- Goal: prove $\dim_{H,P}(\mathcal{D}(\vec{f})) \geq \overline{\delta}(\vec{f})$ for an arbitrary template $\vec{f}$.

  here $\mathcal{D}(\vec{f}) = \{ A : \vec{h}_A \asymp_+ \vec{f} \}$

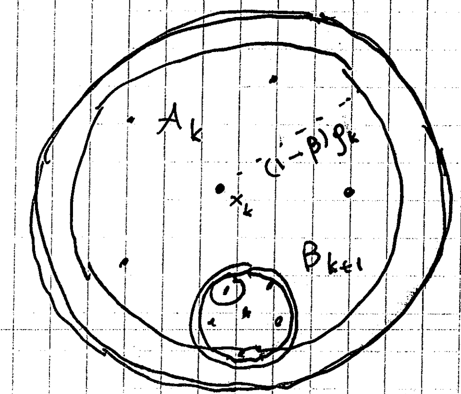
  to prove var. princ.: let S be a Borel collection of ~~fns~~ templates $[0,\infty) \to \mathbb{R}^d$ closed under f. p., and

  $$\mathcal{D}(S) := \{ A : h_A \in S \}.$$

  Then $\dim_{H,P}(\mathcal{D}(S)) = \sup_{f \in S} \underline{\delta}(\vec{f})$

- Techniques: δ-dim Haus. game.

- Given $0 < \beta < 1$ and $\delta > 0$,
- Alice chooses $\rho_0 > 0$ $\quad \mathbb{R}^D$
- on kth turn Alice chooses finite $3\rho_k$-sep. set A_k, Bob chooses ball $B_k = B(x_k, \rho_k)$ with $x_k \in A_k$ and $\rho_k = \beta^k \rho_0$
- Alice must choose $A_{k+1} \subset B(x_k, (1-\beta)\rho_k)$.



$B_{k+1} \supset B_k \cdots$

$x_\infty = \bigcap B_k$

- score of $\mathcal{A} = (A_k)$: $\underline{\delta}(\mathcal{A}) = \liminf_{k\to\infty}^{\sup} \frac{1}{k} \sum_{i=0}^{k} \frac{\log \# A_i}{-\log \beta}$
- $S \subset \mathbb{R}^D$ is δ-dim Haus. (pack) β-winning if Alice has strategy s.t. $x_\infty \in S$ and $\underline{\delta}(\mathcal{A}) \geq \delta$.

  winning if β-winning $\forall$ suff. small β.

- Thm: $\dim_{H,P}(S) = \sup\{ \delta : S \text{ is } \delta\text{-dim } H(P) \text{ winning} \}$

- Modified game: pull back everything under $T_k(\vec{z}) = \vec{x}_k + \varrho_k \vec{z}$ to origin.

  make a few slight modifications, such as

  $$A_{k+1} \subset B(\vec{0}, 1-\beta).$$

  outcome: $\vec{x}_\infty = \vec{x}_0 + \sum_{k=1}^{\infty} \beta^k \varrho_{-1} \vec{x}_k$.

- Target set: $D(\vec{f}) = \{ A : \vec{h}_A \asymp_+ \vec{f} \}$

  so we want to insure that as Alice plays on the space of matrices, $X_1, X_2, \ldots$ her choices stay "close" to the original template $\vec{f}$.

- Measuring this "closeness": unimodular lattice Λ_k.

  $$\Lambda_{k+1} = g_\gamma u_{X_k} \Lambda_k, \quad \gamma = -\underbrace{\frac{mn}{m+n}}_{\alpha} \log\beta > 0.$$

  explicitly $\Lambda_{k+1} = g_{-\alpha \log(\beta^{k+1} \varrho_{-1})} u_{Y_k} \mathbb{Z}^d$, $Y_k = X_0 + \sum_{i=1}^{k} \beta^i \varrho_{-1} X_i$

- Importance of this sequence:

  Lemma: define $\vec{j}(k\gamma) = \vec{h}(\Lambda_k)$ $\quad h_i(\Lambda) = \log \lambda_i(\Lambda)$ on $\gamma\mathbb{N}$, extend to $[0,\infty)$ by linear interpolation. Then $\vec{j} \asymp_+ \vec{h}_{X_\infty}$. So $\vec{h}(\Lambda_k)$ needs to stay close to $\vec{f}$ for $\vec{h}_{X_\infty} \asymp \vec{f}$.

