Lecture – The initial dimension bound

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- Let $$\sigma = \sum_{i=1}^{m} d_i \phi_{i*}\sigma$$

  where $\phi_i(x) = r_i x + b_i$, $r_i > 0$,

  $$\sum_{i=1}^{m} d_i = 1,$$

  be a borel probability self-similar measure on $\mathbb{R}$.

- Denote
$$P = \{a(t)\,u(s) \mid t>0, s \in \mathbb{R}\} \subseteq G = SL_2(\mathbb{R}),$$
the subgroup of upper triangular matrix with positive diagonal.

For every $g \in P$, we can write uniquely

$$\boxed{g = a(r_g)^{-1} \cdot u(b_g) = \begin{pmatrix} r_g^{-1/2} & r_g^{-1/2} b_g \\ 0 & r_g^{1/2} \end{pmatrix}}$$

$r_g \in \mathbb{R}_+$, $b_g \in \mathbb{R}$.

Let $\mathrm{Aff}^+(\mathbb{R})$ denote the group of orientation preserving affine transformation on $\mathbb{R}$.

We identify P with $\mathrm{Aff}(\mathbb{R})^+$ by anti isomorphism,

$$\Phi : P \longrightarrow \mathrm{Aff}(\mathbb{R}^+)$$

$$\Phi(g)(s) = r_g s + b_g,$$

$$\Phi(g_1 g_2) = \Phi(g_2)\cdot \Phi(g_1).$$

- Let λ be a probability measure on $\mathrm{Aff}^+(\mathbb{R})$ and μ corresponding measure on P i.e., $\mu = \Phi^{-1}_* \lambda$.

- Denote

$$\lambda = \sum_{i=1}^{m} \lambda_i \delta_{\phi_i}, \text{ then we have}$$

$$\sigma = \lambda * \sigma$$

$$\Rightarrow \sigma \text{ is } \lambda\text{-stationary.}$$

For every test function f

$$\int f(x)\, d(\lambda * \sigma)(x) = \sum_{i=1}^{m} \lambda_i \int f(\phi_i(x))\, d\sigma(x)$$

$$= \sum_{i=1}^{m} \lambda_i \int f(x)\, d(\phi_{i*}\sigma)(x)$$

$$= \int f(x)\, d\left(\sum_{i=1}^{m} \lambda_i \phi_{i*}\sigma\right)(x)$$

$$= \int f(x)\, d\sigma(x).$$

- Define $\sigma^{(m)}$ as the image of μ^{*m} under under the map

$$g \longrightarrow bg$$

$\Rightarrow$ for any Borel set $A \subseteq \mathbb{R}$

$$\sigma^{(m)}(A) = \mu^{*m}\left(\{g \in P : bg \in A\}\right).$$

Hence

$$\boxed{\sigma^{(m)} = \lambda^{*m} * \delta_0}$$

Proposition (3.1) There exist $A, \kappa > 0$ such that for every $x \in X$, $\rho > 0$, $n \geq |\log \rho| + A|\log \mathrm{inj}(x)|$, we have

We will not assume X compact.

$$\forall y \in X, \quad \mu^{*n} * \delta_x (B_\rho y) \ll \rho^\kappa. \qquad \text{--- (1)}$$

Proof

STEP (1) Let $\kappa > 0$ be a parameter (small). Let $0 < \rho < \frac{1}{10}$ (we will see later $\rho \leq_\kappa 1$), $n \geq |\log \rho|$

$x, y \in X = G/\Lambda = SL_2(\mathbb{R})/\Lambda$ and assume that

$$\boxed{\mu^{*n} * \delta_x (B_\rho y) \geq \rho^\kappa} \qquad \text{--- (2)}$$

Let $\alpha = \frac{1}{10(\lambda+1)}$ and $m = \lfloor \alpha |\log \rho| \rfloor$, where

$$\lambda = -\int_P \log r_g \, d\mu(g) > 0.$$

Now write

$$\mu^{*n} * \delta_x = \mu^{*m} * \mu^{*(n-m)} * \delta_x.$$

Therefore

$$\mu^{*n} * \delta_x (B_\rho y) = \int_X \mu^{*m} * \delta_z (B_\rho y) \, d\left(\mu^{*(n-m)} * \delta_x\right)(z)$$

Define

$$\boxed{Z = \left\{ z \in X : \mu^{*m} * \delta_z (B_\rho y) \geq \rho^{2\kappa} \right\}}$$

