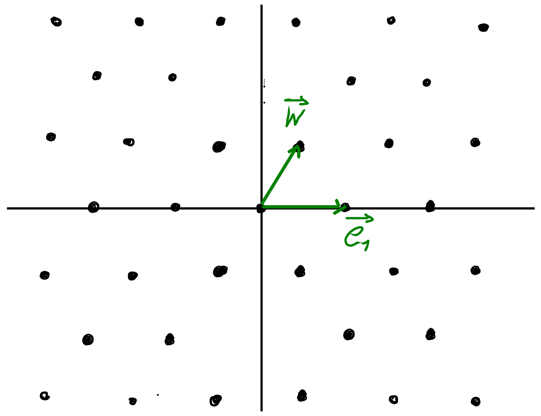
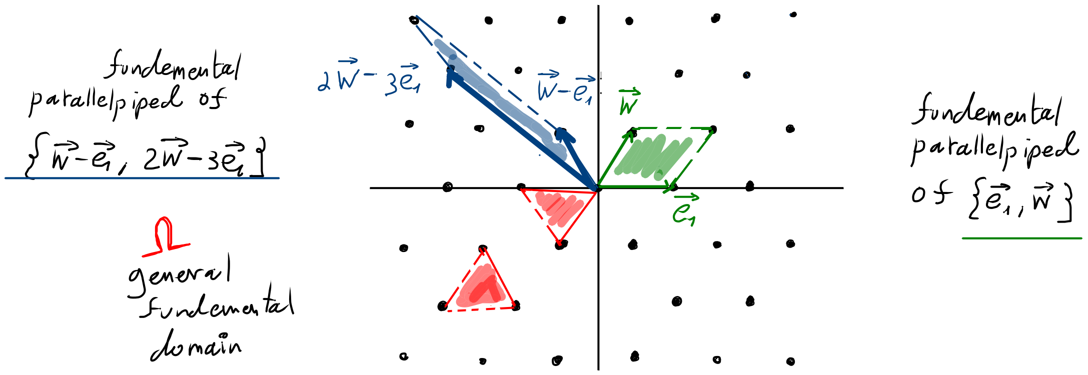
picture 1



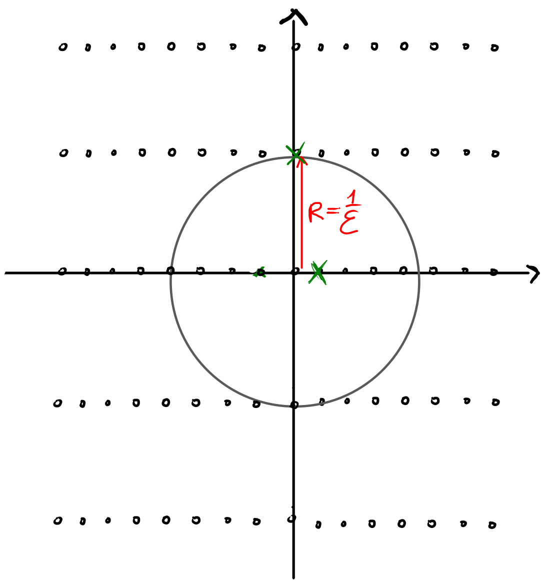
$L = \mathrm{span}_{\mathbb{Z}}(\vec{e}_1, \vec{w})$

$\vec{w} = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$

picture 2



picture 3



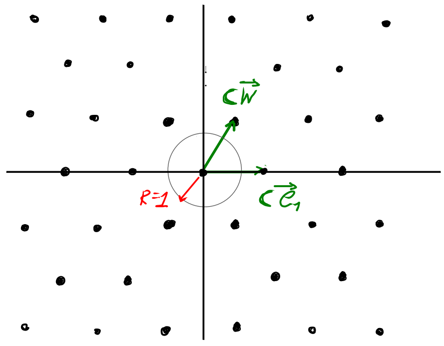
$L = \mathrm{span}_{\mathbb{Z}}(\varepsilon\vec{e}_1, \frac{1}{\varepsilon}\vec{e}_2)$

$0 < \varepsilon < 1$

$\lambda_1 = \varepsilon$

$\lambda_2 = \frac{1}{\varepsilon}$

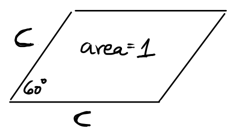
picture 4



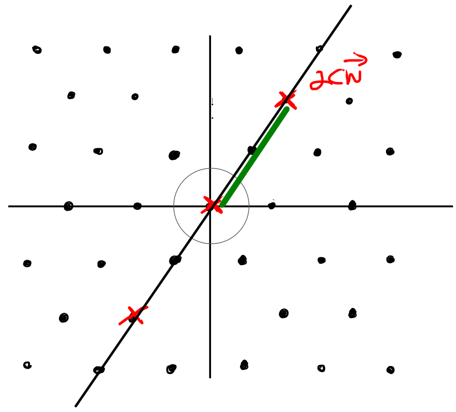
$\Lambda = \mathrm{span}_{\mathbb{Z}}(c\vec{e}_1, c\vec{w})$

