9.3.2022

Variant DFSU of Schmidt Games

Definition: Given $\beta \in (0,1)$ and $\delta > 0$ we define the δ-dimensional Hausdorff β-game:

(Packing)

0th step

a $3\rho_0$-separated set

or a doubling space

Alice plays first and choses $\rho_0 > 0$ and some $A_0 \subseteq \mathbb{R}^D$.

Bob responds by choosing $B_0 = B(x_0, \rho_0)$ such that $x_0 \in A_0$.

closed ball

$B_k \subseteq B_{k-1}$

$\Uparrow$

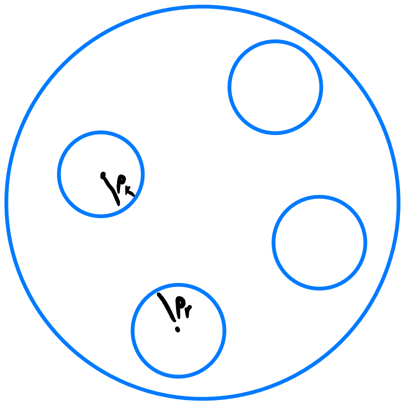
$= \beta^{k-1}\rho_0 - \beta^k \rho_0$

$= (1-\beta)\beta^{k-1}\rho_0 =$

kth step $(k \in \mathbb{N} \setminus \{0\})$

$d(x,y) \geq 3 \cdot \rho_k$ for every $x,y \in A_k$

Alice chooses a nonempty finite $3 \cdot \rho_k$-seperated set $A_k \subseteq B(x_{k-1}, (1-\beta)\rho_{k-1})$ and Bob responds by choosing a ball $B_k = B(x_k, \rho_k)$ such that $x_k \in A_k$ and $\rho_k = \beta^k \cdot \rho_0$.



$$X_\infty = \lim_{k\to\infty} X_k \in \bigcap_{k=0}^{\infty} B_k .$$

We define

$$\underline{\delta}\left((A_k)_{k\in\mathbb{N}}\right) = \liminf_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k} \frac{\ln(|A_i|)}{-\ln(\beta)} = \frac{\liminf_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \ln(|A_i|)}{-\ln(\beta)}$$

$$\overline{\delta}\left((A_k)_{k\in\mathbb{N}}\right) = \limsup_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k} \frac{\ln|A_i|}{-\ln(\beta)} = \frac{\limsup_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k} \ln|A_i|}{-\ln(\beta)}$$

Alice goal: $X_\infty \in S$ and score above δ, i.e. $\underline{\delta}((A_k)_{k\in\mathbb{N}}) \geq \delta$.

(We can think of it as if $X_\infty \notin S \Rightarrow \underline{\delta} = 0$. $\quad \overline{\delta}((A_k)_{k\in\mathbb{N}}) \geq \delta$.

The set S is a δ-dimensional Hausdorff winning set
packing
if it is a δ-dimensional Hausdorff β-game winning set for
packing
all $\beta \in (0, \varepsilon)$ for some $\varepsilon > 0$.

Theorem (28.2 in DFSU)

Let $S \subseteq \mathbb{R}^d$ be a Borel set. Then:

$$H(S) = \sup\{\delta \in \mathbb{R}_{\geq 0} \mid S \text{ is a } \delta\text{-dimensional Hausdorff winning set}\}$$

$P(S) =$ " " " " " " packing " "

True also for:

Doubling space:

A metric space is "doubling" if there exists constants C, r_0 such that every ball of radius $0 < r \leq r_0$ can be covered by at most C balls of radius $\frac{r}{2}$.

Preliminaries

Reminder

We define the outer-measure on a metric space:

$$\mathcal{H}_s(A) = \lim_{\varepsilon \to 0}\left(\inf\left\{\sum_{j=1}^{\infty}(\operatorname{diam} A_j)^s \,\middle|\, A \subseteq \bigcup_{j=1}^{\infty} A_j \text{ and } \operatorname{diam} A_j \leq \varepsilon\right\}\right)$$

and

$$\dim_{\mathcal{H}}(A) = \inf\{s \geq 0 \mid \mathcal{H}_s(A) = 0\} = \sup\{s \geq 0 \mid \mathcal{H}_s(A) = \infty\}$$

$$P_s(A) = \inf\left\{\sum_{i=1}^{\infty} \widetilde{P}_s(A_i) \,\middle|\, A \subseteq \bigcup_{i=1}^{\infty} A_i\right\} \quad \text{where}$$

