\[ L = \text{span}_Z \left( \overrightarrow{e_1}, \overrightarrow{w} \right) \]
\[ \overrightarrow{w} = (\cos \frac{\pi}{2}, \sin \frac{\pi}{2}) \]

**Picture 2**

Fundamental parallelepiped of \( \{ \overrightarrow{w}, \overrightarrow{w} - 2\overrightarrow{e_1}, \overrightarrow{w} - 3\overrightarrow{e_1} \} \)

General fundamental domain

**Picture 3**

\[ L = \text{span}_Z \left( \overrightarrow{e_1}, \frac{1}{\varepsilon} \overrightarrow{e_2} \right) \]
\[ 0 < \varepsilon < 1 \]
\[ \lambda_1 = \varepsilon \]
\[ \lambda_2 = \frac{4}{\varepsilon} \]
\( J = 0 \quad J = 1 \quad J = 2 \quad J = 3 \)

Profile of \( L \)
(no intersecting vertex)

\[ L = \lambda \oplus 2 \vec{e}_3 \]
\( \lambda_1 = \lambda_2 = 1 \)

\[ L = (1-\delta) \lambda \oplus \frac{\lambda}{(1-\delta)} \cdot 2 \vec{e}_3 \]
\( \lambda_1 = \frac{\lambda}{1-\delta} > 1 \)
\( \lambda_2 < 1 \)

\[ L = \frac{\lambda}{(1-\delta)} \lambda \oplus (1-\delta) \cdot 2 \vec{e}_3 \]
\( \lambda_1 = 1-\delta < 1 \)
\( \lambda_2 > 1 \)
Crash course on Lattices

based on lectures 1-5 from "Geometry of numbers" course given by Barak Weiss, fall 2020

**Definition:** \( \mathbb{L} \subset \mathbb{R}^n \) is called a lattice if there exists \( \vec{v}_1, \vec{v}_2, ..., \vec{v}_n \) linearly-independent vectors such that
\[
\mathbb{L} = \left\{ \sum_{j=1}^{n} a_j \vec{v}_j : a_j \in \mathbb{Z} \right\} = \text{span}_\mathbb{Z} \{ \vec{v}_1, ..., \vec{v}_n \}.
\]

The set \( \{ \vec{v}_1, ..., \vec{v}_n \} \) will be called a basis for \( \mathbb{L} \).

**Observation:** \( \mathbb{L} \) is a lattice if and only if \( \exists A \in \text{GL}_n(\mathbb{R}) \) such that \( \mathbb{L} = A \mathbb{Z}^n = \text{span}_\mathbb{Z} \{ A\vec{v}_1, ..., A\vec{v}_n \} \). In this case, we say that \( A \) is a basis of \( \mathbb{L} \).

Define \( \text{GL}_n(\mathbb{Z}) := \{ B \in \text{M}_n(\mathbb{Z}) \mid \det B = \pm 1 \} \)

and notice this is a group, i.e., if \( B \in \text{GL}_n(\mathbb{Z}) \) then \( B^{-1} \in \text{GL}_n(\mathbb{Z}) \). \( B^{-1} = (\det B)^{-1} \cdot C^t \), \( C \) is a co-Sapthors matrix \( \in \text{M}_n(\mathbb{Z}) \).

**Claim:** Let \( A, \tilde{A} \in \text{GL}_n(\mathbb{R}) \). Then
\[
\mathbb{L} = A \mathbb{Z}^n = \tilde{A} \mathbb{Z}^n \iff \exists B \in \text{GL}_n(\mathbb{Z}) \text{ s.t. } A = \tilde{A} B
\]

particularly, if \( \mathbb{L} = A \mathbb{Z}^n = \tilde{A} \mathbb{Z}^n \), then
\[
| \det A | = | \det \tilde{A} |
Def: for a lattice \( L \), we define the
\[
\operatorname{covol}(L) := |\det A| \quad \text{where} \quad A \in \text{GL}_n(\mathbb{R}) \quad \text{st} \quad L = A\mathbb{Z}^n
\]
(by the last claim, this is independent of the choice of \( A \)).

\[
= \operatorname{vol} \left( A([0,1]^n) \right)
\]
calculating so define \( A([0,1]^n) := \text{The fundamental parallelipiped associated to } A \)
and get \( \operatorname{covol}(L) = \operatorname{vol} \left( A([0,1]^n) \right) \).

more generally, if \( L \subset \mathbb{R}^n \) is a Borel set s.t.
\[
\mathbb{R}^n = \bigcup_{\ell \in L} \ell + \mathbb{Z}^n
\]
so we say that \( L \) is
\[
\text{a fundamental domain of } L.
\]
in terms of additive groups point of view
\( L \subset \mathbb{R}^n \), and so \( L \) is a collection of representatives of \( \mathbb{R}^n/L \).

