GEOMETRIC AND ARITHMETIC ASPECTS OF APPROXIMATION VECTORS – introduction

January 25, 2025

step 1: equid of $\left(disp\,(\vec{\theta},\vec{v}_k),\,\left[\pi_{\mathbb{R}^d}^{\vec{v}_k}(\mathbb{Z}^n)\right]\right)$ and of $\left(disp\,(\vec{\theta},\vec{w}_k),\,\left[\pi_{\mathbb{R}^d}^{\vec{w}_k}(\mathbb{Z}^n)\right]\right)$.

Seperate between the Lebesgue a.e beaviour [case 1 (Alon)] θ with special algebraic properties [case 2 (Vika)].

step 2: extend to triples $\left(disp\,(\vec{\theta},\vec{v}_k),\,\left[\pi_{\mathbb{R}^d}^{\vec{v}_k}(\mathbb{Z}^n)\right],\,\vec{v}_k\right)$ and $\left(disp\,(\vec{\theta},\vec{w}_k),\,\left[\pi_{\mathbb{R}^d}^{\vec{w}_k}(\mathbb{Z}^n)\right],\,\vec{w}_k\right)$ [Rishi].

Get in total 8 cases.

PICTURE

In this talk

- define S_{r_0} – it is μ-cross section for any $\{a_t\}$-inv measure
- reasonable for $m_{\mathcal{X}_n}$ the Haar measure

Chapter 8.

Given $W \subset \mathbb{R}^n$ and $k \geq 1$ denote

$$\mathcal{X}_n(W,k) := \{\Lambda \in \mathcal{X}_n \,|\, \#(\Lambda_{prim} \cap W) \geq k\,\}$$
$$\mathcal{X}_n(W) := \mathcal{X}_n(W,1).$$

$$\mathcal{X}_n^{\#}(W) := \{\Lambda \in \mathcal{X}_n \,|\, \#(\Lambda_{prim} \cap W) = 1\,\}.$$

So we have $\mathcal{X}_n^{\#}(W) \subset \mathcal{X}_n(W)$.

Theorem (Lemma 8.1). *Let $W \subset \mathbb{R}^n$ compact set, $V \subset W$ open in W and $k \geq 1$.*

1. *The set $\mathcal{X}_n(W,k)$ is closed in $\mathcal{X}_n$.*
2. *The set $\mathcal{X}_n^{\#}(W) \cap \mathcal{X}_n(V)$ is open in $\mathcal{X}_n(W)$.*
3. *The map $v : \mathcal{X}_n^{\#}(W) \to W$ defined by $v(\Lambda) = \Lambda_{prim} \cap W$ is continuous.*

proof of 1: Let $(\Lambda_i)_{i=1}^{\infty} \subset \mathcal{X}_n(W,k)$ with $\Lambda_i \longrightarrow \Lambda$.

Choose $g_i \in SL_n(\mathbb{R})$ and $h \in SL_n(\mathbb{R})$ such that $g_i\mathbb{Z}^n = \Lambda_i$, $h\mathbb{Z}^n = \Lambda$ and $g_i \longrightarrow h$.

(g_i) is converging and W is compact, so there exists M compact such that $g_i^{-1}W \subset M$ for all i.

Since $g_i\mathbb{Z}^n \in \mathcal{X}_n(W,k)$ we have for all i that

$$\#(g_i^{-1}(W) \cap \mathbb{Z}^n_{prim}) \geq k.$$

So, after passing to a subsequence (which we omit from indices), there exists $v_1, ..., v_k \in \mathbb{Z}^n_{prim}$ such that $g_i v_j \in W$ for all i and all $1 \leq j \leq k$.

W is compact, and $g_i v_j \longrightarrow hv_j$, so $hv_j \in W$ for all $1 \leq j \leq k$ with $hv_1, ... hv_k$ primitive in $h\mathbb{Z}^n = \Lambda$.

So we get $\#(\Lambda \cap W) \geq k$, hence $\Lambda \in \mathcal{X}_n(W,k)$.