$$\widetilde{P}_s(A) = \inf_{\varepsilon > 0} \sup\left\{\sum_{j=1}^{\infty}(\operatorname{diam} B_j)^s \,\middle|\, (B_j)_{j=1}^{\infty} \text{ is a disjoint collection of balls with centers in } A \text{ such that } \operatorname{diam} B_j \leq \varepsilon \text{ for all } j \in \mathbb{N}\setminus\{0\}\right\}.$$

$$\dim_P(A) = \inf\{s \geq 0 \mid P_s(A) = 0\} = \ldots$$

For each $x \in \mathbb{R}^d$ we define the lower and the upper pointwise dimensions of a measure μ at x by:

$$\underline{\dim}_x(\mu) = \liminf_{\rho \to 0} \frac{h(\mu(B(x,\rho)))}{h(\rho)} = \liminf_{n \to \infty} \frac{h(\mu(B(x,\rho_n)))}{h(\rho_n)}$$

where $\rho_n \to 0$ and $\frac{\rho_n}{\rho_{n+1}}$ is bounded.

and

$$\overline{\dim}_x(\mu) = \limsup_{\rho \to 0} \frac{h(\mu(B(x,\rho)))}{h(\rho)} = \limsup_{n \to \infty} \frac{h(\mu(B(x,\rho_n)))}{h(\rho_n)}$$

By Rogers-Taylor-Tricot density theorem for the Hausdorff and packing dimension we get:

Theorem (27.1 in DFSU)

Let $D \in \mathbb{N} \setminus \{0\}$ and μ a locally finite Borel measure on $\mathbb{R}^D$. Then for every Borel set $A \subseteq \mathbb{R}^D$ we have:

- If $\underline{\dim}_x(\mu) \geq s$ for all $x \in A$ and $\mu(A) > 0$ then $\dim_H(A) \geq s$.
- If $\underline{\dim}_x(\mu) \leq s$ for all $x \in A$ then $\dim_H(A) \leq s$.
- If $\overline{\dim}_x(\mu) \geq s$ for all $x \in A$ and $\mu(A) > 0$ then $\dim_P(A) \geq s$.
- If $\overline{\dim}_x(\mu) \leq s$ for all $x \in A$ then $\dim_P(A) \leq s$.

Proof (for Hausdorff dimension case)

Let $S \subset \mathbb{R}^d$ be a Borel set.

Suppose that S is a δ-dimensional Hausdorff winning set. ($\Rightarrow \dim_H(S) \leq \delta$)

Let $\beta > 0$ such that S is a δ-dimensional Hausdorff β-game winning set. Let σ_A^β be a winning strategy for Alice in this δ-dimensional Hausdorff β-game.

For each $k \in \mathbb{N}$, denote by E_k the union of all sets A_k that alice might choose according to her strategy in response to some possible sequence of moves that Bob could play.

E_k is $3\rho_k$-seperated for every $k\in\mathbb{N}$, since every A_k is $3\rho_k$-seperated, and every A_k lies in a ball $B(x_{k-1},\rho_{k-1})$, and the centers of these balls are $3\rho_{k-1}$ seperated.

Thus, the set

$$C=\bigcap_{k\in\mathbb{N}}\bigcup_{x_k\in E_k} B(x_k,\rho_k)$$

is the set of all possible outcomes of the game when Alice plays σ_A^β. It is closed and totally disconnected, and of course contained in S.

Consider the following scenario: On the kth turn, Bob chooses the point x_k uniformly at random independently of all previous choices.

This yields a random game whose outcome is distributed according to some probability μ over C. Let $x \in C$. Let $(x_k)_{k=0}^{\infty}$ be a sequence such that $x \in B(x_k, \rho_k)$ and $x_k \in E_k$ for every $k \in \mathbb{N}$. Since E_k is $3\rho_k$-seperated, if Bob plays such that the final outcome is in $B(x, \rho_k)$, then on the kth turn he must choose the ball $B(x_k, \rho_k)$.