*Picture 2*

Claim: fundamental parallelipiped
is a fundamental domain.
Indeed, if \( x \in \mathbb{R}^n \) then can write
\[
A^{-1} x = \ell + y \quad \text{for unique } \ell \in \mathbb{Z}^n, \ y \in [0,1]^n
\]
\[
\Rightarrow \quad x = A(\ell) + A(y)
\]
\[
L = A([0,1]^n)
\]
by uniqueness at the start, get
\[
\mathbb{R}^n = \bigcup_{\ell \in L} \ell + A([0,1]^n)
\]
Claim: Volume invariance extends from fundamental parallelepipeds to fundamental domains, i.e., if \( \Lambda_1, \Lambda_2 \) are two fundamental domains of \( L \), then \( \text{vol}(\Lambda_1) = \text{vol}(\Lambda_2) \).

This is true for the general case of \( L \)
sub-group of \( G \) where \( \Gamma \) is a discrete sub-group, \( G \) is a nice-enough topological group, \( \text{vol} \) is Haar-measure on \( G \).

\( S \subseteq \mathbb{R}^n \) is called **discrete** if \( \forall \mathbf{s} \in S, \mathbf{s} \neq \mathbf{0} \) is not a limit point for other points of \( S \), i.e., the topology inherited on \( S \) is discrete. It is called **additive** if \( \forall \mathbf{x}, \mathbf{y} \in S \) we have \( \mathbf{x} + \mathbf{y} \in S \).

**Theorem:** TFAE

1. \( S \subseteq \mathbb{R}^n \) is
2. \( S \) is an additive sub-group
3. \( S \) is discrete
4. \( S \) contains a basis of \( \mathbb{R}^n \)

Example of use:

\[
L_1 = \left\{ \mathbf{v} \in \mathbb{Z}^3 \mid \sum_{j=1}^{3} v_j = 0 \text{ mod } 4 \right\}
\]

\[
L_2 = \text{span}_\mathbb{Z} \left\{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \sum_{j=1}^{4} \mathbf{e}_j \right\} \subseteq \mathbb{R}^4
\]

Both are lattices. Hard to find a basis, easy to verify (i), (ii), (iii).
Summary

- We have a good way of deciding whether $\mathbb{C} \subset \mathbb{R}^n$ is a lattice.
- Each lattice has a geometric parameter attached to it - the volume of those sets one can fill the space with, using translations of L-elements.

Question: Given a lattice $L$ and $K \subset \mathbb{R}^n$, when can we know $KL \neq \emptyset$?

Def: $K \subset \mathbb{R}^n$ is centrally symmetric convex bdy (CS) if
1. $CS \Rightarrow K = -K$
2. $Cb \Rightarrow$ convex & non-empty interior

**Minkowski's 1st Thm**

$L \subset \mathbb{R}^n$, $K \subset Cs$. So

$$2^n \text{ conv}(L) \leq \text{ vol}(K) \Rightarrow KL \neq \{0\}.$$ 

Remark: This is sharp. Take $L = \mathbb{Z}^n$, $K = (-\frac{1}{2}, \frac{1}{2})^n$

Strengthening: If $K$ is also compact then

$$2^n \text{ conv}(L) \leq \text{ vol}(K) \Rightarrow KL \neq \{0\}$$
Application in diophantine approximations

Generalised Dirichlet Thm

$||.||$ norm on $\mathbb{R}^n$. so for all $x \in \mathbb{R}^n$ $\forall T \in \mathbb{R}$ (large) $\exists (q, \tilde{\nu}) \in N \times \mathbb{Z}^n$ with $1 \leq q \leq T$ and

$$||q_{\tilde{\nu}} - \tilde{\nu}|| \leq \frac{2}{\sqrt{T}} \left( \frac{\text{vol}(B_{||.||}(\tilde{\nu}, 1))}{\sqrt{T}} \right)^{\frac{1}{n}} = \frac{C}{\sqrt{T}}$$

Proof: define a cylinder in $\mathbb{R}^{n+1}$

$$K := \left\{ (\tilde{y}, t) \in \mathbb{R}^{n+1} \left| \begin{array}{c} 1 \leq t \leq T \\ ||t\tilde{x} - \tilde{y}|| \leq \frac{C}{\sqrt{T}} \end{array} \right. \right\}$$

$k$ is $cscb$ and compact,

$$\text{Vol}(k) = 2T \cdot \frac{C^n}{T} \cdot \text{vol}(B(0, 1)) = 2^{n+1}$$

$$\Rightarrow k \cap \mathbb{Z}^{n+1} \neq \{\tilde{\nu}\} \text{ since } \text{vol}(\mathbb{Z}^{n+1}) = 1$$

If $T$ is large enough (in terms of $||.||$ and $n$) get that $\exists (q, \tilde{\nu}) \in N \times \mathbb{Z}^n$ sat the statement holds.