Theorem (Lemma 8.2). *If $W \subset \mathbb{R}^n$ is open then for any $k \geq 1$, $\mathcal{X}_n(W,k)$ is open in $\mathcal{X}_n$.*

The Cross section S_{r_0}

$$Cylinder(r,s) \ = C_r(s) := \{\vec{x} \in \mathbb{R}^n \mid \|\pi_{\mathbb{R}^d}(\vec{x})\| \leq r \ |x_n| \leq s\,\}$$
$$C_r := C_r(1)$$

where for $\vec{x} = (x_1, ..., x_d, x_{d+1}) \in \mathbb{R}^n$ we have $\pi_{\mathbb{R}^d}(\vec{x}) = (x_1, ..., x_d)$.

norm dependent

Choose r_0 such that $vol(C_{r_0}) \geq 2^n$. So by Minkowski's thm, we have $\mathcal{X}_n(C_{r_0}) = \mathcal{X}_n$.

For $r > 0$ define

$$D_r := \{\vec{x} \in \mathbb{R}^n \mid \|\pi_{\mathbb{R}^d}(\vec{x})\| \leq r \ |x_n| = 1\,\}$$
$$S_r := X_n(D_r).$$

Theorem (Lemma 8.4). *Let μ be any $\{a_t\}$-invariant probability measure on $\mathcal{X}_n$. Then S_{r_0} is a μ-cross-section for $(\mathcal{X}_n, \mu, \{a_t\})$. Furthermore, the cross-section measure satisfies*

$$\mu_{S_{r_0}}(\mathcal{X}_n(D_{r_0}, 2)) = 0.$$

(notice $\mathcal{X}_n(D_{r_0}, k) \subseteq \mathcal{X}_n(D_{r_0}, 2)$ for any $k \geq 2$. So particularly $\mu_{S_{r_0}}(\mathcal{X}_n(D_{r_0}, k)) = 0$ for any $k \geq 2$.)

proof: First we show S_{r_0} is a μ-cross section. So we need to show

- S_{r_0} is Borel (which holds by Lemma 8.1)

- there exists an $\{a_t\}$-invariant $X_0 \subset \mathcal{X}_n$ such that
  - $\mu(\mathcal{X}_n \setminus X_0) = 0$
  - S_{r_0} is a Borel cross section for X_0:
    1. For any $\Lambda \in X_0$ the set of visit times of Λ in S_{r_0} is discrete and totally unbounded.
    2. The return time function $\tau_{S_{r_0}}$ is measurable.

Define
$$X_0 := \{\Lambda \in \mathcal{X}_n \,|\, \Lambda \cap \left(\pi_{\mathbb{R}^d}(\mathbb{R}^n) \cup span\{\vec{e_n}\}\right) = \vec{0}\,\}.$$

Notice if $\Lambda \notin X_0$ then Λ has a point in $\pi_{\mathbb{R}^d}(\mathbb{R}^n)$ or in $span(\vec{e}_n)$, hence Λ diverges by Mahler compactness criterion. So by Poincare recurrence, $\mu(\mathcal{X}_n \setminus X_0) = 0$.

(almost every point is recurrent, i.e. returns to any nbd of the point infinitely many times).

Let $\Lambda \in X_0$. We show that $\{t \in \mathbb{R} \,|\, a_t\Lambda \in S_{r_0}\}$ is discrete and unbounded from below and from above.

First note that $a_t\Lambda \in S_{r_0}$ if and only if Λ_{prim} contains a vector in $a_{-t}(D_{r_0})$.

Discreteness – $(a_t\Lambda)_{prim}$ is a discrete set in $\mathbb{R}^n$, and how a_t acts (cannot change a vector in parallel to the horizontal space).

Suppose by contradiction that there is $T > 0$ such that for all $s \geq T$, $a_s\Lambda \notin S_{r_0}$. I.e. for all s large enough, $(a_s\Lambda)_{prim} \cap D_{r_0} = \varnothing$.

The set $a_T\Lambda \cap C_{r_0}$ is finite, and since $\Lambda \in X_0$, all vectors in $a_T\Lambda \cap C_{r_0}$ have non-zero horizontal component. It follows that for all large enough $t > T$ we have

$$a_t(\Lambda \cap a_{-T}C_{r_0}) \cap C_{r_0} = a_t\Lambda \cap a_{t-T}C_{r_0} \cap C_{r_0} = \varnothing. \tag{1}$$

Fix t with $T < t$ large enough s.t the above holds.

