1. Reminder

A Borel probability measure σ on the real line $\mathbb{R}$ is called *self-similar* if it satisfies

$$\sigma = \sum_{i=1}^{\mathtt{m}} \lambda_i \, \phi_{i\star}\sigma \tag{1.1}$$

for some integer $\mathtt{m} \geq 1$, some probability vector $(\lambda_1, \cdots, \lambda_{\mathtt{m}}) \in \mathbb{R}^{\mathtt{m}}_{>0}$, and some invertible affine maps $\phi_1, \dots, \phi_{\mathtt{m}} : \mathbb{R} \to \mathbb{R}$ without common fixed point.

Definition of λ.

We let $\mathrm{Aff}(\mathbb{R})$ denote the affine group of $\mathbb{R}$. For every $\phi \in \mathrm{Aff}(\mathbb{R})$, we let $\mathtt{r}_\phi \in \mathbb{R}^*$, $\mathtt{b}_\phi \in \mathbb{R}$ denote the unique numbers such that

$$\phi(t) = \mathtt{r}_\phi t + \mathtt{b}_\phi, \quad \forall t \in \mathbb{R}. \tag{1.2}$$

Let $\mathrm{Aff}(\mathbb{R})^+$ denote the group of orientation preserving affine transformations of the real line. Denote by

$$P = \{\, a(t)u(s) : t > 0,\ s \in \mathbb{R} \,\} \subseteq G$$

the subgroup of upper triangular matrices with positive diagonal entries. For every $g \in P$, we let $\mathtt{r}_g \in \mathbb{R}_{>0}$ and $\mathtt{b}_g \in \mathbb{R}$ be the unique numbers such that

$$g = a(\mathtt{r}_g)^{-1} u(\mathtt{b}_g) = \begin{pmatrix} \mathtt{r}_g^{-1/2} & \mathtt{r}_g^{-1/2}\mathtt{b}_g \\ 0 & \mathtt{r}_g^{1/2} \end{pmatrix}.$$

We identify P with $\mathrm{Aff}(\mathbb{R})^+$ by mapping $g \in P$ with the similarity $s \mapsto \mathtt{r}_g s + \mathtt{b}_g$. This is an anti-isomorphism between the two groups.

Definition 1.1. Let G be a group acting on X, $\mu \in \mathrm{prob}(G)$, $\nu \in \mathrm{prob}(X)$. A convolution $\mu * \nu$ is defined as

$$\mu * \nu = \int_G g_* \nu d\mu(g).$$

It is a probability measure on X.

Fix a probability measure λ on $\mathrm{Aff}(\mathbb{R})^+$ with finite support $\mathrm{supp}\,\lambda$ that is not a single point. Denote by μ the corresponding probability measure on P via the above anti-isomorphism. Throughout this paper, λ and μ determine each other in this way. For $n \in \mathbb{N}$, we write $\lambda^{*n} = \lambda * \cdots * \lambda$ to denote the n-fold convolution of λ with itself, we define μ^{*n} similarly.

2. Preliminaries

Self-similar measure σ.

Given a finitely supported probability measure λ on $\mathrm{Aff}^+(\mathbb{R})$ we let σ denote a probability measure on $\mathbb{R}$ that is λ-stationary, which means

$$\sigma = \int_{\mathrm{Aff}^+(\mathbb{R})} \phi_\star \sigma \, \mathrm{d}\lambda(\phi) = \sum_{i=1}^{\mathtt{m}} \lambda_i \, \phi_{i\star}\sigma.$$

In Alon's talk we saw this lemma.