So on $X - Z$, we have

$$\mu^{*m} * \delta_z (B_\rho y) < \rho^{2\kappa} \qquad \text{--- (3)}$$

By (2), we have

$$\rho^K \le \mu^{*m} * \mu^{*(n-m)} * \delta_x (B_\rho Y)$$

$$= \int_Z \mu^{*m} * \delta_z (B_\rho Y) \, d\left(\mu^{*(n-m)} * \delta_x\right)(z)$$

$$+ \int_{X-Z} \mu^{*m} * \delta_z (B_\rho Y) \, d\left(\mu^{*(n-m)} * \delta_x\right)(z) \quad \text{——— (3.1)}$$

Since $\mu^{*m} * \delta_z$ is a probability measure on X

$$\Rightarrow \quad \mu^{*m} * \delta_z (B_\rho Y) \le 1 \quad \forall\, z \in X \quad \text{—— (4)}$$

By (3), (3.1), (4),

$$\rho^K \le \mu^{*(n-m)} * \delta_x (Z) + \int_{X-Z} \mu^{*(m)} * \delta_z (B_\rho Y) \, d\left(\mu^{*(n-m)} * \delta_x\right)(z)$$

$$\le \mu^{*(n-m)} * \delta_x (Z) + \rho^{2K}$$

$$\Rightarrow$$

$$\mu^{*(n-m)} * \delta_x (Z) \ge \rho^K - \rho^{2K} = \rho^K (1 - \rho^K) \ge \rho^{2K}$$

Provided $\rho \le \left(\frac{1}{2}\right)^{1/K}$

$$\Rightarrow \quad \boxed{\rho << \frac{1}{K}}$$

STEP (2)

In the rest of the proof our aim is to show the set Z does not exist

Fix $z \in Z$. Recall that $m = \lfloor \alpha |\log \rho| \rfloor$.

We have

$$\mu^{*m} * \delta_z(f) = \int_X f(x)\, d(\mu^{*m} * \delta_z)(x)$$
$$= \int_G f(gz)\, d\mu^{*m}(g),$$

For $f = I_{B_\rho Y}$

$$\mu^{*m} * \delta_z(B_\rho Y) = \boxed{\mu^{*m}\big(\underbrace{\{g \in P : gz \in B_\rho Y\}}_{\boxed{A}}\big) \geq \rho^{2k}} \quad \text{——— (3.3)}$$

We will use the set

$A = \{g \in P \mid gz \in B_\rho Y\}$ in the later part of the proof.

Lemma (2.3) (i) There exists $\gamma > 0$ such that

$$\sup_{n \geq 1} \int_{\mathbb{R}} |s|^\gamma \, d\sigma^{(n)}(s) < \infty.$$

Fix $T > 0$ and set

$A = \{ s \in \mathbb{R} \mid |s| > T \}$. Then

$$\int_{\mathbb{R}} |s|^\gamma d\sigma^{(m)}(s) \geq \int_A |s|^\gamma d\sigma^{(m)}(s) \geq T^\gamma \int_A d\sigma^{(m)}(s)$$
$$\geq T^\gamma \sigma^{(m)}(A) \quad \text{——— (3.4)}$$

$$\Rightarrow \quad \sigma^{(m)}(A) = \boxed{\sigma^{(m)}\{|s| > T\} \leq T^{-\gamma} \int_{\mathbb{R}} |s|^\gamma d\sigma^{(m)}(s)}$$

Take $T = \rho^{-\frac{4K}{\gamma}}$.

$$\mu^{*m}\left\{ g \in P : |b_g| > \rho^{-\frac{4K}{\gamma}} \right\} = \sigma^{(m)}\left\{ |s| > \rho^{-\frac{4K}{\gamma}} \right\}$$

$$= \sigma^{(m)}\left\{ |s|^{\gamma} > \rho^{-4K} \right\} \overset{3.4}{\leq} \rho^{4K} \int_{\mathbb{R}} |s|^{\gamma} \, d\sigma^{(m)}(s)$$

$$\underset{\text{Lemma 2.3.1}}{\leq} \rho^{4K} c_0$$

$$\leq \rho^{3K}$$

Hence

$$\boxed{\mu^{*m}\Big(\underbrace{\left\{ g \in P : |b_g| \leq \rho^{-\frac{4K}{\gamma}} \right\}}_{\boxed{B}} \Big) \leq 1 - \rho^{3K} \qquad (3.5)}$$