$c \simeq 1.07$

$\vec{w} = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$



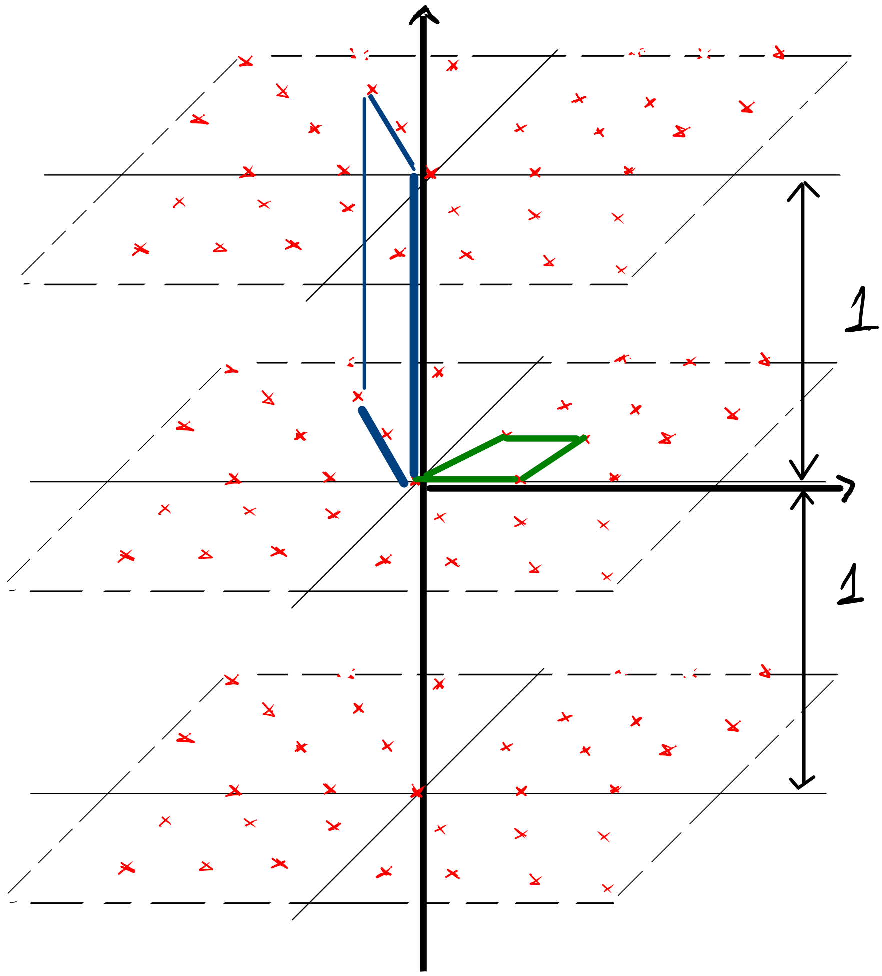
picture 5



$L_0 = \mathrm{span}_{\mathbb{Z}}(2c\vec{w})$

$\boxed{\mathrm{covol}(L_0) = 2c}$

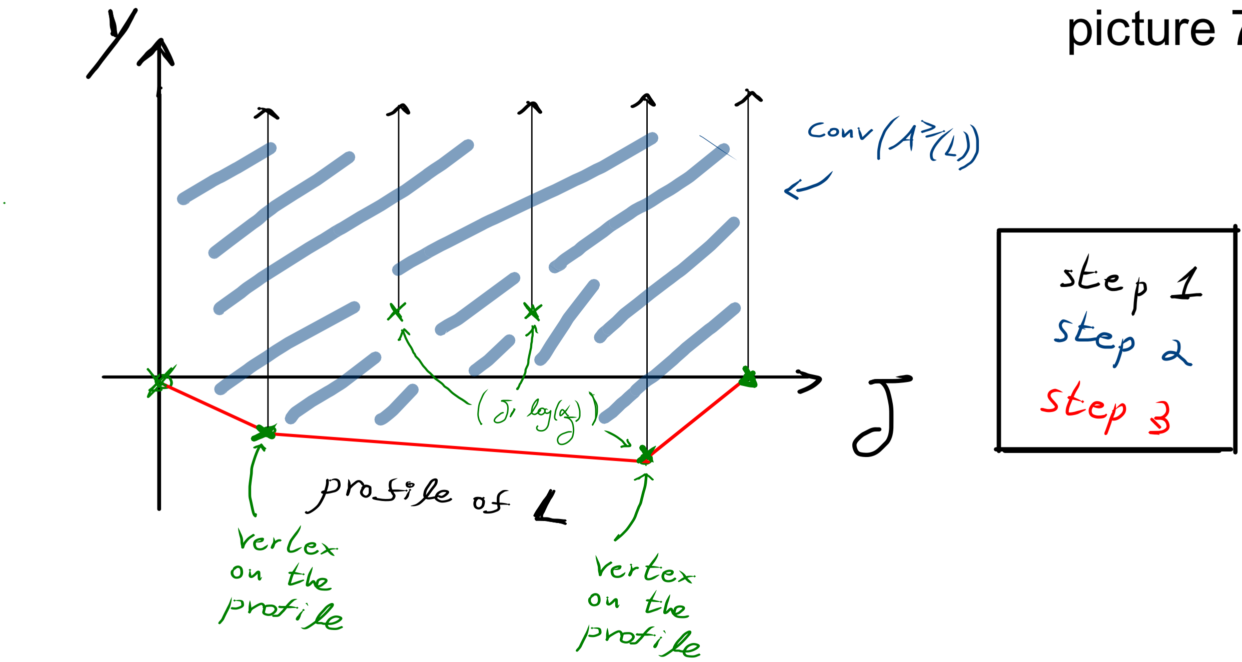
picture 6



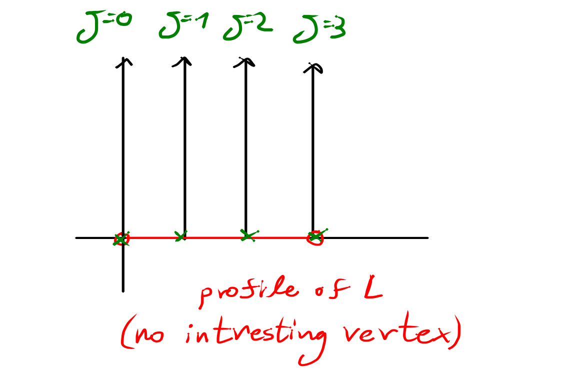
$\mathbb{L} = \Lambda \oplus \mathbb{Z}\vec{e}_3$

