Badly Approximable Vectors on Fractals

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To set the stage for today's discussion, we recall the following

Fact 1. *[Corollary to Dirichlet's approximation theorem] For any real number x there are infinitely many pairs of integers (p, q) such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}$$

Attempts and subsequent struggles to strengthen these results (e.g. increasing the exponent of q or replacing 1 with a smaller constant) lead us to the definition

Definition 2. A number $x \in \mathbb{R}$ is called *badly approximable* if there exists a constant $c > 0$ such that for all pairs of integers (p, q) we have

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^2}$$

Similiarly, in higher dimensions we have

Definition 3. A vector $x \in \mathbb{R}^n$ is called *badly approximable* if there exists a constant $c > 0$ such that for all $p \in \mathbb{Z}^n, q \in \mathbb{N}$ we have

$$\left| x - \frac{p}{q} \right| \geq \frac{c}{q^{\frac{n+1}{n}}}$$

where the vertical bars denote, for instance, the standard Euclidean norm. (We note that by setting $n = 1$ we recover the previous definition).

The set of badly approximable numbers/vectors is denoted BA. Our goal today is to prove the existence of (many) elements of BA in $\mathbb{R}^n$, for any n. We will see this both in general and even when restricting ourselves to vectors which lie on nice enough fractals (and even more than that). To formalize what we mean by 'many' we recall the definition of an (α, β)-Schmidt game from last week: two players, Alice and Bob, are equipped with parameters $0 < \alpha, \beta < 1$ and take turns choosing balls in some complete metric space, Bob begins by choosing a ball $B_0 = B(y_0, r)$ of arbitrary positive radius, Alice then chooses a ball $A_0 = B(x_0, \alpha r) \subset B_0$, play continues with the players alternately choosing balls B_i and A_i such that

$$B_0 \supset A_0 \supset B_1 \supset A_1 \supset \cdots$$

and for each i, the radius of A_i is α times the radius of B_i, while the radius of B_{i+1} is β times the radius of A_i.

Definition 4. A given target set S in the space is called (α, β)-*winning* if Alice has a strategy that forces the intersection of the balls to lie in S, regardless of Bob's choices. If some S is (α, β)-winning for a specific α and all $0 < \beta < 1$ we say the set S is α-*winning*. If S is α-winning for some $0 < \alpha < 1$ we say S is *winning*.

We like winning sets because they are 'large' sets, for example:

Theorem 5. *A winning set is uncountable.*

Theorem 6. *A countable intersection of α-winning sets $S_1, S_2, \ldots$ is also α-winning.*

These properties are analogous to the properties of sets which are large in other senses such as measure or Baire category.

We will begin by showing that BA is a winning set in $\mathbb{R}^1$ (no fractals yet, let's start slow).

Theorem 7. *For $0 < \alpha < \frac{1}{3}$ and any $0 < \beta < 1$, BA in $\mathbb{R}$ is an (α, β)-winning set.*

Remark 8. Idea of the proof: Alice's goal is to direct the intersection of the intervals to a point x that satisifies $\left|x - \frac{p}{q}\right| \geq \frac{c}{q^2}$ for some $c > 0$ and all rationals $\frac{p}{q}$. Alice will successivly 'get away from' rationals of larger and larger denominators, each time dealing with denominators in an exponentially growing chunk

$$R^{k-1} \leq q < R^k$$

for a careful choice of R.

Proof. Alice's strategy begins with arbitrary play until Bob chooses some interval of radius $r \leq \frac{\alpha\beta}{4}$. We 'reset' the move counter by denoting Bob's most recent interval as B_0 and describe Alice's strategy from this point on. We choose $c = \frac{1}{3}r$ and $R = \frac{1}{\sqrt{\alpha\beta}}$. We will show inductively that Alice can choose her intervals such that for all x in the interval B_n and all $0 < q < R^n$ we have $\left|x - \frac{p}{q}\right| \geq \frac{c}{q^2}$ for all integers p.

<u>Base case</u>: For $n = 0$ the required condition holds since there are no $0 < q < 1 = R^0$.