STEP 3

We have

$$r_{g_1 g_2 \cdots g_m} = r_{g_1} \cdot r_{g_2} \cdots r_{g_m}$$

$$\Rightarrow \log r_{g_1 g_2 \cdots g_m} = \log r_{g_1} + \cdots + \log r_{g_m}.$$

Denote

$$X_j = \log r_{g_j}.$$

Then

$$S_m = \log r_{g_1 g_2 \cdots g_m} = X_1 + X_2 + \cdots + X_m$$

Since

$$E[X_1] = \int_P \log r_{g_1} \, d\mu(g_1) = -l$$

$$\Rightarrow E[S_m] = -l \cdot m$$

$\Rightarrow$ The average logarithmic dilation after m-steps is $-l \cdot m$.

By the law of large number, there exist $\varepsilon>0$ depending only on μ, such that

$$\mu^{*m}\left(\{g\in P \mid \log \sigma_g \in [-(\lambda+1)m, -(\lambda-1)m]\}\right) \geq 1-e^{-\varepsilon m}.$$

Since

$$m=\lfloor \alpha|\log\rho|\rfloor, \quad \alpha=\frac{1}{10(\lambda+1)}$$

$$\Rightarrow \quad m \geq \alpha|\log\rho|-1$$

$$\Rightarrow \quad e^{-\varepsilon m} \leq e^{-\varepsilon(\alpha|\log\rho|-1)} = e^{\varepsilon}\cdot e^{\varepsilon\alpha\log\rho}$$

$$= e^{\varepsilon}\cdot\rho^{\varepsilon\alpha}$$

Hence

$$\leq \rho^{\varepsilon\alpha}$$

$$\mu^{*m}\Big(\underbrace{\{g\in P \mid \log\sigma_g \in [-(\lambda+1)m, -(\lambda-1)m]\}}_{\boxed{D}}\Big) \geq 1-\rho^{\varepsilon\alpha} \qquad (3.6)$$

<u>STEP (4)</u> Recall that (3.3), (3.5), (3.6) We denoted

$$A=\{g\in P \mid gz\in B_\rho y\}, \quad B=\{g\in P : |b_g|\leq \rho^{-\frac{4K}{\gamma}}\}$$

and ~~shown that~~

$$D=\{g\in P : \log\sigma_g \in [-(\lambda+1)m, -(\lambda+1)m]\},$$

and shown that

$$\mu^{*m}(A)\geq \rho^{2K}, \quad \mu^{*m}(B)\geq 1-\rho^{3K},$$

$$\mu^{*m}(D)\geq 1-\rho^{\alpha\varepsilon}.$$

Hence

$$\mu^{*m}(A \cap B \cap D) \geq \mu^{*m}(A) - \mu^{*m}(B^c) - \mu^{*m}(D^c)$$

$$\geq \rho^{2K} - \rho^{3K} - \rho^{\alpha\varepsilon}.$$

We take $\boxed{3K < \alpha\varepsilon}$, so for $0 < \rho < 1$

$$\Rightarrow \rho^{3K} \geq \rho^{\alpha\varepsilon}$$

$$\Rightarrow \mu^{*m}(A \cap B \cap D) \geq \rho^{2K} - \rho^{3K} \cdot 2$$

$$\geq \frac{1}{2}\rho^{2K} \qquad \text{—— (1)}$$

- For the set B, the variable bg lies in the interval

$$\left[-\rho^{-\frac{4K}{\gamma}}, \rho^{-\frac{4K}{\rho}}\right]$$

For large $\boxed{c > 1,}$ we cut this interval into subintervals of length ρ^{cK}, then the number of pieces is at most

$$N_b = \left\lceil 2\rho^{-\frac{4K}{\rho}} \cdot \rho^{-cK} \right\rceil$$

- On the set D, the variable $\log rg$ lies in

$$[-(l+1)m, -(l-1)m]$$

Cutting this interval into pieces of length ρ^{cK},

the number of pieces is at most

$$N_r = \lceil 2m\rho^{-cK} \rceil.$$

By pigenhole principal, there exists $(b_0, r_0) \in \mathbb{R}^2$ with

$$|b_0| \leq \rho^{\frac{-4K}{\gamma}}, \quad r_0 \in \left[\bar{\iota}^{(\iota+1)m}, \bar{\iota}^{(\iota-1)m}\right]$$