$= \mathrm{span}_{\mathbb{Z}}(c\vec{e}_1, c\vec{w}, \vec{e}_3)$

picture 7

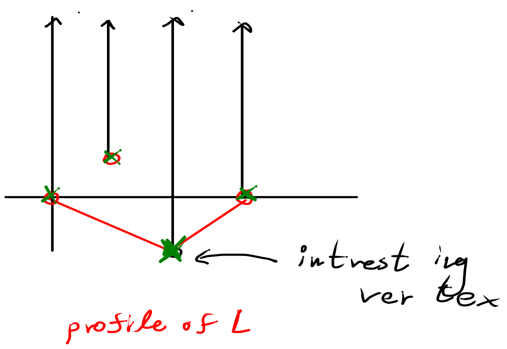


picture 8



$$L = \Lambda \oplus \mathbb{Z}\vec{e}_3$$

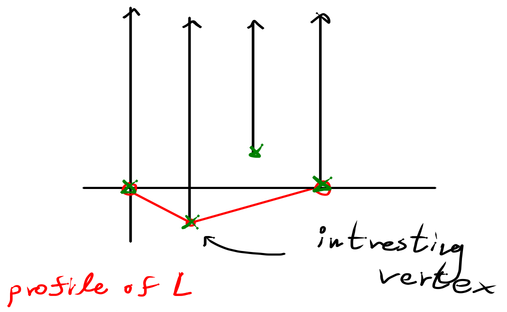
$$\alpha_1 = \alpha_2 = 1$$



$$L = (1-\varepsilon)\Lambda \oplus \frac{1}{(1-\varepsilon)} \mathbb{Z}\vec{e}_3$$

$$\alpha_1 = \frac{1}{1-\varepsilon} > 1$$

$$\alpha_2 < 1$$



$$L = \frac{1}{(1-\varepsilon)} \Lambda \oplus (1-\varepsilon) \mathbb{Z}\vec{e}_3$$

$$\alpha_1 = 1-\varepsilon < 1$$

$$\alpha_2 > 1$$

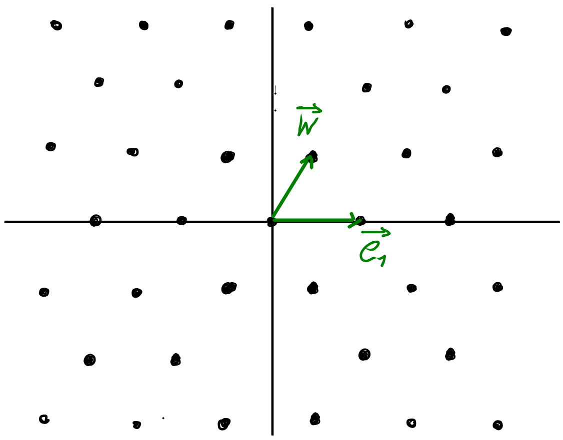
Crash course on Lattices

based on lectures 1-5 from "Geomtry of numbers" course given by Barak Weiss, fall 2020

Def: $L \subset \mathbb{R}^n$ is called a Lattice if there exists $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ linearly-independent vectros s.t

$$L = \left\{ \sum_{j=1}^{n} a_j \vec{v}_j \;\middle|\; a_j \in \mathbb{Z} \right\} = \mathrm{span}_{\mathbb{Z}} \{ \vec{v}_1, \dots, \vec{v}_n \}.$$

the set $\{\vec{v}_1, \dots, \vec{v}_n\}$ will be called a basis for L.



$L = \mathrm{span}_{\mathbb{Z}} \{ \vec{e}_1, \vec{w} \}$

where

$\vec{w} = \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3} \right)$

Observation: L is a lattice iff $\exists A \in GL_n(\mathbb{R})$ s.t $L = A\mathbb{Z}^n = \mathrm{span}_{\mathbb{Z}} [A\vec{e}_1, \dots, A\vec{e}_n]$. in this conditions, we say that A is a basis of L

Define $GL_n(\mathbb{Z}) := \{ B \in M_n(\mathbb{Z}) \mid \det B = \pm 1 \}$ and notice this is a group. i.e - if $B \in GL_n(\mathbb{Z})$ then $B^{-1} \in M_n(\mathbb{Z})$. ($B^{-1} = (\det B)^{-1} \cdot C^t$, $C =$ co-factors matrix $\in M_n(\mathbb{Z})$)

claim let $A, A' \in GL_n(\mathbb{R})$. then

$$L = A\mathbb{Z}^n = A'\mathbb{Z}^n \iff \exists B \in GL_n(\mathbb{Z}) \text{ s.t } A = A'B$$

partiaularly, if $L = A\mathbb{Z}^n = A'\mathbb{Z}^n$, then

$$|\det A| = |\det A'|$$

Def: for a lattice L, we define the covol$(L) := |\det A|$ where $A \in GL_n(\mathbb{R})$ s.t $L = A\mathbb{Z}^n$

(by the last claim, this is independent on the choice of A.)

$= \mathrm{vol}(A([0,1)^n))$

calculus

so define $A([0,1)^n) :=$ The fundemental parallelpiped Associated to A

and get $\mathrm{covol}(L) = \mathrm{vol}(A([0,1)^n))$.

more generally, if $\Omega \subset \mathbb{R}^n$ is a Borel set s.t $\mathbb{R}^n = \bigcupdot_{\vec{l} \in L} \vec{l} + \Omega$ so we say that Ω is a Fundemental Domain of L.

in terms of additive groups point of view $L < \mathbb{R}^n$, and so Ω is a collection of representatives of $\mathbb{R}^n / L$.

(Picture 2)

claim: fundemental Parallelpiped is a fundemental domain.

indeed, if $\vec{x} \in \mathbb{R}^n$ then can write

$A^{-1}\vec{x} = \vec{l} + \vec{y}$ for unique $\vec{z} \in \mathbb{Z}^n$, $\vec{y} \in [0,1)^n$

$$\Rightarrow \vec{x} = \underbrace{A(\vec{z})}_{\in L} + \underbrace{A(\vec{y})}_{\in A([0,1)^n)}$$

by uniqueness at the start, get that $\mathbb{R}^n = \bigcupdot_{\vec{l} \in L} \vec{l} + A([0,1)^n)$

claim: Vol-invariance extends from fundamental-parallelepipeds to fundamental domains. i.e, if Ω_1, Ω_2 are two fundamental domains of L, then $vol(\Omega_1) = vol(\Omega_2)$.

[true for the general case of Ω fund-dom of Γ where $\Gamma < G$ is a discrete sub-group, G is a nice-enough topological-group, vol is Haar-measure on G]

$S \subseteq \mathbb{R}^n$ is called discrete if $\forall \vec{s} \in S$, $\vec{s}$ is not a limit point for other points of S, i.e - the topology inherited on S is discrete.

it is called Additive if $\forall \vec{x}, \vec{y} \in S$ we have $\vec{x} + \vec{y} \in S$.

Theorem TFAE

(1) $S \subset \mathbb{R}^n$ is (i) S is an additive sub-group
(ii) S is discrete
(iii) S contains a basis of $\mathbb{R}^n$

(2) $S \subset \mathbb{R}^n$ is a lattice.