Lemma 2.1 (Moment and Hölder-regularity of σ). *There exists $\gamma > 0$ such that*

$$(i) \int_{\mathbb{R}} |s|^\gamma \, \mathrm{d}\sigma(s) < \infty, \qquad (ii)\ \forall r > 0,\ \sup_{s \in \mathbb{R}} \sigma(s + [-r, r]) \ll r^\gamma.$$

Given an integer $n \in \mathbb{N}$, we denote

$$\sigma^{(n)} = \lambda^{*n} * \delta_0,$$

where δ_0 denotes the Dirac measure at $0 \in \mathbb{R}$. We can rewrite it as

$$\sigma^{(n)} = \sum_{1 \leq i_1, \ldots, i_n \leq m} \lambda_{i_1} \ldots \lambda_{i_n} (\phi_{i_1} \circ \cdots \circ \phi_{i_n})_* \delta_0.$$

We will write

$$\lambda_I = \lambda_{i_1} \ldots \lambda_{i_n} \quad \text{and} \quad \phi_I = \phi_{i_1} \circ \cdots \circ \phi_{i_n}.$$

We show that the measures $\sigma^{(n)}$ have a uniformly finite positive moment, and uniform positive dimension above an exponentially small scale. For this, we first observe that $\sigma^{(n)}$ converges toward σ at exponential rate. We denote by $\mathrm{Lip}(\mathbb{R})$ the space of bounded Lipschitz functions on $\mathbb{R}$ with the norm $\|f\|_{\mathrm{Lip}} = \|f\|_\infty + \sup_{s \neq t} \frac{|f(s)-f(t)|}{|s-t|}$.

Lemma 2.2. *There exists $\varepsilon > 0$ such that for all $n \geq 0$, all $f \in \mathrm{Lip}(\mathbb{R})$, we have*

$$|\sigma^{(n)}(f) - \sigma(f)| \ll e^{-\varepsilon n} \|f\|_{\mathrm{Lip}}.$$

Proof. We have

$$\begin{aligned}
|\sigma^{(n)}(f) - \sigma(f)| &= |\lambda^{*n} * \delta_0(f) - \lambda^{*n} * \sigma(f)| \\
&= \left| \sum_I \lambda_I \cdot \int f(x) d\phi_{I*}\delta_0(x) - \sum_I \lambda_I \int f(x) d\phi_{I*}\sigma(x) \right| = \\
&= \left| \sum_I \lambda_I \cdot f(\phi_I(0)) - \sum_I \lambda_I \int f(\phi(x)) d\sigma(x) \right| = \\
&= \left| \sum_I \lambda_I \cdot \int \big(f(\phi_I(0)) - f(\phi(x)) \big) d\sigma(x) \right| \leq \\
&\leq \left| \sum_I \lambda_I \cdot \int \|f\|_{\mathrm{Lip}} |r_I x| d\sigma(x) \right|.
\end{aligned}$$

Because σ is compactly supported, $\int |x| d\sigma(x) = c$, where c is some constant. Let $r_{\max} = \max\{|r_1|, \ldots, |r_m|\}$. Then

$$\begin{aligned}
|\sigma^{(n)}(f) - \sigma(f)| &\leq \left| \sum_I \lambda_I \cdot \int \|f\|_{\mathrm{Lip}} |r_I x| d\sigma(x) \right| \leq \\
&\leq r_{\max}^n c \|f\|_{\mathrm{Lip}} \cdot \left| \sum_I \lambda_I \right| = e^{n \ln r_{\max}} c \|f\|_{\mathrm{Lip}}.
\end{aligned}$$

Because $r_{\max} < 1$, $\ln r_{\max} < 0$. So we set $\varepsilon = -\ln r_{\max}$ and the proof is complete. □

We now deduce our claim on the measures $\sigma^{(n)}$.

Lemma 2.3 (Moment and Hölder-regularity of $\sigma^{(n)}$)**.** *There exists $\gamma > 0$ such that*

$$(i) \quad \sup_{n \geq 1} \int_{\mathbb{R}} |s|^\gamma \, \mathrm{d}\sigma^{(n)}(s) < \infty$$

and

$$(ii) \quad \forall n \geq 1, \forall r > e^{-n}, \quad \sup_{s\in\mathbb{R}} \sigma^{(n)}(s+[-r,r]) \ll r^{\gamma}.$$

Proof. (i) We claim that there exist a compact K such that for any n $\operatorname{supp}\sigma^{(n)} \subseteq K$. Then for a continuous function on a compact set the claim is trivial.