Such that, the set

$$\boxed{E = \{ g \in P \mid gz \in B_\rho y \}, \ |b_g - b_0| \leq \rho^{CK}, \ |\log r_g - \log r_0| < \rho^{CK} \}}$$

has measure

$$\mu^{*m}(E) \geq \frac{\rho^{2K} - \rho^{3K} - \rho^{\kappa\varepsilon}}{\lceil 2\rho^{\frac{-4K}{\gamma}} \cdot \bar{\rho}^{CK} \rceil \cdot \lceil 2m\bar{\rho}^{CK} \rceil}$$

$$\overset{(1)}{\geq} \frac{\frac{1}{2}\rho^{2K}}{\lceil 2\rho^{\frac{-4K}{\gamma}} \cdot \bar{\rho}^{CK} \rceil \cdot \lceil 2m\bar{\rho}^{CK} \rceil}$$

$$\geq \frac{\frac{1}{2}\rho^{2K}}{4\rho^{\frac{-4K}{\gamma}} \cdot \bar{\rho}^{CK} \cdot 4m\bar{\rho}^{CK}} = \frac{\frac{1}{2}\rho^{2K}}{16\rho^{\frac{-4K}{\gamma}} \cdot m\bar{\rho}^{CK \cdot 2}}$$

$$= \frac{1}{32} \frac{\rho^{(2+\frac{4}{\gamma}+2C)K}}{m} \qquad (2)$$

Since $m \sim |\log \rho|$, so for fixed $\delta > 0$

$$m \leq \bar{\rho}^{\delta}$$

$$\overset{(2)}{\Longrightarrow} \quad \mu^{*m}(E) \geq \frac{1}{32}\rho^{(2+\frac{4}{\gamma}+2C)K+\delta}$$

Now choose C large enough and $K <\!\!<_C 1$ so that

$$(2+\tfrac{4}{\gamma})K + \delta \leq 2CK \implies C \geq \tfrac{4}{\gamma}$$

Hence

$$\mu^{*m}(E) \geq \rho^{4CK}$$

provided $C \geq \frac{4}{\gamma}$, $K <\!\!<_C 1$, $\rho <\!\!<_K 1$

STEP 5 Idea of contradiction

From here they choose $g_1, g_2 \in E$ and shown that

We will recall dist in G and $G/\Lambda = X$, in Step 6.

$$\mathrm{dist}(z, g_1^{-1} g_2 z) \ll \rho^{1/2} \quad \text{—— (1)}$$

$$\rho^{\frac{1}{4}} \leq \mathrm{dis}(g_1^{-1} g_2, id) \ll \rho^{\frac{ck}{2}} \quad \text{—— (2)}$$

$$\Rightarrow \quad \mathrm{inj}(z) \leq \mathrm{dist}(id, g_1^{-1} g_2) + \mathrm{dist}(z, g_1^{-1} g_2 z)$$
$$\ll \rho^{\frac{1}{2}} + \rho^{\frac{ck}{2}}$$
$$\ll \rho^{\frac{ck}{4}} \quad \text{—— (3)}$$

For compact case, inj radius is bounded below

$$\Rightarrow \quad \mathrm{inj}(x) > a_0 > 0 \ \forall x \in X$$

which is a contradiction of (3)

$\boxed{\text{In the rest of proof, we will find } g_1, g_2 \in E \text{ satisfying (1) and (2)}}$

STEP ~~2~~ 6

$G = SL_2(\mathbb{R})$, $\Lambda \subseteq G$ is a lattice and $X = G/\Lambda$.

We equip G with the right invariant Riemannian metric comming from the orthonormal basis

$$\left\{ u_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ u_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ u_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

Lemma (A) For any $u, v \in G$

$$d(gu, gv) \leq \|Ad(g)\| \, d(u,v),$$

where

$$Ad(g): T_1 G \longrightarrow T_1 G$$

$$Ad(g)(X) = gXg^{-1}$$

After identifying

$$T_h G \cong T_1 G \cong T_{gh} G \qquad \text{——— (1)}$$

$$(dL_g)_h = Ad(g).$$

Proof ÷

Let $Xh \in T_h G$, $X \in T_1 G$ be a tangent vector, then

$$(dL_g)_h(Xh) = \frac{d}{dt}\Big|_{t=0} g\,(e^{tX} \cdot h) = gXh$$

Now, we rewrite gXh as a tangent vector at gh. Since tangent vector at gh have the form $Y(gh)$, $Y \in T_1 G$.