Example of use:

$$L_1 = \left\{ \vec{v} \in \mathbb{Z}^8 \;\middle|\; \sum_{j=1}^{8} v_j \equiv 0 \mod 4 \right\}$$

$$L_2 = \mathrm{span}_{\mathbb{Z}} \left\{ 2\vec{e}_1, 2\vec{e}_2, 2\vec{e}_3, 2\vec{e}_4, \sum_{j=1}^{4} \vec{e}_j \right\} \subset \mathbb{R}^4$$

both are lattices. Hard to find a basis, easy to varify (i), (ii), (iii).

Summary

- we have a good way of deciding weather $S \subset \mathbb{R}^n$ is a lattice.
- each lattice has a geometric paramater attachd to it – the volume of those sets one can fill the space with, using translations of L-elements.

<u>Question</u> given a lattice L and $K \subset \mathbb{R}^n$, when can we know $K \cap L \neq \phi$?

<u>Def:</u> $K \subset \mathbb{R}^n$ is centrally simmetric convex body (cscb) if (i) cs $\Rightarrow$ $K = -K$

(ii) cb $\Rightarrow$ convex & non-empty interior

Minkwoski's 1^{st} Thm

$L \subset \mathbb{R}^n$, K cscb. so

$$2^n \operatorname{covol}(L) < \operatorname{vol}(K) \Rightarrow K \cap L \neq \{\vec{0}\}.$$

<u>remark:</u> this is sharp. take $L = \mathbb{Z}^n$, $K = (-1,1)^n$

<u>Strengthening</u>: if K is also <u>compact</u> then

$$2^n \operatorname{covol}(L) \leq \operatorname{vol}(K) \Rightarrow K \cap L \neq \{\vec{0}\}$$

Application in diophantine approximations

Generalised Dirichlet Thm

$\|\cdot\|$ norm on $\mathbb{R}^n$. so for all $\vec{x} \in \mathbb{R}^n$ $\forall T \in \mathbb{R}$ (large enough)

$\exists (q, \vec{p}) \in \mathbb{N} \times \mathbb{Z}^n$ with $1 \leq q \leq T$ and

$$\| q\vec{x} - \vec{p} \| \leq \frac{2\left(vol\left(B_{\|\cdot\|}(\vec{0},1)\right)\right)^{\frac{-1}{n}}}{\sqrt[n]{T}} = \frac{C}{\sqrt[n]{T}}$$

Proof define a cylinder in $\mathbb{R}^{n+1}$

$$K := \left\{ (\vec{y}, t) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} |t| \leq T \\ \|t\vec{x} - \vec{y}\| \leq \frac{C}{\sqrt[n]{T}} \end{array} \right\}$$

in $\mathbb{R}^n$

K is cscb and compact.

$$Vol(K) = 2T \cdot \frac{C^n}{T} \cdot vol\left(B(\vec{0},1)\right) = 2^{n+1}$$

$\Rightarrow K \cap \mathbb{Z}^{n+1} \neq \{\vec{0}\}$ since $covol(\mathbb{Z}^{n+1}) = 1$

if T is large enough (in terms of $\|\cdot\|$ and n)

get that $\exists (q, \vec{p}) \in \mathbb{N} \times \mathbb{Z}^n$ s.t the statement holds.

remark: for $\|\cdot\|_{max}$ this gives a sharp result (cant improve $C=1$), not true for general norm.

For $n=2$, $\|\cdot\|_2$ get $C = \frac{2}{\sqrt{\pi}}$.

best possible is $\sqrt{\frac{2}{\sqrt{3}}} = \sqrt{\text{hermite constant of } \|\cdot\|_2 \ n=2}$

(generally open question what is C)

for $\|\cdot\|_2$ known for $1 \leq n \leq 8$ or $n=24$

so we know that if K is large w.r.t to L, K contains a point of L.

what about the opposite? - Fix $K = \overline{B_{\|\cdot\|}(\vec{0}, 1)}$ to get info on L.

define $\lambda_1 = \lambda_1(\|\cdot\|, L) = \inf\{\|\vec{v}\| \mid 0 \neq \vec{v} \in L\}$

we will see shortly $\leftarrow$ $= \min\{ \quad " \quad \}$

choose $c > 0$ s.t $2^n \cdot \mathrm{covol}(cL) = \mathrm{vol}(\overline{B(\vec{0},1)})$

$$\left|\det\begin{pmatrix} c & & \\ & \ddots & \\ & & c\end{pmatrix} A\right| = c^n \cdot \mathrm{covol}(L)$$

$\lambda_1(cL) \leq 1$ (by minkowski 1st)

$$\lambda_1(L) \leq \frac{1}{c} = \sqrt[n]{\frac{\mathrm{covol}(L) \cdot 2^n}{\mathrm{vol}(B(\vec{0},1))}}$$

if $\mathrm{covol}(L) = 1$ we get $\lambda_1(L) < \frac{2}{\sqrt[n]{\mathrm{vol}(B(\vec{0},1))}}$

$\Rightarrow$ shortest vector cannot be too large.

<u>Def</u>: $L \subset \mathbb{R}^n$ is a lattice. For $j \in \mathbb{N}$ $1 \leq j \leq n$ define Minkowski's j^{th}-successive-minima

$$\lambda_j = \lambda_j(\|\cdot\|, L) = \inf\left\{\lambda > 0 \;\middle|\; L \cap \overline{B(\vec{0}, \lambda)} \text{ contains at least } j \text{ lin-ind vectors}\right\}$$

claim $\rightarrow$ $= \min\{\lambda > 0 \mid \quad " \quad \}$

(picture 3)

<u>Proof</u>: $\lambda_1 > 0$ by discreteness.

$$\forall \vec{x} \neq \vec{y} \in L \quad \|\vec{x}-\vec{y}\| \geq \lambda_1 > 0$$

since distances are bounded below, each Ball can contain only finitely many points of L.

so pick shortest option in $B(\vec{0}, 2\lambda_j)$

(must have at least j lin-ind by definition of λ_j)

<u>remark 1</u>: $\{\vec{u}_j\}_{j=1}^{n}$ realising $\{\lambda_j\}$ not necesserly a basis of L!