- Reduction. • Template is <u>η-integral</u> if

  (i) corner pts are $\in \eta\mathbb{N}$,

  (ii) $\forall t \in \eta\mathbb{N}$, $f_i(t) \in \frac{\eta}{mnd!}\mathbb{Z}$ $\quad \forall i = 1, \ldots, d$.

  think "grid"

  • Template is <u>simple</u> if splits, merges, transfers are all pairwise disjoint.

- Thm: $\forall \eta > 0$ and template $\vec{f}$, $\exists$ simple η-integral template $\vec{g}$ s.t. $\vec{g} \asymp_+ \vec{f}$, implied constant dep. on η not on $\vec{f}$. <u>and</u> $\underline{\delta}(\vec{g}) \geq \underline{\delta}(\vec{f})$, $\overline{\delta}(\vec{g}) \geq \overline{\delta}(\vec{f})$.

- <u>C-match</u>: Let $\vec{g}$ be a γ-integral partial template, a lattice $\Lambda \subset \mathbb{R}^d$ is a C-match for $\vec{g}$ at $t \in \gamma\mathbb{N}$ if:

  (i) $\|\vec{h}(\Lambda) - \vec{g}(t)\| < C$ — i.e. $\Lambda \cap V$ lattice in V.

  (ii) $\exists$ family of nested Λ-rational subspaces $(V_q)_{q \in Q(t)}$

  $$Q(t) = \{q : g_q(t) < g_{q+1}(t)\},$$ — Λ can "switch" but not g.

  $$\dim V_q = q \quad \forall q \in Q(t),$$

  g "looks like" $\vec{h}(\Lambda)$ up to some error.

  $$|\log \lambda_i(\Lambda \cap V_q) - h_i(\Lambda)| \leq C$$

  $$\dim(V_q \cap \mathcal{L}) \geq L_-(\vec{g}, I, q)$$ — $\{\vec{0}\} \times \mathbb{R}^n \subset \mathbb{R}^d$ contracted by flow

  where I is an interval of linearity for $\vec{g}$ whose left endpt. is t

  $$G_q' = \frac{L_+}{m} - \frac{L_-}{n} \text{ on } I.$$

- Mini-strategy: Fix $k_1 < k_2 \in \mathbb{N}$, $t_i = k_i\gamma$, $\vec{g} : [t_1, \infty) \to \mathbb{R}^d$ a γ-int. partial template, and N = max # of intervals of linearity of $\vec{g}|_{(t_1, t_2)}$.

  - Suppose on k_1^{th} turn of game, Λ_{k_1} is a C_1-match for $\vec{g}$ at t_1. Then Alice has a strategy for $k_1, \ldots, k_2 - 1$ s.t.

    (i) $\forall k \in [k_1, k_2]_{\mathbb{Z}}$, $\vec{h}(\Lambda_k) \asymp_{+C_1, N, \beta} \vec{g}(k\gamma)$

    (ii) final lattice Λ_{k_2} is a C_2-match for $\vec{g}$ at time t_2, where $C_2 = C_2(C_1, N, \beta)$

    (iii)
    $$\Delta(\mathcal{A}, [k_1, k_2]) := \frac{1}{k_2 - k_1} \sum_{k=k_1}^{k_2} \frac{\log \# A_k}{-\log \beta} = \delta(\vec{g}, [t_1, t_2]) + O\left(\frac{1}{\gamma} + \frac{1}{k_2 - k_1}\right)$$

    where implied constant depends on C_1, N but <u>not</u> β.