$$gXh = gXg^{-1} \cdot gh$$

$$\Rightarrow (dL_g)_h(Xh) = Ad(g) \cdot (X)(gh)$$

$$\Rightarrow (dL_g)_h \overset{(1)}{=} Ad(g).$$

Let $\gamma: [0,1] \longrightarrow G$ be a smooth curve. Its length is

$$l(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| \, dt.$$

By the chain rule

$$\frac{d}{dt}(L_g \circ \gamma)(t) = (dL_g)_{\gamma(t)}(\dot{\gamma}(t))$$

Thus

$$\left\| \frac{d}{dt}(L_g \circ \gamma)(t) \right\| \leq \|Ad(g)\| \, \|\dot{\gamma}(t)\|$$

Integrating

$$l(L_g \circ \gamma) \leq \|Ad(g)\| \, l(\gamma).$$

For any $u, v \in G$, taking the infimum over all curves joining them gives

$$\boxed{d(gu, gv) \leq \|Ad(g)\| \, d(u,v).}$$

STEP (7) Let $g_1 \in E$. We will show that

$$\|Ad(g_1^{-1})\| \leq \rho^{-\frac{1}{2}}, \text{ provided } \kappa \ll_C 1.$$

We have

$$g_1^{-1} = u(-b_g)\, a(r_{g_1})$$

$$\Rightarrow \quad \|Ad(g_1^{-1})\| \leq \|Ad(u(-b_g))\| \cdot \|Ad(a^{\cdot}(r_{g_1}))\|.$$

Now we will bound each factor separately.

(1) Upper bound of $\|Ad(u(-b_{g_1}))\|$

We have

$$Ad(u(s))\, l_- = u(s)\, l_- \, u(s)^{-1}$$

$\Longrightarrow$

$$Ad(u(s))\,l_- = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} s & -s^2 \\ 1 & -s \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + s\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - s^2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= l_- + s\,l_0 - s^2 l_+$$

$$Ad(u(s))\,l_0 = l_0 - 2s\,l_+$$

$$Ad(u(s))\,l_+ = l_+$$

Hence there is a universal constant $c_1 > 0$ such that

$$\|Ad(u(s))\| \le c_1 (1+|s|)^2 \quad \text{—— (1)}$$

$\Longrightarrow$

$$\|Ad(u(-b_{g_1}))\| \le c_1 (1+|b_{g_1}|)^2 \quad \text{—— (2)}$$

Now, we use the defination of E, we have

$$|b_{g_1} - b_0| \le \rho^{cK}, \qquad |b_0| \le \rho^{-\frac{4K}{\gamma}}$$

$\Longrightarrow$

$$|b_{g_1}| \le |b_0| + \rho^{cK} \le 2\rho^{-\frac{4K}{\gamma}} \qquad \left(c > \frac{4}{\gamma}\right)$$

By (2), we get for some constant $c_2 > 0$

$$\boxed{\|Ad(u(-b_{g_1}))\| \le c_2\, \rho^{-\frac{8K}{\gamma}}} \quad \text{—— (3)}$$

(2) <u>The diagonal part $Ad(a(\tau_{g_1}))$</u>

Recall that

$$E = \left\{ g \in P : gz \in B_\rho Y : |b_{g_1} - b_g| \le \rho^{cK},\ |\log \tau_g - \log \tau_0| \le \rho^{cK} \right\}$$

where $|b_0| \le \rho^{-\frac{4K}{\gamma}}$, $\tau_0 \in [\lambda^{-(4+1)m}, \lambda^{-(4-1)m}]$. $|1 - \tau_g \tau_0^{-1}|$

We have

$$\mathrm{Ad}(a(r_g))\,\ell_- = r^{-1}\ell_-$$

$$\|$$

$$\begin{pmatrix} r^{-1/2} & 0 \\ 0 & r^{1/2} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r^{-1/2} & 0 \\ 0 & r^{1/2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ r^{-1} & 0 \end{pmatrix} = r^{-1}\ell_-.$$

and

$$\mathrm{Ad}(a(r_g))\,\ell_0 = 0$$

$$\mathrm{Ad}(a(r_g))\,\ell_+ = r\,\ell_+$$

Hence

$$\|\mathrm{Ad}(a(r_g))\| = \max(r_g, r_g^{-1})$$

Since $g_1 \in E$, we have

$$|1 - r_1 r_0^{-1}| \leq \rho^{\epsilon k}$$

$$\Longrightarrow \quad \max(r_1, r_1^{-1}) \leq 2\max(r_0, r_0^{-1})$$

$$\leq 2\,e^{(\ell+1)m}$$

$$\Longrightarrow \quad \|\mathrm{Ad}(a(r_{g_1}))\| \leq 2\,e^{(\ell+1)m} \qquad \text{—— (4)}$$