Remark: for $||.||_{\text{max}}$ this gives a sharp result (can't improve $C = 1$), not true for general norms.

For $n=2$, $||.||_2$ get $C = \frac{2}{\sqrt{\pi}}$.

Best possible is $\sqrt{\frac{2}{\sqrt{3}}} = \sqrt{\text{hemite constant}}$ of $||.||_2$, $n=2$.

(Generally open question what is $C$)

For $||.||_2$, known for $1 \leq n \leq 8$ or $n=24$.
so we know that if $K$ is large w.r.t. to $L$, $K$ containing a point of $L$.

What about the opposite? - Fix $K = B(0, 1)$ to get info on $L$.

Define $\lambda_1 = \lambda_1(1\cdot l, L) = \inf \left\{ \|v\| \mid \text{for some } v \in L \right\}$

we will see shortly $\lambda_1 = \min \left\{ \frac{1}{C} \right\}$

Choose $C > 0$ s.t. $2^n \cdot \text{covol}(c L) = \text{vol}(B(0, 1))$

$|\det (c \cdot c^t) A| = C^n \cdot \text{covol}(L)$

$\lambda_1(cL) \leq 1$ (by Minkowski 1st)

$\lambda_1(L) \leq \frac{1}{C} = \sqrt{\frac{\text{covol}(L) \cdot 2^n}{\text{vol}(B(0, 1))}}$

If $\text{covol}(L) = 1$ we get $\lambda_1(L) < \frac{2}{\sqrt{\text{vol}(B(0, 1))}}$

$\Rightarrow$ shortest vector cannot be too large.

**Def:** $L \subset \mathbb{R}^n$ is a lattice. For $j \in \mathbb{N}$ define Minkowski's $j$-th successive minima

$\lambda_j = \lambda_j(1\cdot l, L) = \inf \left\{ \lambda > 0 \mid L \cap B(\bar{0}, \lambda) \text{ contains at least } j \text{ linearly independent vectors} \right\}$

$\lambda_j = \min \left\{ \lambda > 0 \mid \right\}$

claim

(Picture 3)
Proof: \( \lambda_1 > 0 \) by discreteness.

\[ \forall \hat{x}, \hat{y} \in \mathbb{L}, \| \hat{x} - \hat{y} \| \geq \lambda_1 > 0 \]

since distances are bounded below.

Each \( \mathbb{L} \) can contain only finitely many points.

so pick shortest option in \( \mathbb{B}(\hat{0}, 2\lambda_1) \)

(must have at least \( J \) lin. ind. by)

Definition of \( \lambda_J \)

**remark 1**: \( \{ \hat{u}_j \}_{j=1}^n \) realising \( \{ \lambda_j \} \)

not necessarily a basis of \( \mathbb{L} \)!

\( L_2 = \text{span} \{2\hat{e}_i, 2\hat{e}_2, 2\hat{e}_3, 2\hat{e}_4, (1, 1, 1)\} \) with \( \| \cdot \|_2 \)

easy to see \( \lambda_1 - \lambda_2 = \lambda_3 - \lambda_4 = 2 \) with \( 2\hat{e}_1 \) as a realiser, but this is not a basis for the lattice.

**remark 2**: can have many realisers.

\[ L_4 = \left\{ \hat{v} \in \mathbb{Z}^4 \mid \frac{1}{4} \sum \hat{v}_i = 0 \text{ mod } 4 \right\} \]

which has 240 vectors realising \( \lambda_4 \)

**remark 3**: how to find realisers?

\( \Rightarrow \) greedy algorithm works.

Pick \( \hat{u}_4 \) satisfying \( \| \hat{u}_4 \| = \lambda_4 \).

After choosing \( \hat{u}_1, \ldots, \hat{u}_J \), choose \( \hat{u}_{J+1} \) s.t. \( \hat{u}_{J+1} \) is shortest vector in

\[ \{ \hat{w} \in \mathbb{L} \mid \hat{u}_1, \ldots, \hat{u}_J, \hat{w} \text{ are lin. ind.} \} \]
Question 4: if we have a basis \( \{ \mathbf{e}_j \}_{j=1}^n \) of \( \mathbf{L} \), can we relate \( \| \mathbf{e}_j \| \) and \( \lambda_j \)?