- Error correction: Fix $\eta > 0$, let $\vec{f}$ be a simple η-t.f. template, fix $t_0 \in \eta \mathbb{N}$,
  - let $\vec{b} \in \mathbb{R}^d$ with $b_i \leq b_{i+1}$ $\forall i$ s.t. $f_i(t_0) = f_{i+1}(t_0)$. perturbation vector of $\vec{f}$ at t_0.
  - let $t_k = t_0 + k\eta$, define $\vec{a} : \mathbb{N} \cup \{-1\} \to \mathbb{R}^d$:
    - $\vec{a}(-1) = \vec{b}$
    - let $I_k = (t_k, t_{k+1})$. if $(p,q]_{\mathbb{Z}}$ is an interval of equality for $\vec{f}$ on I_k, then $\forall\, i = p+1, \ldots, q$, set

$$a_i(k) = \begin{cases} a_i(k-1) & \text{if } f'_{p+1} = \cdots = f'_q \in \{\frac{1}{m}, -\frac{1}{n}\} \\ \frac{1}{q-p} \sum_{j=p+1}^{q} a_j(k-1) & \text{else.} \end{cases}$$

Consequence: $\|\vec{a}(k)\|_\infty \leq \|\vec{b}\|_\infty$ $\forall k$.

also $a_i(k) \leq a_{i+1}(k)$ whenever $f_i = f_{i+1}$ on I_k.

- Lemma 31.14. <u>With</u> <u>notation</u> <u>as</u> <u>above</u>, if $\|\vec{b}\|_\infty < c_\eta = \frac{\eta}{2mnd!}$, there exists partial template $\vec{g} : [t_0, \infty) \to \mathbb{R}^d$

$$\vec{g}(t_0) = \vec{f}(t_0) + \vec{b}$$

and s.t. $\forall k$,

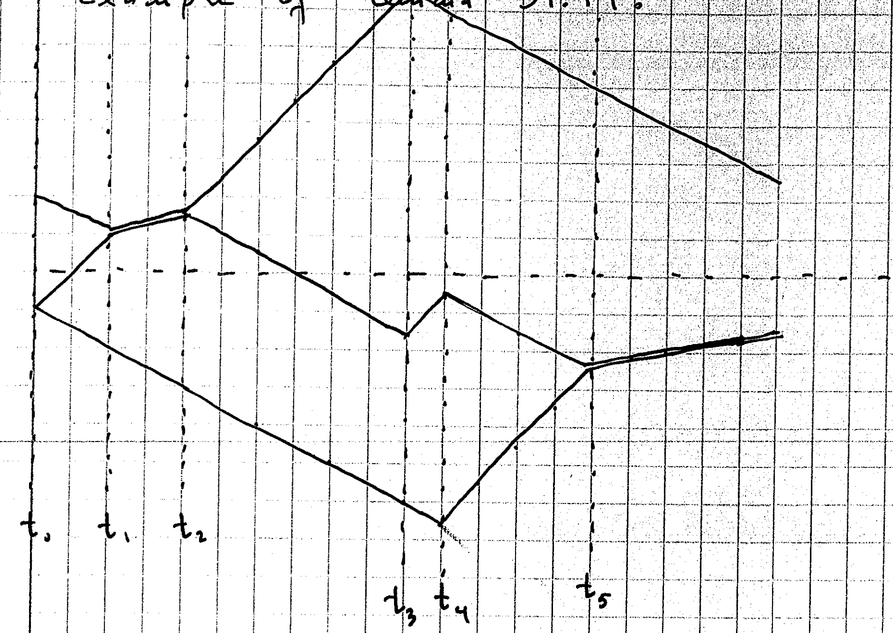
$$\vec{g} = \vec{f} + \vec{a}(k) \quad \text{on} \quad \underline{\tilde{I}_k = (t_k + s, t_k - s)}$$