$L_2 = \mathrm{span}_{\mathbb{Z}}\{2\vec{e}_1, 2\vec{e}_2, 2\vec{e}_3, 2\vec{e}_4, (1,1,1,1)\}$ with $\|\cdot\|_2$

easy to see $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 2$ with $2\vec{e}_i$ as a realiser, but this is not a basis for the lattice.

<u>remark 2</u>: can have many realisers.

$$L_1 = \left\{ \vec{v} \in \mathbb{Z}^8 \mid \sum_1^8 v_i = 0 \bmod 4 \right\}$$

$\Rightarrow$ has 240 vectors realising λ_1

<u>remark 3</u>: how to find realisers?

$\Rightarrow$ greedy algorithm works.

Pick $\vec{u}_1$ s.t $\|\vec{u}_1\| = \lambda_1$.

after cohoosing $\vec{u}_1, \dots, \vec{u}_j$ choose $\vec{u}_{j+1}$ s.t $\vec{u}_{j+1}$ is shortest vector in

$$\left\{ \vec{w} \in L \;\middle|\; \begin{array}{c} \vec{u}_1, \dots, \vec{u}_j, \vec{w} \\ \text{are lin-ind} \end{array} \right\}$$

Question 4: if we have a basis $\{\vec{v}_j\}_{j=1}^n$ of L, can we relate $\|\vec{v}_j\|$ and λ_j?

$\Rightarrow$ not for all basis, but we can find Korkin-Zolotarev basis - this basis has $\frac{1}{C}\cdot\lambda_j \leq \|\vec{v}_j\| \leq C\cdot\lambda_j$

(if $\|\cdot\| = \|\cdot\|_2$, can show $C = \frac{\sqrt{j+3}}{2}$)

Question 5: can we do something like Minkowski's 1st theorem? - can we bound $\lambda_1, \lambda_2, \dots \lambda_j$ in terms of $\mathrm{covol}(L)$?

$$\Downarrow$$

Minkowski 2nd thm

$$\frac{1}{n!}\cdot\frac{2^n\cdot \mathrm{covol}(L)}{\mathrm{Vol}(B(\vec{0},1))} \leq \prod_{j=1}^{n}\lambda_j(L) \leq \frac{2^n\cdot \mathrm{covol}(L)}{\mathrm{Vol}(B(\vec{0},1))}$$

remark: this is sharp for some norms.

$L=\mathbb{Z}^n$, $\|\cdot\|_\infty$ get equality on RHS

$L=\mathbb{Z}^n$, $\|\cdot\|_1$ get equality on LHS

(open question: given $\|\cdot\|$, does there exist $L\subset\mathbb{R}^n$ to have equality in Mink 2nd?)

(from now on $\|\cdot\| = \|\cdot\|_2$)

useful example from now on (picture 4)

$\Lambda = \mathrm{span}_{\mathbb{Z}}\{c\vec{w}, c\vec{e_1}\}$
where $c \simeq 1.07$ s.t
$\mathrm{covol}(\Lambda) = 1$

<u>motivation</u>: find other parameters beside λ_j which gives information on the geometry of the lattice

if $L_0 < L$ then $\exists$ <u>lin-ind</u> $\vec{u}_1, \vec{u}_2, \dots \vec{u}_j \in L$
$(1 \le j \le n)$ s.t $L_0 = \mathrm{span}_{\mathbb{Z}}\{\vec{u}_1, \dots, \vec{u}_j\}$.
in this case we say L_0 is a <u>sub-lattice of rank j</u>. (if $j<n$, not a lattice in $\mathbb{R}^n$!)

write $\mathrm{span}_{\mathbb{R}}\{\vec{u}_1, \dots, \vec{u}_j\} = V$ and fix $\ell : V \xrightarrow{\sim} \mathbb{R}^j$
s.t $\mathrm{Vol}_{\mathbb{R}^j}\left(\ell\left(\left\{\sum_{i=1}^{j} a_i \vec{z}_i \;\middle|\; \begin{array}{l} a_i \in [0,1) \\ \{\vec{z}_i\} \text{ orthonormal basis of } V \end{array}\right\}\right)\right) = 1$

for all $S \subset V$ define <u>$\mu_V(S)$</u> $:= \mathrm{vol}_{\mathbb{R}^j}(\ell(S))$.

using this, for $L_0 < L$ define <u>$\mathrm{Covol}(L_0)$</u> $:= \mu_V(\Omega)$
where Ω is fundemental domain for L_0.
(i.e Ω is measurable & choice of representatives of V/L_0)

<u>Example</u>: take $L_0 = \mathrm{span}_{\mathbb{Z}}\{2c\vec{w}\}$.
so $\mathrm{covol}(L_0) = \mathrm{length}(2c\vec{w}) = 2c$

(Picture 5)

for $1 \leq j \leq n$ we say $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_j\} \subset L$
is a primitive set.
if $\exists\, \vec{u}_{j+1}, \dots, \vec{u}_n \in L$ which completes
$\vec{u}_1, \dots, \vec{u}_j$ into a basis of L

claim $\Longleftrightarrow$ $\mathrm{span}_{\mathbb{R}}\{\vec{u}_1, \dots, \vec{u}_j\} \cap L = L_0$

$$\left(\begin{array}{l} \text{for } \Rightarrow \\ \text{write } \vec{x} = \sum_1^n a_i \vec{u}_i = \sum_1^j b_i \vec{u}_i \quad b_i \in \mathbb{R},\ a_i \in \mathbb{Z} \\ \text{by lin-ind, } b_i \in \mathbb{Z}. \end{array} \right)$$

if $L_0 = \mathrm{span}_{\mathbb{Z}}\{\vec{u}_1, \dots, \vec{u}_j\}$ and $\{\vec{u}_1, \dots, \vec{u}_j\}$
is a primitive set, we say L_0 is
a primitive sub-lattice of rank j.

Example: $L_0 = \mathrm{span}_{\mathbb{Z}}\{2 \vec{w}\}$ not-primitive.
(Picture 5)

For $L \subset \mathbb{R}^n$ lattice define
$$\alpha_j(L) := \inf \left\{ \mathrm{covol}(L_0) \;\middle|\; \begin{array}{l} L_0 \text{ is sub-lattice} \\ \text{of rank } j \end{array} \right\}$$
claim $= \min \{ \quad \text{"} \quad \text{"} \quad \}$

"Proof": assume $\{L_k\}_{k=1}^{\infty}$ sub-lattices of rank j
s.t $\mathrm{covol}(L_k) \xrightarrow{k \to \infty} \alpha_j(L)$.