Since $m = \lfloor \alpha |\log \rho| \rfloor$, $\alpha = \frac{1}{10(\ell+1)}$

$$\Longrightarrow \quad e^{(\ell+1)m} \leq e^{(\ell+1)\cdot\alpha|\log\rho|}$$

$$\leq e^{\frac{1}{10}|\log\rho|}$$

$$\leq \rho^{-\frac{1}{10}}.$$

Hence

$$\|\mathrm{Ad}(a(r_{g_1}))\| \leq C_4\,\rho^{-\frac{1}{10}} \qquad \text{—— (5)}$$

By (3), and (5), $\|\mathrm{Ad}(g_1^{-1})\| \leq \|\mathrm{Ad}(u(-b_{g_1}))\| \cdot \|\mathrm{Ad}(a(\cdot b_{g_1}))\|$

$$\leq C_5\,\rho^{-\frac{8k}{8}} \cdot \rho^{-\frac{1}{10}}.$$

Now choose $K>0$ small enough so that

$$\frac{8K}{\gamma}+\frac{1}{10}<\frac{1}{2}$$

Hence

$$\|Ad(g_1^{-1})\| << \rho^{-\frac{1}{2}}$$

STEP (8) For any $g_1, g_2 \in E$. Fix $z \in Z$, we will prove

$$dist(z, g_1^{-1}g_2 z) << \rho^{1/2}.$$

Since both $g_1, g_2 \in E$, we have

$$g_1 z, g_2 z \in B_\rho Y$$

$$\Rightarrow \quad dist(g_1 z, g_2 z) \le dist(g_1 z, Y) + dist(g_2 z, Y) \le 2\rho$$

$$\Rightarrow \quad dist(g_1 z, g_2 z) << \rho. \quad \text{— (1)}$$

Now applying left multiplication by g_1^{-1}. Since left multiplication by g_1^{-1} is $\|Ad(g_1^{-1})\|$ Lipschitz, we get

$$\begin{aligned} dist(g_1^{-1}g_2 z, g_1^{-1}g_2 z) &\le \|Ad(g_1^{-1})\| \, dist(g_1 z, g_2 z) \\ &\overset{(1)}{<<} \|Ad(g_1^{-1})\| \, \rho \\ &\overset{\text{Step (7)}}{<<} \rho^{-\frac{1}{2}} \cdot \rho = \rho^{1/2} \end{aligned}$$

Hence

$$\boxed{dist(z, g_1^{-1}g_2 z) << \rho^{1/2}}$$

Step (9) We will choose $g_1, g_2 \in E$, such that

$$\rho^{\frac{1}{4}} \ll \operatorname{dist}(g_1^{-1}g_2, id) \ll \rho^{\frac{ck}{2}}.$$

Recall that

$$g_i = a(r_{g_i})^{-1} u(b_i), \quad i = 1,2.$$

We can rewrite

$$g_1^{-1} \cdot g_2 = u(-b_{g_2}) \cdot h\, u(b_{g_2})$$

where

$$h = u(b_{g_2} - b_{g_1})\, a(r_{g_1} r_{g_2}^{-1})$$

Claim: We can choose $g_1, g_2 \in E$, such that

$$|b_{g_2} - b_{g_1}| > \rho^{\frac{5ck}{\gamma}} = r \text{ (say)}.$$

Proof of claim

We know that

$$\mu^{*m}(E) \geq \rho^{4ck} \qquad (1)$$

Suppose for contradiction that all the values b_g with $g \in E$ lie in a single interval

$$I = s + [-r, r]$$

Then

$$E \subseteq \{g : b_g \in I\}.$$

$$\Rightarrow \mu^{*m}(E) \leq \mu^{*m}\{g \in P : b_g \in I\}$$
$$= \sigma^{(m)}(I) \qquad (1')$$