<u>Inductive step</u>: Assume $B_0, \ldots, B_{k-1}$ are already such that for $x \in B_i$ and $0 < q < R^i$ we have $\left|x - \frac{p}{q}\right| \geq \frac{c}{q^2}$ for all integers p. In her next move Alice has to worry only about fractions $\frac{p}{q}$ where $R^{k-1} \leq q < R^k$. The **surprising and important fact** is that Alice has to worry over at most **one** such fraction: assume, for the sake of contradiction, that there are two fractions $\frac{p}{q} \neq \frac{p'}{q'}$ with $R^{k-1} \leq q, q' < R^k$ that are close to points x, x' in B_{k-1},

respectively. Our contradiction will come from the fact that on one hand, these fractions must be close to each other, since the intervals are very small and the denomintors are not too small, and on the other hand the fractions must be far apart, since their denominators are not too large either.

Formally, if we have $\left|x-\frac{p}{q}\right|<\frac{c}{q^2}, \left|x'-\frac{p'}{q'}\right|<\frac{c}{q'^2}$, then from the triangle inequality we get

$$\begin{aligned}\left|\frac{p}{q}-\frac{p'}{q'}\right| &\leq \frac{c}{q^2}+\underbrace{2\left(\alpha\beta\right)^{k-1}r}_{\text{length of } B_{k-1}}+\frac{c}{q'^2} \\ &\leq 2cR^{2-2k}+2\left(\alpha\beta\right)^{k-1}r \\ &= 2cR^{2-2k}+2R^{2-2k}r \\ &= 2\left(c+r\right)R^{2-2k} \\ &< 4rR^{2-2k} \\ &\leq \alpha\beta R^{2-2k} \\ &= R^{-2k}\end{aligned}$$

while on the other hand,

$$\begin{aligned}\left|\frac{p}{q}-\frac{p'}{q'}\right| &= \left|\frac{pq'-p'q}{qq'}\right| \\ &\geq \frac{1}{qq'} \\ &> R^{-2k}\end{aligned}$$

And so, in the case that such a problematic $\frac{p}{q}$ exists, Alice only needs to make sure subsequent intervals do not intersect a subinterval I of B_{k-1} of length at most $2\cdot\frac{c}{q^2}\leq 2cR^{2-2k}$ centered at $\frac{p}{q}$. If I has its center to the left of or on the center b of B_{k-1}, then I is contained in the halfline

$$\begin{aligned}x &\leq b+\frac{c}{q^2} \\ &\leq b+\mathrm{c}R^{2-2k} \\ &= b+c\left(\alpha\beta\right)^{k-1}\end{aligned}$$

Alice will then choose her next interval to be as far right as possible:

$$A_{k-1}=\left(b+\left(\alpha\beta\right)^{k-1}r-2\alpha\left(\alpha\beta\right)^{k-1}r, b+\left(\alpha\beta\right)^{k-1}r\right)$$

The leftmost end of this interval is

$$b+\left(\alpha\beta\right)^{k-1}r-2\alpha\left(\alpha\beta\right)^{k-1}r=b+\left(\alpha\beta\right)^{k-1}r\left(1-2\alpha\right)$$

but

$$r\left(1-2\alpha\right)>\frac{r}{3}$$

$$= c$$

so Alice's choice and all subsequent intervals will not intersect I. If I has its center to the right of b, Alice will similarly choose her next interval to be as far left as possible and evade intersection. $\square$

A key idea in the previous proof is that a given interval contained at most one problematic rational point, in other words - all problemtic points were contained in a 0-dimensional affine hyperplane of $\mathbb{R}$. It turns out that this phrasing is the one that allows for useful generalization to higher dimensions. We have the following result in $\mathbb{R}^n$:

Lemma 9. *[Simplex lemma] For $\theta \in (0,1)$ let $R = \theta^{-\frac{n}{n+1}}$. Let*

$$U_k = \left\{ \frac{p}{q} : q \in \mathbb{N}, p \in \mathbb{Z}^n \text{ and } R^{k-1} \leq q < R^k \right\}$$

Denote by V_n the (Lebesgue) volume of the n-dimensional unit ball. Then for every $r > 0$ such that

$$r^n < \frac{\theta^n}{n! V_n}$$

and for every $x \in \mathbb{R}^n$ there exists an affine hyperplane L such that $U_k \cap B\left(x, \theta^{k-1} r\right) \subset L$.