$\forall k$, Pick KZ-basis of L_k, i.e $\|\vec{v}_i^k\| \asymp \lambda_i(L_k)$
for $i \in \{1, \dots, j\}$. using Minkowski's 2nd, get $\exists C > 0$
$\forall i\ \forall k \quad \lambda_1(L) \leq \|\vec{v}_i^k\| \leq C$
using properties of L get that $\exists$ sub-sequence
s.t $\forall i$ $\vec{v}_i^{k_m} \xrightarrow{m \to \infty} \vec{v}_i^{\infty} \in L$ and that
$\vec{v}_1^{\infty}, \dots \vec{v}_j^{\infty}$ are lin-ind. define $L_\infty = \mathrm{span}_{\mathbb{Z}}(\vec{v}_1^{\infty}, \dots, \vec{v}_j^{\infty})$
and get $\mathrm{covol}(L_\infty) = \lim_{m \to \infty} \mathrm{covol}(L_{k_m}) = \alpha_j(L)$

claim: if $\alpha_J(L) = \mathrm{covol}(L_0)$ for some $L_0 < L$ of rank J, then L_0 is primitive.

Proof: assume $L_0 \subset L$ not primitive

define $L_1 = \mathrm{span}_{\mathbb{R}} L_0 \cap L$ (Primitive)

$$L_0 \subseteq L_1 \Rightarrow \mathrm{covol}(L_1) \leq \mathrm{covol}(L_0)$$

So can look only on primitives

what do we know about the size of $\alpha_J(L)$?

claim $\exists C > 0$ s.t $\forall L$ $\forall J \in \{1, .., n\}$

$$C \prod_{i=1}^{J} \lambda_i(L) \leq \alpha_J(L) \leq \prod_{i=1}^{J} \lambda_i(L)$$

"Proof": if $\vec{u}_1, \vec{u}_2, .. \vec{u}_J$ realises $\lambda_1, \dots, \lambda_J$ and $L_0 = \mathrm{span}_{\mathbb{Z}}\{\vec{u}_1, .., \vec{u}_J\}$ So

$$\alpha_J \leq \mathrm{covol}(L_0) \leq \prod_{i=1}^{J} \|\vec{u}_i\| = \prod_{i=1}^{J} \lambda_J$$

def of $\alpha_J(L)$

parallelpiped vol. get equality if angles are right, otherwise get $\leq$

on the other side, suppose $L_0 = \mathrm{span}_{\mathbb{Z}}\{\vec{u}_1, .., \vec{u}_J\}$

s.t $\frac{1}{2}\mathrm{covol}(L_0) < \alpha_J(L)$

(exist such by definition of $\alpha_J(L)$)

$$\alpha_J(L) > \frac{1}{2}\mathrm{covol}(L_0) \geq C \prod_{i=1}^{J} \lambda_i(L_0) \geq C \prod_{i=1}^{J} \lambda_i(L)$$

get such C from minkowski 2nd for J-dim

$L_0 \leq L$

* an example of claculation:

define $L \coloneqq \Lambda \oplus \mathbb{Z}\vec{e}_3 \subset \mathbb{R}^3$

(Picture 6)

always $\left[\begin{array}{lll} \lambda_1 = 1 & \text{relised by} & \vec{e}_3 \\ \alpha_1 = 1 & \text{relised by} & \operatorname{span}_{\mathbb{Z}}(\vec{e}_3) \end{array}\right.$

$\lambda_2 = c$ realised by $\{c\vec{e}_1, c\vec{w}\}$

$\alpha_2 = 1$ realised by Λ

("Proof" by geometric intuition)

Problems with λ_j, α_j

1) ∘ not easy to find
2) ∘ can have many realisers.

For example if $L = \mathbb{Z}^n$, $\alpha_j(L)$ have at least $\binom{n}{j}$ realisers.

Partial solution for 2 – Harder Narasimhan Filtration.

Harder-Narasimhan Filtration

assume we know $\alpha_J = \mathrm{covol}(L_J)$

for some sub-lattice of rank J, Picture 7

for $0 \le J \le n$. $\boxed{\text{convention } \alpha_0(L) = 1}$

<u>step 1</u>: draw in plain the figure

$$A^{\geq}(L) := \left\{ (J, y) \;\middle|\; \underset{J \in \mathbb{N}}{0 \le J \le n}, \; y \geq \log(\alpha_J) \right\}$$