Thus $B(x,\rho_k)\cap C \subseteq B(x_k,\rho_k)$ and therefore:

$$\mu\big(B(x,\rho_k)\cap C\big) \leq \mu\big(B(x_k,\rho_k)\cap C\big) = \left(\prod_{i=0}^{k}|A_i|\right)^{-1} =$$

$$= \frac{1}{|A_0|}\cdot\frac{1}{|A_1|}\cdot\frac{1}{|A_2|}\cdot\ldots\cdot\frac{1}{|A_k|}$$

The lower pointwise dimension of μ at x is:

$$\underline{\dim}_x(\mu) = \liminf_{\rho\to 0}\frac{\ln(\mu_c(B(x,\rho)))}{\ln(\rho)} = \liminf_{k\to\infty}\frac{\ln(\mu_c(B(x,\rho_k)))}{\ln(\rho_k)} \geq$$

$$\geq \liminf_{k\to\infty}\frac{\ln(\mu(B(x_k,\rho_k)))}{\ln(\rho_k)} = \liminf_{k\to\infty}\frac{-\sum_{i=0}^{k}\ln(|A_i|)}{k\ln(\beta)+\ln(\rho_0)} =$$

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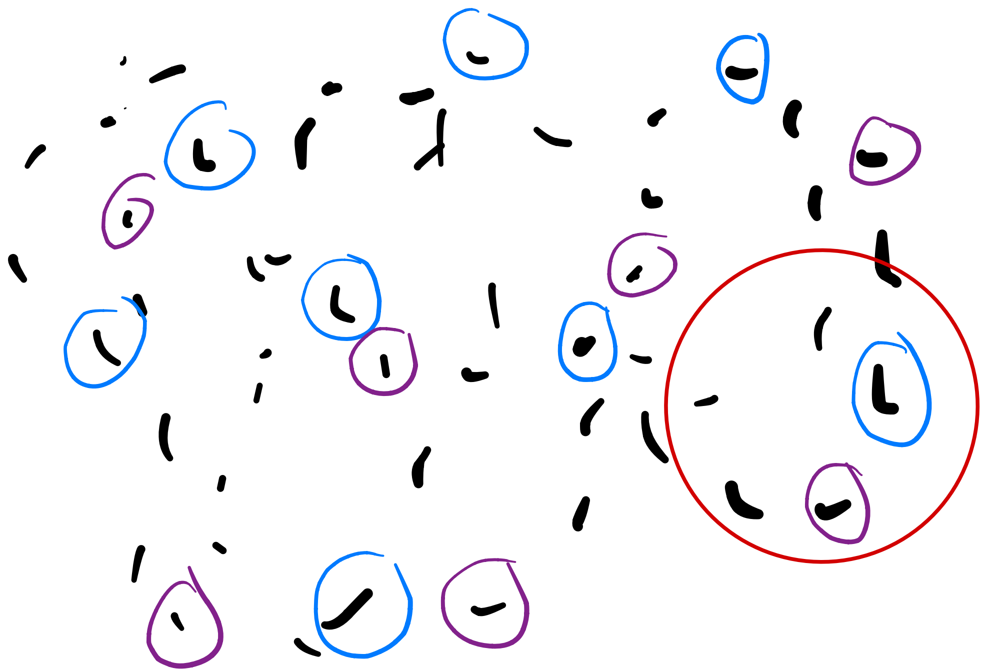
$$= \underline{\delta}(A) \geq \delta \quad \left(\text{since Alice used } \sigma_A^{\beta}\right)$$

Because $x \in C$ was arbitrary and $\mu(C) = 1$, we get by the Rogers-Taylor-Tricot theorem (27.1) that $\dim_H(S) \geq \delta$.

Suppose now that S is not a δ-dimensional Hausdorff winning set. ($\Rightarrow \dim_H(S) \leq \delta$).
Let $\beta \in [0, \frac{1}{2}]$ small enough such that S is not a δ-dimensional Hausdorff β-game winning set.
Then Alice does not have a winning strategy in that β-game, and since S is Borel, by Martin (1975) and (Fishman and Simmons), Bob must have a winning strategy.
Let σ_B^β be a winning strategy for Bob.
Fix $\rho_0 > 0$.

Let E_k be a maximal $\frac{1}{3}\rho_k$-seperated subset of $\mathbb{R}^D$, and let $E_k^{(1)}, E_k^{(2)}, E_k^{(3)}, \dots, E_k^{(p)}$ be disjoint $3\rho_k$-seperated subsets of E_k such that $E_k = \bigcup_{i=1}^{p} E_k^{(i)}$.

Since $\mathbb{R}^D$ is doubling, it is possible to choose p to be independent of k and of β.



