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A PROOF OF OSELEDEC'S MULTIPLICATIVE ERGODIC THEOREM

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ABSTRACT
A new proof of a multiplicate ergodic theorem of Oseledec is presented in this paper.

§1. Introduction

Let (X, S, m) be a probability space and $T\colon X \to X$ a measure-preserving transformation (i.e. for $E \in S$, $T^{-1}(E) \in S$ and $m(T^{-1}(E)) = m(E)$). Let K be a local field and $u : \mathbf{Z} \times X \to M(r, K)$ be a measurable map such that for integers p, $q \geqq 0$ we have

$$(*) \qquad u(p, T^q(x)) \,.\, u(q, x) = u(p + q, x).$$

For a matrix $A \in M(r, K)$, we define its norm $\| A \|$ as follows: if K is archimedean (resp. non-archimedean) $A = \{\Sigma_{1 \leqq i, j \leqq r} | a_{ij} |^2\}^{1/2}$ (resp. $\sup_{1 \leqq i, j \leqq r} | a_{ij} |$) where a_{ij}, $1 \leqq i, j \leqq r$ are the entries of A. As usual for a real-valued function f on X, f^+ is defined by $f^+(x) = \max(0, f(x))$. With this notation we will establish the following.

THEOREM. *Assume that* $\log^+ \| u(1, .) \| \in L^1(X, m)$. *Let* $B = \{(x, v) \in X \times K^r \mid$ *the sequence* $1/n \log \| u(n, x)(v) \|$ *tends to a finite limit or to* $-\infty\}$. *Let* $X' = \{x \in X \mid (x, v) \in B$ *for* all $v \in K^r\}$. *Then* X' *contains a* S*-measurable subset* X'' *such that* $m(X'') = 1$. *Further there exist real valued non-negative measurable functions* $f_1 \leqq f_2 \leqq \cdots \leqq f_r$, *possibly taking the value* $-\infty$, *with the following properties. Let* $I = \{1 = i_1 < i_2 \cdots i_p < i_{p+1} = r + 1\}$ *be any* $(p + 1)$*-tuple of integers and* $X''(I) = \{x \in X'' \mid f_i(x) = f_j(x)$ *for* $i_q \leqq i, j < i_{q+1}$ *and* $f_{i_q}(x) < f_{i_{q+1}}(x)$ *for all* q *with* $1 < q < p\}$. *Then for* $x \in X''(I)$, $1 \leqq q \leqq p$,

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$$E(I,q)(x) = \{v \in K^r \mid \mathrm{Lt}\ 1/n \log \| u(n,x)(v) \| \leqq f_{i_q}(x)\}$$

is a vector subspace of K^r *(of dimension* $i_{q+1}-1$*). Furthermore, if we put* $E(I,0)=\{0\}$, Lt $1/n \log \| u(n,x)(v) \| = f_{i_q}(x)$ *if* $v \in E(I,q)(x) - E(I,q-1)(x)$. *The map* $x \to E(I,q)$ (x) *is a measurable map of* $X(I)$ *into the Grassman manifold. If K is* archimedean *the sequence* $A(n,x) = (u(n,x)^* u(n,x))^{1/2n}$ *converges for* $x \in X''$ *to a limit* $A(x) \in M(r,k)$. *The orthogonal complement of* $E(I,q)(x)$ *in* $E(I,q+1)(x)$, $x \in X(I)$, *is precisely the eigen-space of* $A(x)$ *corresponding to the eigen-value* $\exp f_{i_{q+1}}(x)$.

If T is ergodic, $\det u(1,x) = 1$ *and* $\overline{\mathrm{Lt}} \int_X 1/n. \log \| u(n,x) \| dm > 0$, *then* $f_1 < 0$ *and* $1 < i_2 \leqq r$.

This result is a slight generalisation of the theorem referred to in the title (Oseledec [1]): Oseledec proves the result under the additional assumption that T and $u(1,.)$ are invertible. The proof given in the present paper, I believe, is somewhat more transparent. The Oseledec theorem is deduced here from a weaker result due to Furstenberg and Kesten [2].

I would like to record my thanks to D. Ruelle and J. Tits for their interest in the work: the present version has some modifications suggested by Tits after reading an earlier manuscript; Ruelle pointed out that the proof contained in the earlier version in which T and $u(1,.)$ were assumed invertible goes over to the more general case with minor changes. I should also like to add that I came to know the Oseledec theorem through the beautiful application made of it by G. A. Margulis to solve the question of arithmeticity of discrete uniform subgroups in semisimple Lie groups.

