Definition: Given \( \beta \in (0, 1) \) and \( \delta > 0 \) we define the \( \delta \)-dimensional Hausdorff \( \beta \)-game:

\[ (Packing) \]

0th step

Alice plays first and chooses \( \rho_0 > 0 \) and some \( A_0 \subseteq \mathbb{R}^D \).
Bob responds by choosing \( B_0 = B(x_0, \rho_0) \) such that \( x_0 \in A_0 \).

\( a \) 3\( \rho \)-separated set or a doubling space

1st step \( (k = 1) \)

Alice chooses a non-empty finite 3\( \rho \)-separated set \( A_1 \subseteq B(x_1, (1 - \beta) \rho_1) \) and Bob responds by choosing a ball \( B_1 = B(x_1, \rho_1) \) such that \( x_1 \in A_1 \) and \( \rho_1 = \beta \rho_0 \).

2nd step \( (k = 2) \)

Alice chooses a non-empty finite 3\( \rho \)-separated set \( A_2 \subseteq B(x_2, (1 - \beta) \rho_2) \) and Bob responds by choosing a ball \( B_2 = B(x_2, \rho_2) \) such that \( x_2 \in A_2 \) and \( \rho_2 = \beta \rho_1 \).

kth step \( (k \in \mathbb{N} \setminus \{0\}) \)

Alice chooses a non-empty finite 3\( \rho \)-separated set \( A_k \subseteq B(x_k, (1 - \beta) \rho_k) \) and Bob responds by choosing a ball \( B_k = B(x_k, \rho_k) \) such that \( x_k \in A_k \) and \( \rho_k = \beta \rho_{k-1} \).

\[ d(x, y) \geq 3 \rho \quad \text{for every } x, y \in A_k \]
We define

\[
\sigma \left( (A_k)_{k \in \mathbb{N}} \right) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \frac{\log|A_i|}{-\log(p)} = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{\log|A_i|}{-\log(p)}
\]

\[
\overline{\sigma} \left( (A_k)_{k \in \mathbb{N}} \right) = \limsup_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k} \frac{\log|A_i|}{-\log(p)} = \limsup_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \frac{\log|A_i|}{-\log(p)}
\]

**A goal:** \( x_\infty \in S \) and score above \( \delta \), i.e. \( \sigma (A_k)_{k \in \mathbb{N}} \geq \delta \).

(We can think of it as \( x_\infty \notin S \Rightarrow \overline{\sigma} = 0 \). \( \overline{\sigma} \left( (A_k)_{k \in \mathbb{N}} \right) \geq \delta \).
The set $S$ is a $d$-dimensional Hausdorff winning set if it is a $d$-dimensional Hausdorff $\beta$-game winning set for all $\beta \in (0, \varepsilon)$ for some $\varepsilon > 0$.

**Theorem (28.2 in DFSU)**

Let $S \subseteq \mathbb{R}^d$ be a Borel set. Then:

$$H(S) = \sup \{ \delta \in \mathbb{R}_+ \mid S \text{ is a } d\text{-dimensional Hausdorff winning set} \}$$

$$P(S) = \cdots \cdots \cdots \text{ packing} \cdots \cdots$$

True also for:

**Doubling space:**

A metric space is "doubling" if there exists constants $C, \varepsilon_0$ such that every ball of radius $0 < r \leq \varepsilon_0$ can be covered by at most $C$ balls of radius $\frac{r}{2}$. 
Preliminaries

We define the outer-measure on a metric space:

\[ H_s(A) = \lim_{\varepsilon \to 0} \left( \inf S \left\{ \sum_{j=1}^{\infty} (\text{diam}A_j)^s \mid A \subseteq \bigcup_{j=1}^{\infty} A_j \text{ and } \text{diam}A_j \leq \varepsilon \right\} \right) \]

and

\[ \dim_H(A) = \inf \left\{ S \geq 0 \mid H_s(A) = 0 \right\} = \sup \left\{ S \geq 0 \mid H_s(A) = \infty \right\} \]