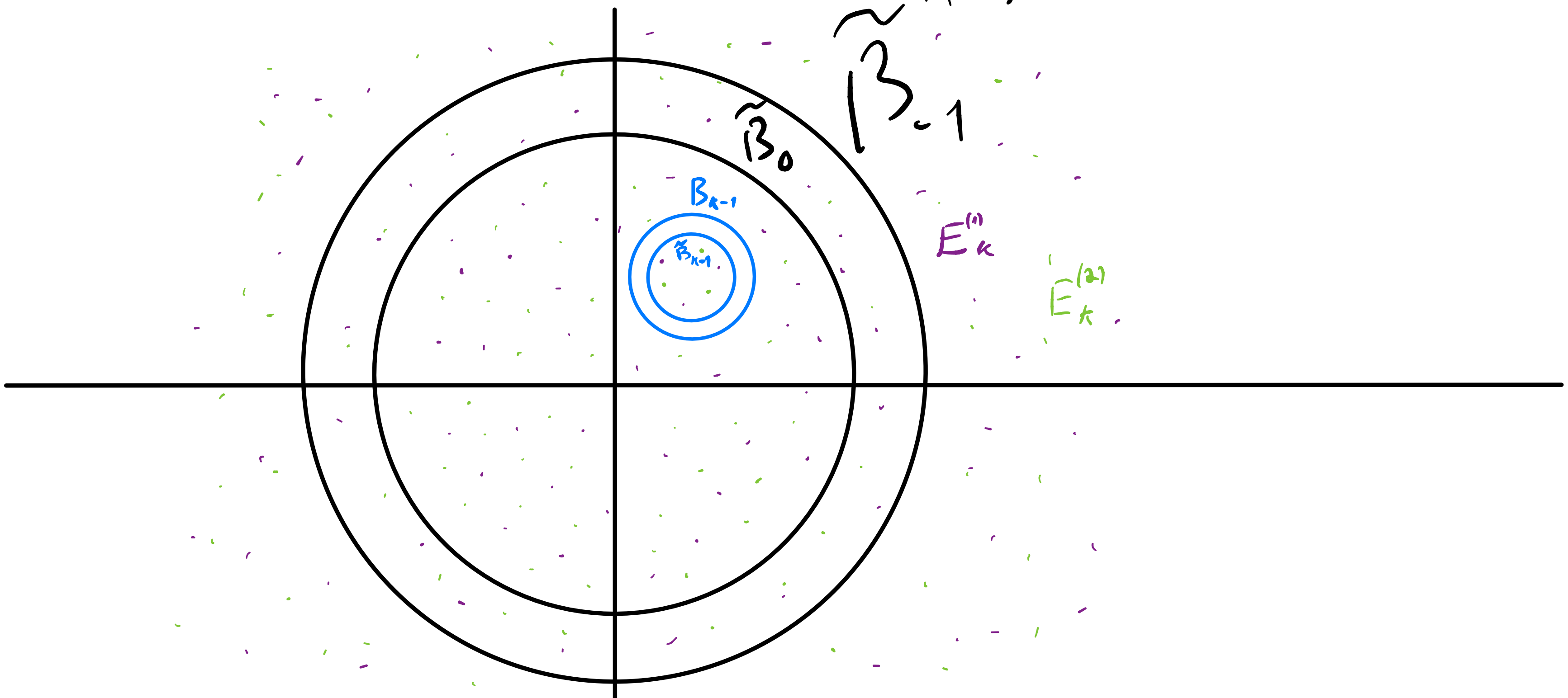
We define a family of strategies for Alice. Denote:

$$\tilde{B}_{k-1} = \begin{cases} B(0, L+\rho_0) & k=0 \\ B(x_{k-1}, (1-\beta)\rho_{k-1}) & k \geq 1 \end{cases}$$

where $L>0$ is a large constant.

Denote:

$$X_k^{(i)} = E_k^{(i)} \cap \tilde{B}_{k-1}, \quad N_k^{(i)} = |X_k^{(i)}| \quad \text{and} \quad A_k^{(i, N_k^{(i)})} = X_k^{(i)}.$$



We now define the moves $A_k^{(i,j)}$ and $B_k^{(i,j)}$ by backward (because it depends on Bob's strategy) recursion:

- if $A_k^{(i,j)}$ is defined for some $j \geq 1$, then $B_k^{(i,j)} = \overbrace{B(x_k^{(i,j)}, \rho_k)}^{\text{Bob's response}}$.
- if $A_k^{(i,j)}$ and $B_k^{(i,j)} = B(x_k^{(i,j)}, \rho_k)$ are both defined for some $j \geq 1$, then $A_k^{(i,j-1)} = A_k^{(i,j)} \setminus \{x_k^{(i,j)}\}$. (What would have been Bob's choice is $x_k^{(i,j)}$ wasn't an option.)