$\alpha_J = \mathrm{covol}(L_J)$

= union of vertical lines

<u>step 2</u>: take convex hull of $A^{\geq}(L)$

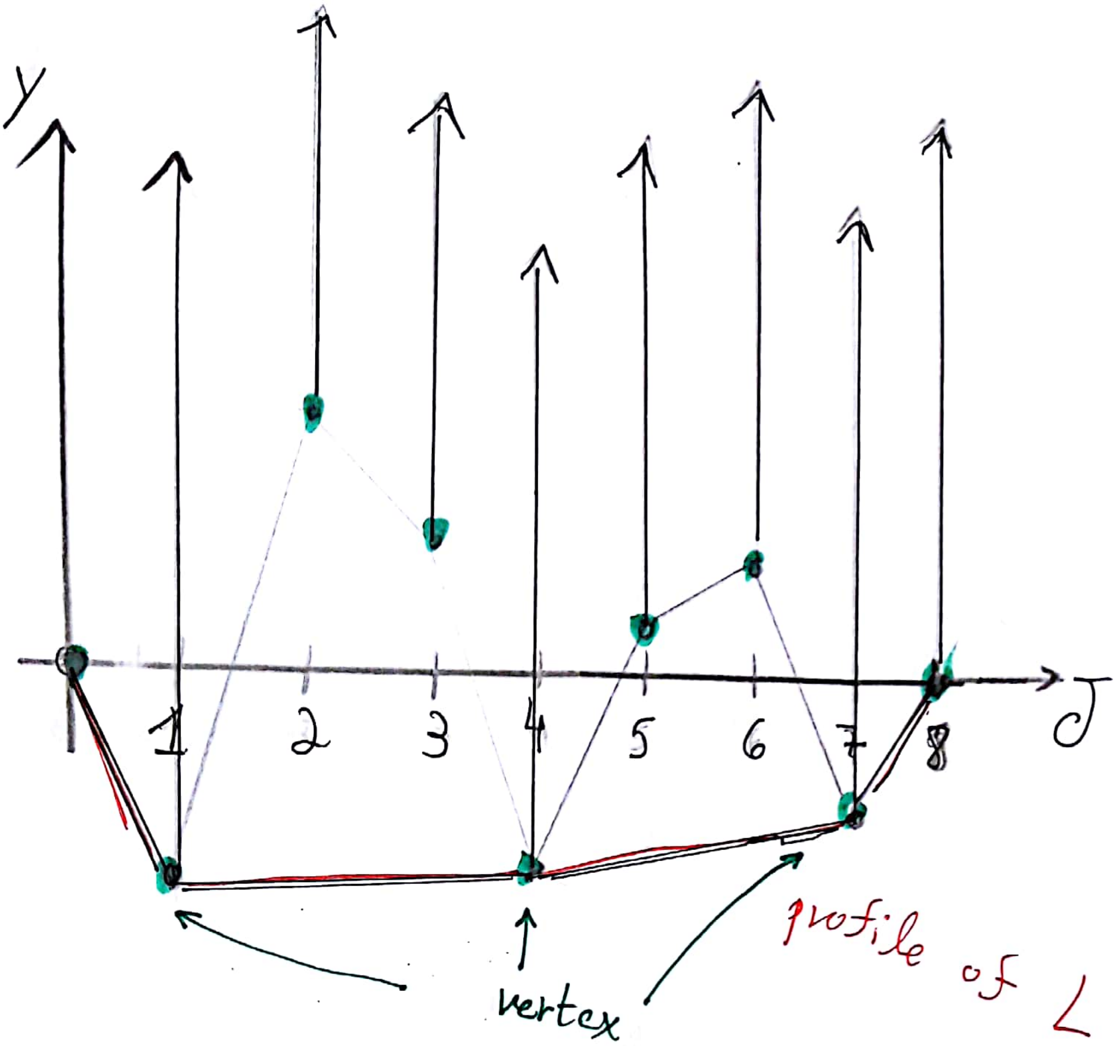
i.e., $\mathrm{Conv}\, A^{\geq}$ = minimal convex set containing $A^{\geq}(L)$

<u>step 3</u>: define <u>Profile of L</u>

to be the bottom polygonal line of $\mathrm{conv}(A^{\geq})$, and mark all vertices which form a $\boxed{\text{vertex}}$!!

<u>on Profile(L).</u>

pictures 7 + 8



remark: for all $L_0 \subset L$ of rank j,

the point $\left(j, \frac{\log \operatorname{covol}(L_j)}{j}\right)$ is above

the profile of L.

by definition of φ!

Thm – HN filtration (1,2,3,4,5)

① if $(J, \log \operatorname{covol}(L_J)) \in$ Vertices

then L_J is unique. i.e - if $L_J' \subset L$

is sub-lattice of rank J, so

$$\operatorname{covol}(L_J) < \operatorname{covol}(L_J')$$

② the sub-lattices for which $(J, \operatorname{covol}(L_J))$

is a vertex are nested. i.e

$$(*) \quad \{0\} \subset \underbrace{L_{J_1} \subset L_{J_2} \subset \dots L_{J_k}}_{k \text{ intersting ones}} \subset L_{J_{k+1}} = L$$

$$0 \leq k \leq n-1$$

$$1 \leq J_i \leq n-1$$

$(*)$ is called the Harder–Narasimhan Filtration of L and we say it has length k.

say it is stable if $k=0$.

3 For each $i \in \{1, \ldots, k\}$

$\{0\} = L_0 \subset L_{j_1} \subset \ldots \subset L_{\boxed{j_i}}$

is the HN-filtration of L_{j_i}

Pf: proof uses heavily the following claim:

if $L_1, L_2 \subset L$ primitive sub-lattices

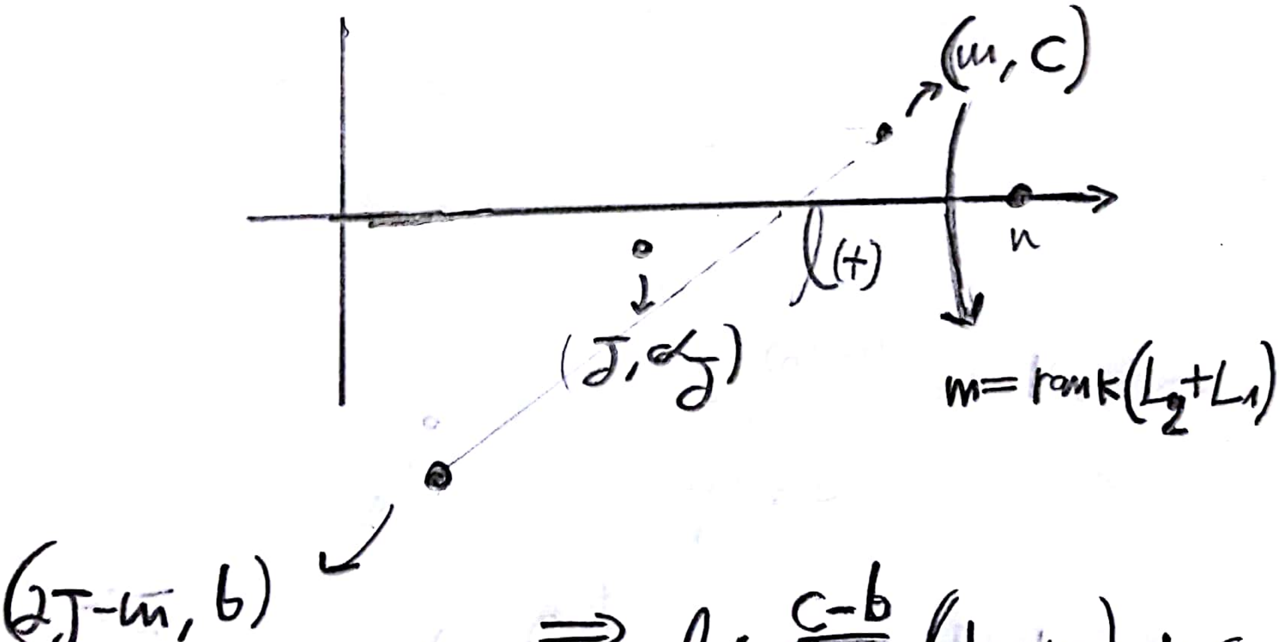
(a) always true, not only for primtives $\mathrm{rank}(L_1 \cap L_2) + \mathrm{rank}(L_1 + L_2) = \mathrm{rank}(L_1) + \mathrm{rank}(L_2)$