Proof. Assuming for the sake of contradiction that there exist $n+1$ affinely independent vectors $\{v_i\}_{i=0}^n$ in $U_k \cap B\left(x, \theta^{k-1} r\right)$, denote their coordinates as $v_i = \left(v_i^j\right)_{j=1}^n$. Let Δ be the n-dimensional simplex that they define. We have

$$0 < \mathrm{Vol}_n(\Delta) = \frac{1}{n!} \det \begin{pmatrix} v_1^1 - v_0^1 & \cdots & v_1^n - v_0^n \\ \vdots & \ddots & \vdots \\ v_n^1 - v_0^1 & \cdots & v_n^n - v_0^n \end{pmatrix}$$

$$= \frac{1}{n!} \det \begin{pmatrix} 1 & v_0^1 & \cdots & v_0^n \\ 0 & v_1^1 - v_0^1 & \cdots & v_1^n - v_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & v_n^1 - v_0^1 & \cdots & v_n^n - v_0^n \end{pmatrix}$$

$$= \frac{1}{n!} \det \begin{pmatrix} 1 & v_0^1 & \cdots & v_0^n \\ 1 & v_1^1 & \cdots & v_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & v_n^1 & \cdots & v_n^n \end{pmatrix}$$

And since $v_i \in U_k$, for appropriate choices of integers $p_{i,j}$,q_i we get

$$= \frac{1}{n!} \det \begin{pmatrix} 1 & \frac{p_{0,1}}{q_0} & \cdots & \frac{p_{0,n}}{q_0} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{p_{n,1}}{q_n} & \cdots & \frac{p_{n,n}}{q_n} \end{pmatrix}$$

$$= \frac{1}{q_0 \cdots q_n \cdot n!} \underbrace{\det \begin{pmatrix} q_0 & p_{0,1} & \cdots & p_{0,n} \\ \vdots & \vdots & \ddots & \vdots \\ q_n & p_{n,1} & \cdots & p_{n,n} \end{pmatrix}}_{\text{Positive integer, and so } \geq 1}$$

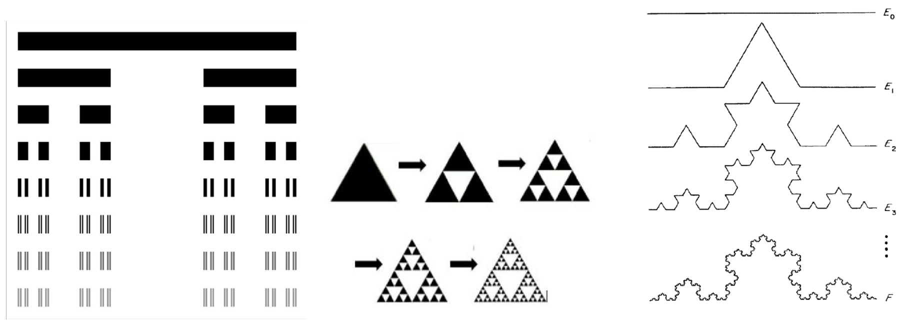
So we have

$$\begin{aligned} \mathrm{Vol}_n\left(\Delta\right) &\geq \frac{1}{q_0 \cdots q_n \cdot n!} \\ &> \frac{1}{R^{k(n+1)} \cdot n!} \\ &= \frac{\theta^{kn}}{n!} \\ &> V_n \theta^{(k-1)n} r^n \\ &= V_n \left(\theta^{k-1} r\right)^n \\ &= \mathrm{Vol}_n\left(B\left(x, \theta^{k-1} r\right)\right) \end{aligned}$$

contradicting $\Delta \subset B\left(x, \theta^{k-1} r\right)$. $\square$

We will use this lemma to follow a similiar strategy to the one dimensional case - at each step Alice will get away from a hyperplane instead of getting away from a point.