(ii) We fix $s \in \mathbb{R}$. We define a function f as

$$f(x) = \begin{cases} f_1(x) = \frac{x-(s-2r)}{r}, & x \in [s-2r, s-r] \\ f_2(x) = 1, & x \in [s-r, s+r] \\ f_3(x) = \frac{-x+(s+2r)}{r}, & x \in [s+r, s+2r] \\ 0, & \text{otherwise.} \end{cases}$$

We can assume that $\sigma^{(n)}([s-r,s+r]) \geq \sigma([s-2r,s+2r])$, because otherwise the claim follows from Lemma 2.1. Lemma 2.2 guarantees that

$$\int f d\sigma^{(n)} - \int f d\sigma \leq e^{-\varepsilon n} c \|f\|_{\mathrm{Lip}}.$$

Then

$$\sigma^{(n)}([s-r,s+r]) + \sigma^{(n)}(f_1) + \sigma^{(n)}(f_3) \leq \sigma([s-r,s+r]) + \sigma(f_1) + \sigma(f_3) + e^{-\varepsilon n} c \|f\|_{\mathrm{Lip}},$$

Function f_1 and f_3 are positive, so we can disregard $\sigma^{(n)}(f_1), \sigma^{(n)}(f_3)$. Because $f_1, f_3 \leq f_2$, we have

$$\sigma([s-r,s+r]) + \sigma(f_1) + \sigma(f_3) \leq \sigma([s-2r,s+2r]).$$

The Lipschitz norm of the function $\|f\|_{\mathrm{Lip}} = 1 + \frac{1}{r}$. So we get

$$\sigma^{(n)}([s-r,s+r]) \leq \sigma([s-2r,s+2r]) + \left(1 + \frac{1}{r}\right) e^{-\varepsilon n} c.$$

For $r > e^{-\varepsilon n/2}$ we have $\frac{1}{r} e^{-\varepsilon n} < e^{-\varepsilon n/2}$, so

$$\sigma^{(n)}([s-r,s+r]) \leq \sigma([s-2r,s+2r]) + e^{-\varepsilon n} c + e^{-\varepsilon n/2} c.$$

By Lemma 2.1

$$\sigma^{(n)}([s-r,s+r]) \leq c_1 r^{\gamma} + e^{-\varepsilon n} c + e^{-\varepsilon n/2} c.$$

We can always find such $\gamma_1 > \gamma$ and $M > 0$ such that for all $r > e^{-\varepsilon n/2}$

$$\sigma^{(n)}([s-r,s+r]) \leq c_1 r^{\gamma} + e^{-\varepsilon n} c + e^{-\varepsilon n/2} c \leq M r^{\gamma_1}.$$

□

Finally, we derive from Lemma 2.3 that $\sigma^{(n)}$ satisfies a non-concentration estimate with respect to polynomials of degree 2.

Lemma 2.4 (Regularity of $\sigma^{(n)}$ for quadratic polynomials)**.** *There exists $\gamma > 0$ such that for every $n \geq 1$, $r > e^{-n}$ and $(a,b,c) \in \mathbb{R}^3$ with $\max(|a|,|b|,|c|) \geq 1$, we have*

$$\sigma^{(n)}\{s : |as^2 + bs + c| \leq r\} \ll r^{\gamma}.$$

Proposition 2.5 (Effective recurrence on X)**.** *There exist constants $c, c' > 0$ depending on μ only such that for every $x \in X$, $n \in \mathbb{N}$, and $\rho > 0$, we have*

$$\mu^{*n} * \delta_x\{\mathrm{inj} < \rho\} \ll (\mathrm{inj}(x)^{-c} e^{-c'n} + 1)\rho^c.$$