Recall Lemma (2.3) (iii) — There exists $\gamma>0$ such that

$$\forall\, n\geq 1\ \forall\, r>\bar{\iota}^{\,n}$$

$$\sup_{s\in\mathbb{R}} \sigma^{(n)}\left(s+[-r,r]\right) \ll r^{\gamma}.$$

By ① and ①', provided our $r=\rho^{\frac{5cK}{\gamma}}>\bar{\iota}^{\,m}$,

$$\rho^{4cK} \leq \mu^{*m}(E) = \sigma^{(m)}(I) \ll r^{\gamma}$$
$$= \rho^{5cK} \qquad \text{——— ②}$$

Contradiction. Hence E can not fit inside one interval of length $r=\rho^{\frac{5cK}{\gamma}}$.

Hence there must be two points $g_1, g_2\in E$ with

$$|bg_2-bg_1| > r=\rho^{\frac{5cK}{\gamma}}$$

To complete the proof of claim, it remain to show $r=\rho^{\frac{5cK}{\gamma}}>\bar{\iota}^{\,m}$.

We have $m=\lfloor \alpha|\log\rho|\rfloor$, $\alpha=\frac{1}{10(\ell+1)}$ $\Rightarrow m\leq \alpha|\log\rho|$

$$\Rightarrow \bar{\iota}^{\,m}\geq \bar{\iota}^{\,\alpha|\log\rho|}=\rho^{\alpha}$$

and also

$$m\geq \alpha|\log\rho|-1 \Rightarrow \bar{\iota}^{\,m}\leq \iota\,\rho^{\alpha}$$

Hence $\boxed{\bar{\iota}^{\,m}\cong\rho^{\alpha}}$

So it is enough to make $\rho^{\frac{5cK}{\gamma}}>\rho^{\alpha} \Rightarrow \frac{5cK}{\gamma}<\alpha$

$$\Rightarrow \boxed{K \underset{c}{<<} 1}$$

Let $\Delta b = b_{g_2} - b_{g_1}$, $t = \pi_{g_1}\pi_{g_2}^{-1}$.

We know that

$$|\Delta b| = |b_{g_2} - b_{g_1}| > \rho^{\frac{5CK}{\gamma}}$$

We know that

$$|b_{g_i} - b_0| < \rho^{CK}, \quad i = 1,2.$$

$$\Longrightarrow \quad |1 - \pi_{g_i}\pi_0^{-1}| < \rho^{CK}, \quad i = 1,2$$

also $\quad |b_0| \leq \rho^{-\frac{4K}{\gamma}}$.

We have

$$g_1^{-1}g_2 = U(-b_{g_2}) \cdot h \cdot U(b_{g_2}) \qquad \text{--- (2)}$$

$$h = U(\Delta b) \cdot a(t)$$

So

$$h = \begin{pmatrix} t^{1/2} & \Delta b\, t^{1/2} \\ & t^{-1/2} \end{pmatrix}$$

$$\Longrightarrow \quad \boxed{\begin{aligned} \mathrm{dist}(h, id) &\simeq |\Delta b| + |1-t| \quad \text{using } t \simeq 1 \\ &\in \left[\rho^{\frac{5CK}{\gamma}}, 4\rho^{CK}\right] \end{aligned}} \qquad \text{--- (2')}$$

We have seen that

$$\|Ad(U(s))\| \leq (1+|s|)^2 \qquad \text{--- (3)}$$

By (2), (2'), and (3), we will bound $\mathrm{dist}(g_1^{-1}g_2, id)$ from above and below.

We have seen in step (7)

$$\|Ad(U(-b_{g_2}))\| \leq \rho^{-\frac{8K}{\gamma}}.$$

1. ## lower bound for $\mathrm{dist}(g_1^{-1}g_2, id)$.