§2. Proof of the theorem

By a standard argument we may assume that T is *ergodic.* We make this assumption in the sequel.

The following result is the Furstenberg–Kesten theorem referred to in the Introduction.

Proposition. *Let* (X,S,m), T *and* u *be as in the statement of the main theorem. Let*

$$Y = \{y \in X \mid \textit{the sequence } 1/n \log \|u(n,x)\| \textit{ tends to } -\infty \textit{ or a finite limit}\}.$$

Then $m(X - Y) = 0$.

A simple proof of this result can be found in [3]. For an element $A \in M(r,K)$ we denote by $E^p(A)$, the natural extension of A to the pth exterior power of K^r.

Evidently for each p, $E^p(u)$ again satisfy equations similar to $(*)$ so that we have the following

COROLLARY. *The notation is as in the proposition. Then if Y is the set of $x \in X$ for which the sequences $1/n. \log \| E^p(u(n,x)) \|$ converge for all p to $-\infty$ or a finite limit, $m(X-Y)=0$.*

The following result is well known.

LEMMA 1. *Let $M = \{A \in M(r,K) \mid \| Av \| = \| v \| \text{ for all } v \in K^r\}$, and D the set of all diagonal matrices in $M(r,K)$ with the diagonal entries in non-decreasing order of absolute values. Then there exist measurable maps $\boldsymbol{M}^*: M(r,K) \to M$, $\boldsymbol{M}'^*: M(r,K) \to M$ and $\boldsymbol{D}^*: M(r,K) \to D$ such that for $g \in M(r,K)$, we have $g = \boldsymbol{M}'^*(g)\, \boldsymbol{D}^*(g)\, \boldsymbol{M}^*(g)$.*

Let $\boldsymbol{M} = \boldsymbol{M}^* \circ u$, $\boldsymbol{M}' = \boldsymbol{M}' \circ u$ and $\boldsymbol{D} = D^* \circ u$. We denote by $d_i(n,x)$ the ith diagonal entry of $\boldsymbol{D}(n,x)$. Then we have evidently $\| u(n,x) \| = \| \boldsymbol{D}(n,x) \| = | \tilde{d}_r(n,x) |$. More generally $\| E^p(u(n,x)) \| = \Pi_{r-p<i\leqq r} | \tilde{d}_i(n,x) |$. From the Corollary above we know that each of the sequences $1/n\ \Sigma_{r-p<i<r} \log | \tilde{d}_i(n,x) |$ converge to a finite limit or tend to $-\infty$ for $x \in Y$ as in the corollary. Since $| \tilde{d}_i(n.x) | \leqq | \tilde{d}_{i+1}(n,x) |$, we conclude that each sequence $1/n. \log | \tilde{d}_i(n,x) |$, $1 \leqq i \leqq r$, converges to a finite limit or tends to $-\infty$. Let $\mathrm{Lt}_{n\to\infty} 1/n \log | \tilde{d}_i(n,x) | = d_i(x)$. Since we have assumed that T is ergodic the limits in question being T-invariant are constants almost everywhere (w.r.t. m). Thus we have a T-stable subset of X of measure 1 on which $d_i(x) = d_i$ is independent of x. We then have a unique partition $I = (1 = i_1 < i_2 \cdots i_p < i_{p+1} = r+1)$ such that $d_{i_q} = d_{i_{q+1}-1} < d_{i_{q+1}}$. In the sequel we take X itself to be the T-stable subset described above. Let $E(I,q,n)$ be the measurable map of X into the Grassman manifold which associates to each $x \in X$, the K-linear space $\boldsymbol{M}(n,x)^{-1}(\Sigma_{1\leqq i<i_{q+1}} K e_i)$ where $\{e_i \mid 1 \leqq i \leqq r\}$ is the standard basis of K^r.

CLAIM I. *$E(I,q,n)$ converges to a limit function $E(I,q)$ almost everywhere (w.r.t. m). Moreover if $v \in E(I,q)(x) - E(I,q-1)(x)$,*

$$\mathop{\mathrm{Lt}}_{n\to\infty} 1/n \log \| u(n,x)v \| = d_{i_q}.$$

We will need

LEMMA 2. *Let $B \subset \mathrm{GL}(r,K)$ be a compact subset. Then there is a constant $C > 0$ with the following property. Let $b = (b_{ij})_{1\leqq i,j\leqq r} \in B$ be any element such that*