The strategy will be suitable not only for the case of a game played on $\mathbb{R}^n$ but also on a game where the choices of centers for both players are restricted to lie on certain fractals, such as the Cantor ternary set, the Sierpinski triangle and the Koch curve:



Usually when we tell somebody about these sets for the first time we describe them as starting with, for example, a line segment or a solid triangle, repeatedly deleting their 'centers' and finally taking the intersection of all the intermediate steps.

I remember taking a real analysis course and working on some homework problem about the Sierpinski triangle, I had a hard time visualizing all its intricacies and so I opened up MS Paint on my computer, as you do, and tried to follow the procedure: I drew a big triangle and then I tried to delete smaller and smaller triangles inside it. This turned out to be both very time consuming and inaccurate, and so eventually I found a better way - I again started from my initial triangle, but instead of deleting pieces, I shrunk it down and made 3 copies of the result, selected the whole thing and again shrunk it down and made 3 copies. After a few iterations I could no longer see any difference and was satisfied. Let's formalize this process:

Definition 10. An affine map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a *similiarity map* if it can be written as $\phi(x) = \alpha Bx + c$ with $\alpha \in \mathbb{R}$, $B \in \mathrm{Mat}_n(\mathbb{R})$, $c \in \mathbb{R}^n$ and B is orthogonal . It is *contracting* if, in addition, $0 < |\alpha| < 1$.

It is known that for any finite family of contracting similiarity maps $\phi_1, \ldots, \phi_m$ there exists a unique non-empty compact set K called the *attractor* of the family, such that

$$K = \bigcup_{i=1}^{m} \phi_i(K)$$

(this follows from the Banach fixed-point theorem on the space of non-empty compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff distance).

Example 11. The Cantor ternary set is the attractor for the family ϕ_1, ϕ_2:

$$\phi_1(x) = \frac{1}{3}x$$
$$\phi_2(x) = \frac{1}{3}x + \frac{2}{3}$$

Example 12. Closed cubes in $\mathbb{R}^n$ are attractors for a family of 2^n maps of scaling factor $\frac{1}{2}$, and so if we describe a strategy that shows BA is winning on these kinds of fractals, we are also proving as a special case that BA is winning in $\mathbb{R}^n$, since after the first step of the Schmidt game on $\mathbb{R}^n$, we can assume we are inside such a cube.

These attractors can, in general, still be too wild to work with conveniently, so we impose further restrictions:

Definition 13. A family $\phi_1, \ldots, \phi_m$ as above is said to satisfy *the open set condition* if there is an open set U such that for any i, $\phi_i(U) \subset U$ and for any $i \neq j$, $\phi_i(U) \cap \phi_j(U) = \varnothing$.

Lastly, since we will want all our bad points to be contained in a hyperplane, and subsequently wish to get away from that hyperplane, we impose the additional condition:

Definition 14. A family $\{\phi_i\}$ is *irreducible* if there is no finite collection of proper affine subspaces which is invariant under each ϕ_i.

The previous examples are all attractors of irreducible families that satisfy the open set condition.

Definition 15. The support $\operatorname{supp}\mu$ of a measure μ is the set of points x such that all balls centered at x have positive measure.

Definition 16. The d-dimensional Hausdorff measure of a set S is given by

$$\mathcal{H}^d(S) := \liminf_{r\to 0} \left\{ \sum_i r_i^d : \text{ there is a countable cover of } S \text{ by balls with radii } 0 < r_i < r \right\}$$

Definition 17. The Hausdorff dimension of a set S is

$$\begin{aligned}\dim_{\mathrm{H}}(S) &:= \inf\{d \geq 0 : \mathcal{H}^d(S) = 0\} \\ &= \sup\{d \geq 0 : \mathcal{H}^d(S) = \infty\}\end{aligned}$$

Fact 18. *Let δ be the Hausdorff dimension of K as above and μ be the δ-dimensional Hausdorff measure restricted to K. Hutchinson showed that $\mu(K)$ is positive and finite and that there exist constants $a, b > 0$ such that for every $0 < r < 1, x \in \operatorname{supp}(\mu)$:*

$$ar^\delta \leq \mu(B(x,r)) \leq br^\delta$$

such a measure is said to satisfy a power law.