Note that $|A_k^{(i,j)}| = j$ for all $0 \leq j \leq N_k^{(i)}$.

Consider now the scenario where Bob plays according to σ_B^β and Alice plays randomly: on the kth turn, Alice chooses a move $A_k^{(i_k, j_k)}$ where the integers i_k and j_k are chosen independently of previous turns with respect to a probability distribution satisfying

$$\mathbb{P}(i_k = i, j_k = j) \geq c_{\varepsilon,p}\, j^{-(1+\varepsilon)}$$

where $\varepsilon > 0$ is fixed and $c_{\varepsilon,p}$ depends on ε and on p.

By Kolmogorov extension theorem, this yields a random sequence of plays whose outcome is distributed according to some probability μ

Let $x \in S \cap B(0,L)$.

For each $k \in \mathbb{N}$ there exists $x_k \in E_k$ such that $d(x,x_k) \leq \frac{1-\beta}{1+\beta}\rho_k$ (because since $\beta < \frac{1}{2}$ we have $\frac{1}{3} < \frac{1-\beta}{1+\beta}$).

Note that $x_0 \in B(0,L+\rho_0)$ and $x_{k+1} \in B(x_k,(1-\beta)\rho_k)$ for all $k \in \mathbb{N}$. It follows that Alice can guarantee that the outcome is equal to x by playing the move $A_k^{(I_k,J_k)}$ on the kth turn for some sequences $(I_k)_{k \in \mathbb{N}}, (J_k)_{k \in \mathbb{N}}$ because:

$$d(x_{k+1}, x_k) \leq d(x_k, x) + d(x, x_{k+1}) \leq \frac{1-\beta}{1+\beta} \cdot \rho_k + \frac{1-\beta}{1+\beta} \rho_{k+1} =$$

$$= \frac{1-\beta}{1+\beta} \left(\rho_k + \beta \rho_k \right) = \frac{1-\beta}{1+\beta} \cdot \rho_k (1+\beta) = (1-\beta) \rho_k .$$

Denote this strategy of Alice by $\hat{\sigma}_A^\beta$.

Since Bob plays a winning strategy, Alice's score must be less than δ, i.e. $\underline{\partial}(A) < \delta$. (We did not use μ)

Now consider the game where Alice plays in the l first turns according to $\hat{\sigma}_A^\beta$. In such a case the outcome of the game will lie in $B(x, 2\rho_l)$. If Alice plays randomly according to $\mathbb{P}$, this happens in probability $\mathbb{P}(i_k = I_k, j_k = J_k \text{ for all } k \leq l)$. Therefore:

$$\mu(B(x,2\rho_\ell)) \geqslant \mathbb{P}(i_k = I_k,\ j_k = J_k \text{ for all } k \le \ell) \geqslant$$

$$\geqslant \prod_{k=0}^{\ell} C_{\varepsilon,\rho} \cdot J_k^{-(1+\varepsilon)} = (C_{\varepsilon,\rho})^{\ell} \cdot e^{(-1+\varepsilon)\cdot \sum_{k=0}^{\ell} h(|A_k|)}$$

and thus:

$$\underline{\dim}_x(\mu) = \liminf_{\ell\to\infty} \frac{h(\mu(B(x,2\rho_\ell)))}{h(2\rho_\ell)} \leqslant$$

$$\leqslant \liminf_{\ell\to\infty} \frac{\ell \cdot h(C_{\varepsilon,\rho}) + (1+\varepsilon)\cdot \sum_{k=0}^{\ell} -h(|A_k|)}{\ell \cdot h(\beta) + h(\rho_0)} \leqslant$$

$$\leqslant \liminf_{\ell\to\infty} \frac{\ell\cdot h(C_{\varepsilon,\beta}) + (1+\varepsilon)\cdot\sum_{k=0}^{\ell} -h(|A_k|)}{\ell\cdot h(\beta)} =$$

$$= \frac{h(C_{\varepsilon,\beta})}{h(\beta)} + (1+\varepsilon)\cdot\underline{\partial}(A) \leqslant \frac{h(C_{\varepsilon,\beta})}{h(\beta)} + (1+\varepsilon)\cdot\overline{\partial}.$$

Since x was arbitrary, applying again the Rogers-Taylor-Tricot theorem we get:

$$\dim_{\mu}(S) \leqslant \frac{h(C_{\varepsilon,\beta})}{h(\beta)} + (1+\varepsilon)\cdot\overline{\partial}.$$

We complete the proof by taking the limits $\beta, \varepsilon \longrightarrow 0$.

Q.E.D