(b) $\mathrm{covol}(L_1 \cap L_2) \cdot \mathrm{covol}(L_1 + L_2) \leq \mathrm{covol}(L_1) \cdot \mathrm{covol}(L_2)$

$$\Rightarrow \underbrace{\log(\mathrm{Co}(\cap))}_{b} + \underbrace{\log(\mathrm{Co}(+))}_{c} \leq \log(\underset{\alpha_J}{C(L_1)}) + \log(\underset{\alpha_J}{C(L_2)}) \quad (\ast)$$

proof of 1

assume we have two primitive sub-lattices L_1, L_2 s.t $\mathrm{covol}(L_1) = \mathrm{covol}(L_2) = \alpha_J$ and α_J is a vertex of the profile



$$\Rightarrow l: \frac{c-b}{2m-2J}(t-m) + c$$

$$l(j) = \frac{c-b}{2(m-j)}(j-m) + c$$

$$= \frac{b-c}{2} + c = \frac{b+c}{2} \leq d_j$$

violating the fact that $d_j < l(j)$

this is for the case $m > j$.

assume $m = j$

$\Rightarrow \operatorname{rank}(L_1 \cap L_2) = \operatorname{rank}(L_1)$.

L_1, L_2 are primitive, so

cannot have $L_1 \cap L_2 \subset L_1$

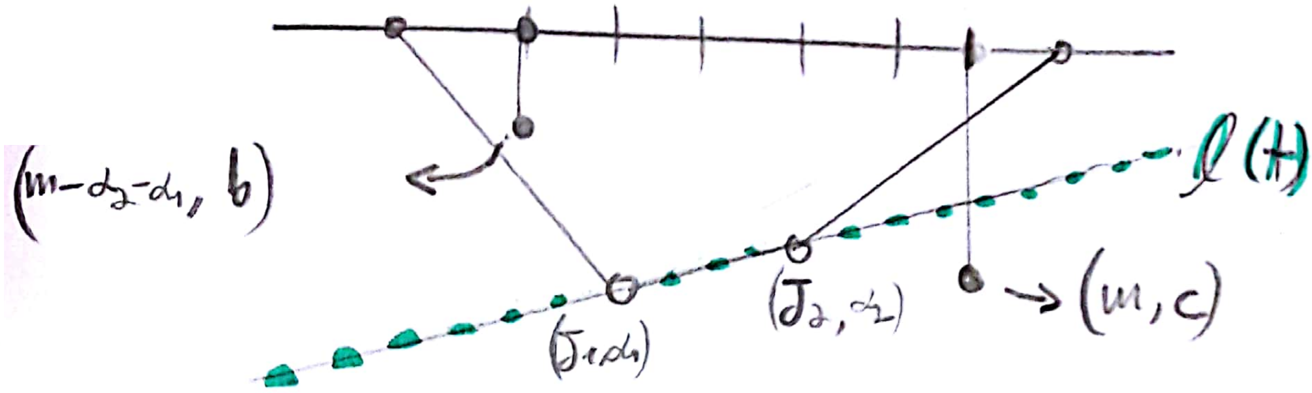
if have same rank

$\Rightarrow L_1 \cap L_2 = L_1$.

$\Rightarrow L_1 \cap L_2 = L_2$ (same argument)

$\Rightarrow L_1 = L_2$.

Pf 2:



assume by contradiction $L_1 \not\subseteq L_2$.

$$(L_1 \cap L_2) \neq L_1$$

$$\Rightarrow \quad \operatorname{rank}(L_1 \cap L_2) < \operatorname{rank} L_1$$

$$\operatorname{rank} L_2 < \operatorname{rank}(L_1 + L_2)$$

writing the equation $\ell(t)$ and using again that $b+c < d_1+d_2$ we get that at least one of point lies <u>beneath</u> $\ell(t)$, contradicting the fact it is in the profile of L.

$\square$

$L_J < L$ primitive subgroup of rank J.

$$\underbrace{\underbrace{\vec{v}_1, \dots \vec{v}_J}_{\text{basis of } L_J}, \vec{v}_{J+1}, \dots \vec{v}_n}_{\text{basis of } L}$$

$$\pi_J : \mathbb{R}^n \xrightarrow{\text{proj}} \left(\mathrm{Span}_{\mathbb{R}}(L_J)\right)^{\perp} \cong \mathbb{R}^{n-J}$$

* $\pi_J(L) = \mathrm{span}_{\mathbb{Z}}\left(\pi_J(\vec{v}_{J+1}), \dots, \pi_J(\vec{v}_n)\right)$

* claim : $\pi_J(L)$ is discrete in $\left(\mathrm{Span}_{\mathbb{R}}(L_J)\right)^{\perp}$

$\Rightarrow$ $\pi_J(L)$ is an $n-J$ lattice in $\mathrm{Span}_{\mathbb{R}}(L_J)^{\perp}$

[short disucssion Prior to results 4.5]

$$\{0\} \subset L_{J_1} \subset L_{J_2} \subset \dots \subset L_{J_k} \subset L \quad \text{(reminder)}$$

4 $\forall i \in \{1, \dots, k\}$ define π_{J_i} as before

so

$$\{0\} \subset \pi_{J_i}(L_{J_{i+1}}) \subset \dots \pi_{J_i}(L_k) \subset \pi_{J_i}(L)$$

is the HN-filtration of $\pi_{J_i}(L)$.

5 for all $i \in \{1, \dots, k\}$

$\pi_{J_i}(L_{J_{i+1}})$ is stable.