\[ P_s(A) = \inf \left\{ \sum_{i=1}^{\infty} \widetilde{P}_s(A_i) \mid A \subseteq \bigcup_{i=1}^{\infty} A_i \right\} \quad \text{where} \]

\[ \widetilde{P}_s(A) = \inf_{\varepsilon > 0} \sup \left\{ \sum_{j=1}^{\infty} (\text{diam}B_j)^s \mid (B_j)_{j=1}^{\infty} \text{ is a disjoint collection of balls with centers in } A \text{ such that } \text{diam}B_j \leq \varepsilon \text{ for all } j \right\} \]

\[ \dim_P(A) = \inf \left\{ S \geq 0 \mid P_s(A) = 0 \right\} = \ldots \]
For each \( x \in \mathbb{R}^d \) we define the lower and the upper pointwise dimensions of a measure \( \mu \) at \( x \) by:

\[
\dim_x(\mu) = \lim_{\rho \to 0} \frac{h(\mu(B(x, \rho)))}{h(\rho)} = \lim_{n \to \infty} \frac{h(\mu(B(x, \rho_n)))}{h(\rho_n)}
\]

where \( \rho_n \to 0 \) and \( \frac{\rho_n}{\rho_{n+1}} \) is bounded.

and

\[
\overline{\dim_x}(\mu) = \limsup_{\rho \to 0} \frac{h(\mu(B(x, \rho)))}{h(\rho)} = \limsup_{n \to \infty} \frac{h(\mu(B(x, \rho_n)))}{h(\rho_n)}
\]

By Rogers-Taylor-Tricot density theorem for the Hausdorff and packing dimension we get:
Theorem (27.1 in DFSU)

Let $D \subset \mathbb{R}^d$ and $\mu$ a locally finite Borel measure on $\mathbb{R}^d$. Then for every Borel set $A \subset \mathbb{R}^d$ we have:

- If $\overline{\dim}_x(\mu) > s$ for all $x \in A$ and $\mu(A) > 0$ then $\dim_H(A) \leq s$.
- If $\underline{\dim}_x(\mu) > s$ for all $x \in A$ and $\mu(A) > 0$ then $\dim_\mu(A) \leq s$.
- If $\overline{\dim}_x(\mu) \leq s$ for all $x \in A$ then $\dim_H(A) \leq s$.
- If $\underline{\dim}_x(\mu) \leq s$ for all $x \in A$ then $\dim_\mu(A) \leq s$. 
Proof (for Hausdorff dimension case)

Let \( S \subset \mathbb{R}^d \) be a Borel set.

Suppose that \( S \) is a \( J \)-dimensional Hausdorff winning set. \((\Rightarrow \dim_H(S) \leq J)\)

Let \( p > 0 \) such that \( S \) is a \( J \)-dimensional Hausdorff \( \beta \)-game winning set. Let \( \mathcal{A}_b^p \) be a winning strategy for Alice in this \( J \)-dimensional Hausdorff \( \beta \)-game.

For each \( k \in \mathbb{N} \), denote by \( E_k \) the union of all sets \( A_k \) that Alice might choose according to her strategy in response to some possible sequence of moves that Bob could play.
Ek is $3p_k$-separated for every $k \geq 0$, since every $A_k$ is $3p_k$-separated, and every $A_k$ lies in a ball $B(x_{k-1}, p_{k-1})$, and the centers of these balls are $3p_{k-1}$ separated.

Thus, the set

$$C = \bigcap \bigcup_{k \in \mathbb{N}} B(x_k, p_k)$$

$s_{x_k \in E_k}$

is the set of all possible outcomes of the game when Alice plays $\mathcal{A}^p$. It is closed and totally disconnected, and of course contained in $S$. 
Consider the following scenario: on the kth turn, Bob chooses the point $x_k$ uniformly at random, independently of all previous choices. This yields a random game whose outcome is distributed according to some probability measure $\mu$ over $C$. Let $x \in C$. Let $(x_k)_{k=0}^\infty$ be a sequence such that $x \in B(x_k, \rho_k)$ and $x_k \in E_k$ for every $k \in \mathbb{N}$. Since $E_k$ is $3\rho_k$-separated, if Bob plays such that the final outcome is in $B(x, \rho_k)$, then on the kth turn he must choose the ball $B(x_k, \rho_k)$. 
Thus \( B(x, \rho_k) \cap C \subseteq B(x_k, \rho_k) \) and therefore:

\[
\mathcal{M}(B(x, \rho_k) \cap C) \leq \mathcal{M}(B(x_k, \rho_k) \cap C) = \left( \frac{k}{\prod_{i=0}^{k-1} |A_i|} \right)^{-1} = \frac{1}{|A_0| |A_1| |A_2| \cdots |A_k|}
\]