Example 19. Cantor set with $\delta = \frac{\log 2}{\log 3}$, Sierpienski triangle with $\delta = \frac{\log 3}{\log 2}$, $[0,1]^n$ with $\delta = n$.

Fact 20. *From this result on power laws and from a result by Kleinbock, Lindenstrauss and Weiss, one can establish that μ is a so-called 'absolutely friendly measure':*

Definition 21. We call a measure μ *absolutely friendly* if there exist constants a, C, D such that for $r < 1$ and every $x \in \operatorname{supp}\mu$ we have:

1. For any $0 < \epsilon < r$ and any hyperplane L,

$$\mu\left(B(x,r) \cap L^{(\epsilon)}\right) < C\left(\frac{\epsilon}{r}\right)^a \mu(B(x,r))$$

   where $L^{(\epsilon)}$ is the ϵ neighbourhood of L. (We say μ is (C,a)-decaying).

2. And

$$\mu\left(B\left(x, \frac{5}{6}r\right)\right) > D \cdot \mu(B(x,r))$$

   (We say μ satisfies the Federer property).

From now on we consider a Schmidt game played on some such attractor (it is a compact subset of $\mathbb{R}^n$ so it is still complete). We have the following crucial step in Alice's strategy

Lemma 22. *Let $r < 1$, $0 < \alpha < \frac{1}{12}\left(\frac{D}{C}\right)^{\frac{1}{a}}$ and $\epsilon_0 < \frac{1}{12}\left(\frac{D}{C}\right)^{\frac{1}{a}} r$ then for any affine hyperplane L and $x \in \operatorname{supp}\mu$ there is an $x_0 \in \operatorname{supp}\mu$ such that*

1. $B(x_0, \alpha r) \subset B(x,r)$
2. $d\left(B(x_0, \alpha r), L^{(\epsilon_0)}\right) > \alpha r$
3. $d(B(x_0, \alpha r), \partial B(x,r)) > \alpha r$

The rationale here is that Alice wants to choose a permissible ball that is far from a given hyperplane and any bad points it may contain, and also far from the exterior of the previous ball and bad points that might there, close to the boundary.

Proof. [Sketch]

To interpret the seemingly mysterious conditions in the definition of an absolutely friendly measure, picture a 'no-entry' road sign. The Federer property tells us the shaded region near the boundary has small measure, and the (C, a)-decaying property tells us the shaded region close to a hyperplane has small measure, therefore the remaining unshaded region of 'good choices' for Alice's next ball's center intersects the support of μ in a positive-measure set, and so Alice chooses her next center x_0 in this region. □

Theorem 23. *Let α be as above, and $0 < \beta < 1$. BA is (α, β)-winning on K.*

Proof. [Sketch] Like in the one dimensional case Alice plays arbitrarily until Bob chooses a ball of radius r small enough for the conditions of the Simplex lemma to be satisfied by $\theta = \alpha\beta$ (and so $R = (\alpha\beta)^{-\frac{n}{n+1}}$). On Alice's k'th turn, the simplex lemma tells us all the rational vectors with denominators $R^{k-1} \leq q < R^k$ that are inside the current ball are contained in a hyperplane L. We (or rather Alice) use the previous lemma with a sufficiently small c to find a next center in $\operatorname{supp}\mu$ that is far enough from that hyperplane and also far from any possible bad rational vectors near the boundary of the current ball, in its exterior. □

Corollary 24. *BA is winning on $\mathbb{R}^n$.*

Proof. After the first step of the Schmidt game on $\mathbb{R}^n$, we can assume it is played on a large enough closed cube, which is an attractor for an irreducible family of contracting affine maps that satisfy the open set condition. □

Remark 25. With a small modification of the simplex lemma the same arguments show that for any non singular affine transformation Λ, $K \cap \Lambda\,(\mathrm{BA})$ is α-winning (with α depending only on K), and so we have our final

Corollary 26. *For any countable family Λ_i of non singular affine transformations, the intersection $K \cap \bigcap_i \Lambda_i\,(BA)$ is α-winning.*