$$| b_{ij} | < a_{l(i)}/a_{l(j)} \quad \text{for } l(i) < l(j)$$

where $0 \leqq a_1 < a_2 < a_3 \cdots < a_p$ *are real numbers and* $l(q)$, $1 \leqq q \leqq r$ *is defined by the inequality* $i_{l(q)} \leqq q < i_{l(q+1)}$. *Then if* $c = (c_{ij})$ *is the inverse of* B

$$|c_{ij}| < C a_{l(i)}/a_{l(j)} \quad \text{for} \quad l(i) < l(j).$$

PROOF. Let s be a permutation of $(1, 2, \cdots, r)$ such that $s(j) = i$. Then if we write s as a product of disjoint cycles, one of the cycles takes the form $(i, s(i), \cdots, s^q(i) = j)$ and we have for a suitable $M = M(B) \geqq 1$,

$$\begin{aligned}\prod_{l \neq j} b_{ls(l)} &< M^{r-q} \prod_{u=0}^{q-1} |b_{s^u(i)s^{u+1}(i)}| \\ &\leqq M^{r-q} \prod_{u=0}^{q-1} (a_{l(s^u(i))}/a_{l(s^{u+1}(i))}) \\ &\leqq M^r a_{l(i)}/a_{l(j)}.\end{aligned}$$

Let $D > 0$ be such that $|\det b|^{-1} < D$ for all $b \in B$. Then we have

$$c_{ij} = (\det b)^{-1} \sum_s \pm \prod_{l \neq j} b_{ls(l)}$$

and the required inequality follows if we set $C = r!\, M^r . D$. Hence the lemma.

LEMMA 3. *There exists a* S*-measurable subset* Z *with* $m(Z) = 1$ *such that for all* $x \in Z$, *the sequence* $S_n(x) = 1/n \sum_{0 \leqq q < n} \log^+ \| u(1, T^q(x)) \|$ *converges to a limit* L *(independent of* $x \in Z$*).*

This is just the pointwise ergodic theorem. As an immediate corollary we have

COROLLARY. *Given* $x \in Z$ *and* $1 > \varepsilon > 0$, *there exists* $N = N(\varepsilon, x)$ *such that for all* $n > N$

$$\| u(1, T^n x) \| < \exp n\varepsilon.$$

PROOF. Let $S_n = S_n(x)$, $x \in Z$. Choose $N'(\varepsilon, x)$ such that $|S_{n+m} - S_n| < \varepsilon$ and $|S_n - L| < \varepsilon$ for $m \geqq 0$ and $n \geqq N'(\varepsilon, x)$. Then we have

$$\begin{aligned}\log^+ \| u(1, T^n x) \| &= (n+1) S_{n+1} - n S_n \\ &= n(S_{n+1} - S_n) + (S_{n+1} - L) + L \\ &< (n+1)\varepsilon + L.\end{aligned}$$

Thus $u(1, T^n x) < A \exp n\varepsilon$ for some A independent of ε. If we choose $N''(\varepsilon, x)$ such that

$$A \leqq \exp N''(\varepsilon, x) . \varepsilon$$

and $n \geqq \max(N', N'')$ we have

$$\| u(1, T^n x) \| < \exp 2n\varepsilon,$$

which proves our contention.

LEMMA 4. *Given $1 > \varepsilon > 0$ there exists $N = N(\varepsilon, x)$, $x \in Z$ with the following property. Let $v \in E(I, q, n)$ be a unit vector and set*

$$v = v' + \sum_{i \geqq i_{q+1}} b_i M(n+1, x)^{-1} e_i, \qquad v' \in E(I, q, n+1).$$

Then $| b_i | < \exp\{-n(d_i - d_{i_q} - \varepsilon)\}$ if $n \geqq N$.

PROOF. Applying $u(n+1, x) = u(1, T^n x).u(n, x)$ to the L.H.S., we get (using Lemma 3) for $n \geqq N_1(\varepsilon, x)$

$$\| u(n+1, x)v \| < \exp n\varepsilon . \exp n(d_{i_q} + \varepsilon)$$
$$= \exp n\,(d_{i_q} + 2\varepsilon).$$

On the other hand for $n \geqq N_2(\varepsilon, x)$

$$\| u(n+1, x)v \| \geqq | b_i | \| D(n+1, x)e_i \|$$
$$\geqq | b_i | \exp(n+1)(d_i - \varepsilon)$$

leading to the inequality

$$| b_i | < A \exp\{-n\,(d_i - d_{i_q} - 3\varepsilon)\}$$

and $A > 0$ is a constant independent of ε. If we choose $N_3(\varepsilon, x)$ so large that $A \leqq \exp N_3(\varepsilon, x).\varepsilon$ and take $n \geqq \max(N_1, N_2, N_3)$ we get

$$| b_i | < \exp\{-n\,(d_i - d_{i_q} - 4\varepsilon)\}$$

which gives us the desired result.