Use the conjugation relation

$$h = u(bg_2)\, g_1^{-1} g_2\, u(-bg_2)$$

Then

$$\mathrm{dist}(h, id) \leq \| Ad(u(bg_2)) \| \, \mathrm{dist}(g_1^{-1}g_2, id)$$

$$\Longrightarrow \quad \mathrm{dist}(g_1^{-1}g_2, id) \geq \| Ad(u(bg_2)) \|^{-1} \mathrm{dist}(h, id)$$

By (21) and (3), we get

$$\cancel{\mathrm{dist}(g_1^{-1}g_2, id)} \quad \mathrm{dist}(h, id) \geq \rho^{\frac{5CK}{\gamma}}$$

$$\| Ad(u(bg_2)) \|^{-1} \gg \rho^{\frac{8K}{\gamma}}$$

Hence

$$\mathrm{dist}(g_1^{-1} g_2, id) \gg \rho^{\frac{5CK}{\gamma} + \frac{8K}{\gamma}} > \rho^{1/4}$$

$K \lll 1$

~~For $C > \frac{16}{\gamma}$~~

2. ## Upper bound for $\mathrm{dist}(g_1^{-1}g_2, id)$

Now, we use other direction

$$g_1^{-1} g_2 = u(-bg_2)\, h \cdot u(bg_2)$$

So

$$\mathrm{dist}(g_1^{-1}g_2, id) \leq \| Ad(u(-bg_2)) \| \, \mathrm{dist}(h, id)$$

Using $\| Ad(u(-bg_2)) \| \lll \rho^{-\frac{8CK}{\gamma}}$, $\mathrm{dist}(h, id) \leq 4\rho^{CK}$

We get

$$\mathrm{dist}(g_1^{-1}g_2, id) \leq \rho^{-\frac{8Kc}{\gamma}} \cdot \rho^{CK} = \rho^{(C - \frac{8}{\gamma})K}$$

Take $c > \frac{16}{\gamma}$.

$$\Rightarrow \quad c - \frac{8}{\gamma} > \frac{c}{2}$$

$$\Rightarrow \operatorname{dist}(g_1^{-1}g_2, id) << \rho^{\frac{ck}{2}}.$$

STEP (10) End of the proof.

- Recall that

$$Z = \{ z \in X : \mu^{*m} * \delta_z (B_\rho Y) \geq \rho^{2k} \}.$$

For fix $z \in Z$, in step (8), we have shown for any $g_1, g_2 \in E$

$$\operatorname{dist}(z, g_1^{-1}g_2 z) << \rho^{1/2}$$

in step (9), we have show that, we can choose $g_1, g_2 \in E$ such that

$$\rho^{1/4} << \operatorname{dist}(g_1^{-1}g_2, id) << \rho^{\frac{ck}{2}}$$

Hence

$$\begin{aligned} \operatorname{inj}(z) &\leq \operatorname{dist}(g_1^{-1}g_2, id) + \operatorname{dist}(z, g_1^{-1}g_2 z) \\ &<< \rho^{1/2} + \rho^{\frac{ck}{2}} \\ &<< \rho^{\frac{ck}{4}} \qquad \forall z \in Z \end{aligned}$$

- Let $S = \{ z \in X : \operatorname{inj}(z) \leq \rho^{\frac{ck}{4}} \}$

$\Rightarrow \quad Z \subseteq S$. By defination of Z

$$\Rightarrow \mu^{*(n-m)} * \delta_x (A) \geq \mu^{*(n-m)} * \delta_x (Z) \geq \rho^{2k} \qquad (1)$$

Now apply proposition (25): It say that for some $q, q' > 0$ (21)

$$\mu^{*N} * \delta_x \{ inj < r \} << \left(inj(x)^{-q} \lambda^{-q' N} + 1 \right) r^{q} \quad \forall N \in \mathbb{N}, \; r > 0.$$

- Take $N = n - m$, $r = \rho^{\frac{CK}{4}}$. Then

$$\mu^{*(n-m)} * \delta_x \left\{ inj < \rho^{\frac{CK}{4}} \right\} << \left(inj(x)^{-q} \lambda^{-q'(n-m)} + 1 \right) \rho^{q \cdot \frac{CK}{4}} \quad \text{——— (2)}$$

If $n-m >> |\log inj(x)|$, then the factor

$inj(x)^{-q} \cdot \lambda^{-q'(n-m)} + 1$ is bounded by a constant.

By (2)

$$\left(\mu^{*(n-m)} * \delta_x \right) \left\{ inj < \rho^{\frac{CK}{4}} \right\} << \rho^{q \frac{CK}{4}}$$

Now choose C large

$$\frac{qC}{4} > e$$

$$\Longrightarrow$$

$$\mu^{*(n-m)} * \delta_x \left(inj < \rho^{\frac{CK}{4}} \right\} << \rho^{2K} \quad \text{——— (3)}$$

Which is a contradiction of (1) at page (20).

Hence our earlier assumption is wrong.

So the set Z does not exist.