Also $A_{n-1} \subset h(B_1, \dots, B_{n-1})$.

Thus B_n is also in $\beta(h(B_1,\dots,B_{n-1}))$.

Thus we can indeed define

$$f(B_1,\dots,B_n) \in \frac{\alpha'}{\alpha}[h(B_1,\dots,B_n)).$$

f is clearly a winning strategy since for every $n \in \mathbb{N}\setminus\{0\}$,

$$f(B_1,\dots,B_n) \subset h(B_1,\dots,B_n)$$

and thus

$$\bigcap_{i=1}^{\infty} f(B_1,\dots,B_i) \subset \bigcap_{i=1}^{\infty} h(B_1,\dots,B_i) \subset S$$

because h is optimal.

Q.E.D

Exercises

- **Lemma 9:** Every (α, β)-winning set is also a $(\alpha(\beta\alpha)^k, \beta)$-winning set for every $k \in \mathbb{N}$.

- **Corollary** Let $\alpha'\beta' = (\alpha\beta)^k$ for some $k \in \mathbb{N} \cup \{0\}$ and $\beta' \geq \beta$. Then every (α, β)-winning set is also an (α', β')-winning set.

α-winning set

Let $\alpha \in (0,1)$ and M a complete metric space. S is called an "α-winning" set if for every $\beta \in (0,1)$ it is an (α, β)-winning set.

Lemma 11

Let $0 < \alpha' < \alpha < 1$. Then every α-winning set is also an α'-winning set.

Proof

Let $S \subset M$ be α-winning set and let $\beta \in (0,1)$.

S is an α-winning set, thus it is also an $\left(\alpha, \frac{\alpha'\beta'}{\alpha}\right)$-winning

Set. By lemma 8, S is also an (α', β)-winning set.

Q.E.D

Lemma 12

If $\alpha > \frac{1}{2}$ then the only α-winning set is M.

Proof

There is a $\beta \in (0,1)$ such that $2\alpha \geq 1+\alpha\beta$.
The result is thus clear by lemma 5.

Theorem (2)

The intersection of countably many α-winning sets is itself an α-winning set.

Proof

Let $(S_\ell)_{\ell=1}^{\infty}$ be a sequence of α-winning sets. Denote $S = \bigcap_{\ell=1}^{\infty} S_\ell$. We want to show that S is an α-winning set. Let $\beta \in (0,1)$.

For every $\ell \in \mathbb{N} \setminus \{0\}$ denote f^ℓ to be a winning strategy for Alice in the $(\alpha, \beta(\alpha\beta)^{2^\ell - 1}; S_\ell)$-game.

We now define a strategy for Alice:

Let $(B_1, A_1, B_2, A_2, \dots, B_k)$ be a history of choices in the (α,β)-game. Let $\ell \in \mathbb{N}\setminus\{0\}$ and $t \in \mathbb{N}\setminus\{0\}$ be the unique positive integers such that

$$k = 2^{\ell-1} + (t-1)\cdot 2^{\ell}$$

We thus define:

$$f(B_1, A_1, B_2, \dots, B_k) =$$

$$= f^{\ell}\Big(B_{2^{\ell-1}}, A'_{2^{\ell-1}}, B_{2^{\ell-1}+2^{\ell}}, A'_{2^{\ell-1}+2^{\ell}}, B_{2^{\ell-1}+2\cdot 2^{\ell}}, \dots, B_{\underbrace{2^{\ell-1}+(t-1)\cdot 2^{\ell}}_{k}}\Big) \in \alpha(B_k)$$

for some sequence $\Big(B_{2^{\ell-1}}, A'_{2^{\ell-1}}, B_{2^{\ell-1}+2^{\ell}}, A'_{2^{\ell-1}+2^{\ell}}, B_{2^{\ell-1}+2\cdot 2^{\ell}}, \dots, B_{\underbrace{2^{\ell-1}+(t-1)\cdot 2^{\ell}}_{k}}\Big)$ then is consistent with f^ℓ.

1 3 5 7 9 11 13 15 17 19 21 23

2 6 10 14 18 22

4 12 20

8 24

16

This possible because

$$B_{2^{\ell-1}+j\cdot 2^\ell} \in \beta(\alpha\beta)^{2^{\ell}-1}\left(A_{2^{\ell-1}+(j-1)\cdot 2^\ell}\right)$$

for every ℓ and $0 \le j \le t-1$.

$$A_{2^{\ell-1}+(j-1)\cdot 2^\ell} \in \alpha\left(B_{2^{\ell-1}+(j-1)\cdot 2^\ell}\right)$$

$$\Rightarrow B_{2^{\ell-1}+j\cdot 2^\ell} \in \beta(\alpha\beta)^{2^{\ell}-1}\alpha\left(B_{2^{\ell-1}+(j-1)\cdot 2^\ell}\right).$$

For every $\ell \in \mathbb{N} \setminus \{0\}$ we get that the intersection

$$\bigcap_{t=1}^{\infty} B_{2^{\ell-1}+(t-1)\cdot 2^{\ell}} \subset S_{\ell}$$

since f^{ℓ} is a winning strategy in the $(\alpha, \beta(\alpha\beta)^{2^{\ell}-1})$-game.

Thus the intersection $\bigcap_{k=1}^{\infty} B_k \subset S_{\ell}$ for

every S_{ℓ} when using f.

Thus it is in S, and S is an (α,β)-winning set. So it is an α-winning set.

Q.E.D

Lemma 14 (exercise)

Let M be a banach space with positive dimension. Let S be an α-winning set for $\alpha \in (0, \frac{1}{2}]$. Then $S \setminus \{x_1, \dots, x_n, \dots\}$ is also α-winning (when the set is countable).

Exercise

Prove that there are numbers that are anormal according to every base.