Combining Lemma 4 with Lemma 2, we get (in Lemma 2, take $a_i = \exp(d_i - i\varepsilon)$)

LEMMA 5. *Let $v \in E(I, q, n+1)$ be a unit vector and*

$$v = v'_* + \sum_{i \geqq i_{q+1}} b_{i*} M(n, x)^{-1} e_i, \qquad v'_* \in E(I, q, n).$$

Then for $\varepsilon > 0$ there exists $N = N(\varepsilon, x)$ such that

$$| b_{i*} | < \exp\{-n\,(d_i - d_{i_q} - \varepsilon)\} \quad \textit{if } n \geqq N.$$

It follows from Lemma 4 that we can cr ıstruct inductively sequences $(v_n^i \mid i \leqq i < i_{q+1})$ starting with a basis $(v_N^i \mid 1 \leqq i < i_{q+1})$ of $E(I, q, N)(x)$ with N sufficiently large, and taking v_{N+1}^i to be a unit vector in $E(i, q, n+1)(x)$ such that

$$\| v_{n+1}^i - v_n^i \| \leqq r \exp\{-n(d_{i_{q+1}} - d_{i_q} - \varepsilon)\}.$$

Then (v_n^i) converge to v^i, $1 \leqq i < i_{q+1}$ which are linearly independent. Thus $E(I, q, n)$ converges to a limit $E(I, q)$ for all q. Moreover given $\varepsilon > 0$, it follows from Lemma 5 that we have

$$v_n^i - v^i = w + \sum_{i \geqq i_{q+1}} c_i M(n, x)^{-1} e_i$$

where $w \in E(I, q, n)$ and $\| w \| < \exp\{-n(d_{i_{q+1}} - d_{i_q} - \varepsilon)\}$ while

$$| c_i | < \exp\{-n(d_i - d_{i_q} - \varepsilon)\}.$$

It follows that

$$\| u(n, x) v^i \| \leqq \| u(n, x) v_n^i \| + \| u(n, x)(v^i - v_n^i) \|$$

and $\| u(n, x) v_n^i \|$ is clearly less than or equal to $\exp\{+n(d_{i_q} + \varepsilon)\}$; from the inequalities for $\| w \|$ and c_i above we see that the second term is $O(\exp n(d_{i_q} + \varepsilon))$. We see thus

$$\overline{\mathrm{Lt}} \log \| u(n, x) v^i \| \leqq d_{i_q} \qquad \text{for all } v^i, \quad 1 \leqq i < i_{q+1}.$$

On the other hand if $v^i \notin E(I, q-1)(x)$, one sees that for n *sufficiently large* v^i has a projection v_n^{i*} *off* $E(I, q-1, n)(x)$ with $\| v_n^{i*} \| > c > 0$. It follows then for such a v^i,

$$\underline{\mathrm{Lt}}\, 1/n \log \| u(n, x) v^i \| > d_{i_q}.$$

This completes the proof except for the last statement.

To prove the last statement, let $g(x) = \log \| u(1, x) \|$. Then

$$1/n \,.\, \log \| u(n, x) \| \leqq g_n(x) = 1/n. \sum_{0 \leqq r \leqq n-1} g \circ T^r(x).$$

Now g_n converges almost everywhere to $\int_X g dm$ and $\int g_n dm = \int g dm$ for all n. The sequence $f_n(x) = g_n(x) - 1/n \log \| u(n, x) \|$ is a sequence of non-negative measurable functions so that applying Fatou's lemma we have

$$\underline{\mathrm{Lt}} \int_X f_n dm \geqq \int_X \underline{\mathrm{Lt}}\, f_n\, dm$$

i.e.

$$\overline{\text{Lt}} \int_X 1/n \,.\, \log \| u(n,x) \| \, dm \leqq \int_X \text{Lt}\ 1/n \log \| u(n,x) \| \, dm$$

$$= d_r \quad (= \text{Lt}\ 1/n \log \| u(n,x) \|).$$

If L.H.S. is positive, $d_r > 0$. Since det $u(n,x) = 1$, $d_1 + d_2 + \cdots d_r = 0$ leading to $d_1 < 0$. Thus $E(I,1)(x)$ is a proper subspace of K'.

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