where $\underline{s = 2mnd^2 \|b\|_\infty}$

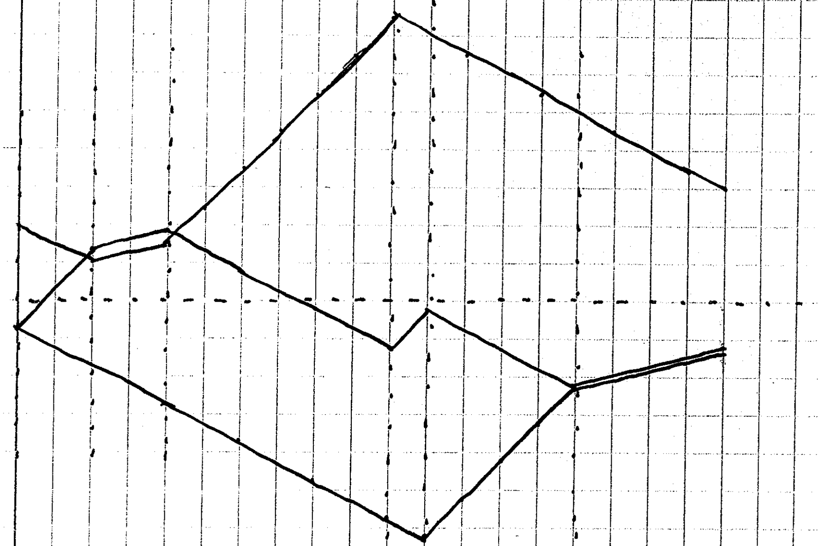
and $\delta(\vec{g}, t) = \delta(\vec{f}, t)$ for $t \in \tilde{I}_k$.

* defn: the template $\vec{g}$ constructed here is the <u>$\vec{b}$-perturbation of $\vec{f}$ at t_0.</u>

5 • Example of lemma 31.14:



graph of $\vec{f}$



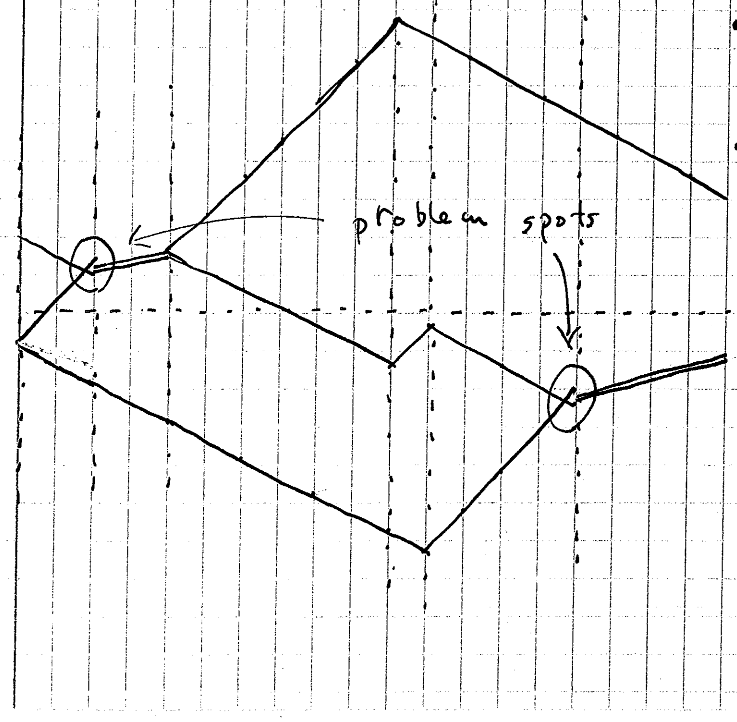
graph of $\vec{f} + \vec{b}$

where $\vec{b} = (1, 1, 0)$

not a template

calculate $\vec{b}$-perturbation of $\vec{f}$:

- $\vec{a}(-1) = \vec{a}(0) = \vec{b}$
- $\vec{a}(1) = \vec{a}(2) = \vec{a}(3) = \vec{a}(4) = \left(1, \frac{1}{2}, \frac{1}{2}\right)$
- $\vec{a}(5) = \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$



• Lemma 31.15: Let η be a sufficiently large multiple of γ, Λ a $C_\eta := \frac{\eta}{2md!}$ – match for an η-integral template $\vec{f}$ at $t_0 \in \eta\mathbb{N}$, and let $\vec{g}$ be the $\vec{b}$-perturbation of $\vec{f}$ at t_0, where $\vec{b} \in (d^3)!\,\gamma\mathbb{Z}^d$ is a perturbation vector such that

$$\|\vec{h}(\Lambda) - [f(t_0) + \vec{b}]\|_\infty < c_1$$

for some constant $c_1 \leq C_\eta$. Suppose that t_0 is not a split for $\vec{f}$. Then $\vec{g}$ is γ-integral and Λ is a c_1-match for $\vec{g}$ at t_0.