The lower pointwise dimension of \( \mathcal{M} \) at \( x \) is:

\[
\dim_x(\mathcal{M}) = \lim_{\rho \to 0} \frac{\mathcal{H}(\mathcal{M}(B(x, \rho)))}{\mathcal{H}(\rho)} = \lim_{k \to \infty} \frac{\mathcal{H}(\mathcal{M}(B(x, \rho_k)))}{\mathcal{H}(\rho_k)} \geq \lim_{k \to \infty} \frac{\mathcal{H}(\mathcal{M}(B(x_k, \rho_k)))}{\mathcal{H}(\rho_k)} = \lim_{k \to \infty} \frac{\mathcal{H}(\mathcal{M}(B(x_k, \rho_k)))}{\mathcal{H}(\rho_k)} = \sum_{i=0}^{k} \frac{\mathcal{H}(|A_i|)}{\mathcal{H}(\rho_k) + \mathcal{H}(\rho)} = \mathcal{O}(A) \geq \delta \quad (\text{since } \mathcal{O}(A) \text{ used } \delta_A)
Because \( x \in C \) was arbitrary and \( M(c) = 1 \), we get by the Rogers-Taylor-Tricot theorem (27.1) that \( \dim_H(S) \geq \delta \).

Suppose now that \( S \) is not a \( \delta \)-dimensional Hausdorff winning set. (\( \Rightarrow \dim_H(S) < \delta \)).
Let \( \beta \in [0,1] \) small enough such that \( S \) is not a \( \delta \)-dimensional Hausdorff \( \beta \)-game winning set.
Then Alice does not have a winning strategy in that \( \beta \)-game, and since \( S \) is Borel, by Martin (1975) and (Fishman and Simmons), Bob must have a winning strategy.
Let \( \Theta_{\beta} \) be a winning strategy for Bob.

Fix \( \rho_0 > 0 \).
Let $E_k$ be a maximal $\frac{1}{3}p_k$-separated subset of $X^k$, and let $E_k^{(1)}, E_k^{(2)}, E_k^{(3)}, \ldots, E_k^{(p)}$ be disjoint $3p_k$-separated subsets of $E_k$ such that $E_k = \bigcup_{i=1}^{p} E_k^{(i)}$. Since $X^k$ is doubling, it is possible to choose $p$ to be independent of $k$ and of $\beta$. 
We define a family of strategies for Alice. Denote:

\[ \tilde{\mathcal{B}}_{k-1} = \begin{cases} \mathcal{B}(0, L+\rho_0) & k = 0 \\ \mathcal{B}(X_{k-1}, (1-\beta)\rho_{k-1}) & k \geq 1 \end{cases} \]

where \( L > 0 \) is a large constant.

Denote:

\[ X_{k}^{(i)} = E_{k}^{(i)} \cap \tilde{\mathcal{B}}_{k-1}, \quad N_{k}^{(i)} = |X_{k}^{(i)}| \quad \text{and} \quad A_{k}^{(i)} = X_{k}^{(i)} \]
We now define the moves \( A_{k}^{(i,j)} \) and \( B_{k}^{(i,j)} \) by backward (because it depends on Bob’s strategy) recursion:

- If \( A_{k}^{(i,j)} \) is defined for some \( j \geq 1 \), then \( B_{k}^{(i,j)} = B_{k}(x_{k}^{(i,j)}, p_{k}) \).
- \( A_{k}^{(i,j)} \) and \( B_{k}^{(i,j)} = B_{k}(x_{k}^{(i,j)}, p_{k}) \) are both defined for some \( j \geq 1 \), then \( A_{k}^{(i,j-1)} = \{ A_{k}^{(i,j)} \setminus \{ x_{k}^{(i,j)} \} \} \).