By Minkowski's thm (and the choice of r_0), there exists $\vec{v} \in (a_t\Lambda)_{prim} \cap C_{r_0}$.

write $\vec{v} = (v_1, ..., v_n)$. Since $\Lambda \in X_0$, we have $v_n \neq 0$, and we can assume without loss of generality that $v_n > 0$.

Write $\vec{v} = a_t\vec{v}_0$ for $\vec{v}_0 \in \Lambda_{prim}$.

By definition of $\{a_s\}$ there is a unique s such that $a_s\vec{v}_0 \in D_{r_0}$, and since the vertical component of v is at most 1, we have $s \leq t$.

This means that $a_s\Lambda \in S_{r_0}$ and by choice of T (our negation assumption) we must have $s \leq T$, so that the vertical component of $a_T\vec{v}_0$ is at most 1. On the other hand

$$\|\pi_{\mathbb{R}^d}(a_T\vec{v}_0)\| = \|\pi_{\mathbb{R}^d}(a_{T-t}\vec{v})\| = e^{T-t}\,\|\pi_{\mathbb{R}^d}(\vec{v})\| \leq r_0,$$

so that $a_T\vec{v}_0 \in C_{r_0}$. This shows that

$$\begin{aligned} a_T\vec{v}_0 &\in a_T\Lambda \cap C_{r_0} \\ a_{T-t}\vec{v} &\in a_T\Lambda \cap C_{r_0} \\ \vec{v} &\in a_t\Lambda \cap a_{t-T}C_{r_0} \end{aligned}$$

As we started with $\vec{v} \in C_{r_0}$ we get in total $\vec{v} \in a_t\Lambda \cap a_{t-T}C_{r_0} \cap C_{r_0}$, contradicting 1.

The argument showing unboundedness from below is similar.

Need to show that the return time function is Borel measurable, or equivalently, that the sub-level sets

$$\begin{aligned} S_{r_0,<\varepsilon} &= \{\Lambda \in S_{r_0} \,|\, \tau_{S_{r_0}}(\Lambda) < \varepsilon\} \\ &= \{\Lambda \in S_{r_0} \mid \min\left(t > 0 \,|\, a_t\Lambda \in S_{r_0}\right) < \varepsilon\} \end{aligned}$$

are Borel.

The set

$$D_{r_0}^{(-\varepsilon,0)} = \bigcup_{t\in(0,\varepsilon)} a_{-t}(D_{r_0})$$

is a Borel subset of $\mathbb{R}^n$ hence

$$\begin{aligned} S_{r_0,<\varepsilon} &= \{\Lambda \in S_{r_0} \,|\, \Lambda_{prim} \cap \bigcup_{t\in(0,\varepsilon)} a_{-t}(D_{r_0}) \neq \varnothing\,\} \\ &= \mathcal{X}_n(D_{r_0}) \cap \mathcal{X}_n(\bigcup_{t\in(0,\varepsilon)} a_{-t}(D_{r_0})) \end{aligned} \tag{2}$$

is also Borel.

To show $\mu_{S_{r_0}}(\mathcal{X}_n(D_{r_0}, 2)) = 0$, recall we showed that for any $E \subset S_{r_0}$ Borel set we have that

$$\mu_{S_{r_0}}(E) = 0 \quad iff \quad \mu(E^{\mathbb{R}}) = 0.$$

so since

$$\mathcal{X}_n(D_{r_0}, 2)^{\mathbb{R}} = \{a_t\Lambda \,|\, t \in \mathbb{R}\ ,\ \#(\Lambda_{prim} \cap D_{r_0}) \geq 2\,\} \subset \mathcal{X}_n \setminus X_0$$

we get the above. □

Parameterizing S_{r_0}.

Let $\mathcal{E}_n = \mathcal{X}_n(\vec{e}_n) = \{\Lambda \in \mathcal{X}_n \,|\, \vec{e}_n \in \Lambda_{prim}\,\}$.

$\overline{B_r} \subset \mathbb{R}^d$ denote the closed ball around the origin w.r.t the norm (omit $\|\cdot\|$ from notation).