What would have been Bob’s choice is \( x_{k}^{(i,j)} \) wasn’t an option.

Note that \( |A_{k}^{(i,j)}| = j \) for all \( 0 \leq j \leq N_{k}^{(i)} \).

Consider now the scenario where Bob plays according to \( \sigma_{0}^{B} \) and Alice plays randomly: on the \( k \)th turn, Alice chooses a move \( A_{k}^{(i_{k},j_{k})} \) where the integers \( i_{k} \) and \( j_{k} \) are chosen independently of previous turns with respect to a probability distribution satisfying

\[
\Pr(i_{k} = i, j_{k} = j) \geq c_{ij}^{-1/2} \epsilon \]

where \( \epsilon > 0 \) is fixed and \( c_{ij} \) depends on \( \epsilon \) and on \( p \).
By Kolmogorov extension theorem, this yields a random sequence of plays whose outcome is distributed according to some probability $\mu$. Let $x \in S \in \mathcal{B}(0, L)$.

For each $k \in N$ there exists $x \in E_k$ such that $d(x, x_k) \leq \frac{2-\beta}{1+\beta} \rho_k$ (because since $\beta < \frac{1}{2}$ we have $\frac{2}{3} < \frac{2-\beta}{1+\beta}$).

Note that $x_0 \in \mathcal{B}(0, L + \rho_0)$ and $x_{k+1} \in \mathcal{B}(x_k, (1-\beta)\rho_k)$ for all $k \in N$. It follows that Alice can guarantee that the outcome is equal to $x$ by playing the move $A_k^{(x_k, j_k)}$ on the $k$th turn for some sequences $(j_k)_{k \in N}$, $(x_k)$.

Because:
Denote this strategy of Alice by $\Theta_A$.

Since Bob plays a winning strategy, Alice's score must be less than $\delta$, i.e. $\Delta(A) < \delta$. (We did not use $\mu$)

Now consider the game where Alice plays in the first turns according to $\Theta_A$. In such a case the outcome of the game will lie in $B(x, 2\epsilon)$. If Alice plays randomly according to $\Pi^*$, this happens in probability $1 - \pi_k = 2^{-k}$. Therefore:
\[ \mu(B(x,2\rho_0)) \geq \prod_{i=1}^{l}(i_k = 1, j_k = 1) \geq \prod_{k=0}^{l} \left(1 + \varepsilon \right) \left(\frac{l}{\sum_{k=0}^{l} h(1A_k)}\right) \]

and thus:

\[ \dim_x(\mu) = \lim_{l \to \infty} \frac{h(\mu(B(x,2\rho_0)))}{h(2\rho_0)} \leq \lim_{l \to \infty} \frac{l \cdot h(\varepsilon) + (1+\varepsilon) \cdot \sum_{k=0}^{l} h(1A_k)}{l \cdot h(\beta) + h(\rho_0)} \]
\[ \lim_{\lambda \to 0^+} \frac{\lambda \cdot h(C_{\lambda}) + (1+\varepsilon) \cdot \frac{\lambda}{\lambda} \cdot h(\lambda A)}{\lambda \cdot h(\beta)} = \]

\[ = \frac{h(C_{\lambda})}{h(\beta)} + (1+\varepsilon) \cdot \frac{\lambda}{\lambda} \cdot h(\beta) + (1+\varepsilon) \cdot \frac{\lambda}{\lambda} \cdot \frac{\lambda}{\lambda} \cdot h(\beta) \cdot h(\beta) \cdot h(\beta) \leq \]

Since \( \lambda \) was arbitrary, applying again the Rogers-Taylor-Trice theorem we get:

\[ \dim_{\lambda} (S) \leq \frac{h(C_{\lambda})}{h(\beta)} + (1+\varepsilon) \cdot \frac{\lambda}{\lambda} \cdot h(\beta) \]

We complete the proof by taking the limits \( \beta, \varepsilon \to 0 \).