Consider the map

$$\begin{aligned} \varphi : \mathcal{E}_n \times \overline{B_{r_0}} &\longrightarrow S_{r_0} \\ \varphi(\Lambda, \vec{v}\,) &:= u(\vec{v}\,)\Lambda \end{aligned}$$

for $u(\vec{v}\,) := \begin{pmatrix} I_d & \vec{v} \\ 0 & 1 \end{pmatrix}$. Notice that $(\vec{v}, 1) \in \big(u(\vec{v}\,)\,\Lambda\big)_{prim}$ (since $\Lambda \in \mathcal{E}_n$). So since $\vec{v} \in \overline{B}_{r_0}$ this is well defined.

Observations:

- φ is onto.
- Furthermore, for any $\Lambda \in S_{r_0}$ we have $\#\,\varphi^{-1}(\Lambda) = \#\,(\Lambda_{prim} \cap D_{r_0})$.

So far – S_{r_0} is a μ-cross section for any $\{a_t\}$-inv probability measure on $\mathcal{X}_n$.

Theorem (Theorem 8.6). *The cross section S_{r_0} is $m_{\mathcal{X}_n}$-reasonable.* [*case* 1]

Recall that in order to say S_{r_0} is a $m_{\mathcal{X}_n}$-reasonable cross-section, need to know/show that

(A) $m_{\mathcal{X}_n}$ is probability on $\mathcal{X}_n$.

(B) $\mu_{S_{r_0}}$ is finite. Recall $\mu_{S_{r_0}} = \mu_{S_{r_0}}(S_{r_0}, m_{\mathcal{X}_n})$, but we omit the dependency on $m_{\mathcal{X}_n}$ from the notation.

(C) S_{r_0} is lcsc.

(D) For all sufficiently small ε, the sets $S_{r_0,\geq\varepsilon}$ are $\mu_{S_{r_0}}$-JM. I.e. $\mu_{S_{r_0}}\big(\partial(S_{r_0,\geq\varepsilon})\big) = 0$.

(E) There exists U an open subset of S_{r_0} such that the following two conditions hold:
   - the map $(0,1) \times U \to X, (t,x) \mapsto a_t x$ is an open map
   - $m_{\mathcal{X}_n}\big((cl(S_{r_0}) \setminus U)^{(0,1)}\big) = 0$.

we have (A) by properties of $m_{\mathcal{X}_n}$, and we have (C) from Lemma 8.1. So in order to prove Thm 8.6 we are left with (B),(D),(E).

We only state the relevant proposotions:

Theorem (Proposition 8.7)(B). *For $m_{\mathcal{X}_n}$ We have that*

$$\mu_{S_{r_0}} = \frac{1}{\zeta(n)}\,\varphi_*\big(m_{\mathcal{E}_n} \times VOL_{|\overline{B}_{r_0}}\big).$$

In particular, $\mu_{S_{r_0}}$ is finite and supp$\,(\mu_{S_{r_0}}) = S_{r_0}$.

Theorem (Proposition 8.9)(D). *For any $\varepsilon > 0$ we have that*

$$\partial_{S_{r_0}}(S_{r_0,<\varepsilon}) \subset \mathcal{X}_n(D_{r_0}, 2) \cup \big(\mathcal{X}_n(D_{r_0}) \cap \mathcal{X}_n(a_{-\varepsilon} D_{r_0})\big) \cup \mathcal{X}_n\Big(\,(D_{r_0} \setminus int(D_{r_0}))^{[-\varepsilon,0]}\,\Big)$$

Furthermore, all of the sets in the union above are $\mu_{S_{r_0}}$ null sets, hence $S_{r_0,<\varepsilon}$ are $\mu_{S_{r_0}}$-JM, hence $S_{r_0,\geq\varepsilon}$ are $\mu_{S_{r_0}}$-JM

Theorem (Proposition 8.10)(E). *Let $U_{r_0} := \mathcal{X}_n^{\#}(D_{r_0}) \cap \mathcal{X}_n(int\,D_{r_0})$.*

(This is just $\mathcal{X}_n^{\#}(int\,D_{r_0})$, but want to use Lemma 8.1 to say its open)

Then (E) holds for U_{r_0}.