⇒ not for all basis, but we can find Korkin-Zolotarev basis − this basis has \( \frac{1}{2} \lambda_j \leq \| \mathbf{e}_j \| \leq C \lambda_j \)

(if \( \| \cdot \| = \| \cdot \|_2 \), can show \( C = \frac{\sqrt{3} + 3}{2} \))

Question 5: can we do something like Minkowski 1st theorem? − can we bound \( \lambda_1, \lambda_2, \ldots, \lambda_n \) in terms of \( \text{covol}(\mathbf{L}) \)?

\[ \frac{1}{n!} \cdot \frac{2^n \cdot \text{covol}(\mathbf{L})}{\text{vol}(B(0,1))} \leq \prod_{j=1}^n \lambda_j(\mathbf{L}) \leq \frac{2^n \cdot \text{covol}(\mathbf{L})}{\text{vol}(B(0,1))} \]

remark: this is sharp for some norms.

\( \mathbf{L} = \mathbb{Z}^n, \| \cdot \|_\infty \) get equality on RHS

\( \mathbf{L} = \mathbb{Z}^n, \| \cdot \|_2 \) get equality on LHS

(open question: given \( \| \cdot \| \), does there exist \( \mathbf{L} \subset \mathbb{R}^n \) to have equality in Minkowski 2nd?)
useful example

\( \Lambda = \text{span}_\mathbb{R} \{ \bar{w}, c\bar{w} \} \)
where \( c \approx 1.07 \) s.t.
\( \text{covol}(\Lambda) = 1 \)

Motivation: find other parameters beside \( \lambda_y \) which gives information on the geometry of the lattice.

If \( L_0 < L \) then \( \exists \) fin.-incl. \( \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_g \in L \)
(1 \( \leq g \leq n \)) s.t. \( L_0 = \text{span}_\mathbb{R} \{ \bar{u}_1, \ldots, \bar{u}_g \} \).

In this case we say \( L_0 \) is a sub-lattice of \( \Lambda \). (If \( g < n \), not a lattice in \( \mathbb{R}^n \).)

Write \( \text{span}_\mathbb{R} \{ \bar{u}_1, \ldots, \bar{u}_g \} = V \) and fix \( \mathcal{L} : V \overset{\cong}{\rightarrow} \mathbb{R}^g \)
s.t. \( \text{vol}_{\mathbb{R}^g}(\mathcal{L}(\{ \bar{u}_1, \ldots, \bar{u}_g \}) \text{ orthonormal basis of } V) = 1 \)
for all \( S \subset V \) define \( \text{Me}(S) = \text{vol}_{\mathbb{R}^g}(\mathcal{L}(S)) \).

Using this, for \( L_0 < L \) define \( \text{Covol}(L_0) : = \text{Me}(\Lambda) \)
where \( \Lambda \) is fundamental domain for \( L_0 \).

(i.e. \( \Lambda \) is measurable & choice of representatives of \( V/\Lambda \)

Example: take \( L_0 = \text{span}_\mathbb{R} \{ 2c\bar{w}, 3 \} \).

So \( \text{Covol}(L_0) = \text{length}(2c\bar{w}) = 2c \)
for $1 \leq j \leq n$ we say \( \{ \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_j \} \subset L \) is a **primitive set**.

is if \( \exists \bar{u}_1, \ldots, \bar{u}_n \in L \) which completes \( \bar{u}_1, \ldots, \bar{u}_n \) into a basis of \( L \).

\begin{align*}
\text{claim} \iff \text{span}_R \{ \bar{u}_1, \ldots, \bar{u}_n \} \cap L &= L_0 \\
\text{for } & \\
\text{write } \quad x = \sum_{i=1}^n a_i \bar{u}_i = \sum_{i=1}^n b_i \bar{u}_i, \quad b_i \in \mathbb{R}, \quad a_i \in \mathbb{Z} \\
\text{by lin-ind, } b_i \in \mathbb{Z}.
\end{align*}

if \( L_0 = \text{span}_Z \{ \bar{u}_1, \ldots, \bar{u}_n \} \) and \( \{ \bar{u}_1, \ldots, \bar{u}_n \} \) is a primitive set, we say \( L_0 \) is a **primitive sub-lattice of rank \( j \)**.

**Example:** \( L_0 = \text{span}_Z \{ 2 \bar{w}_1, \bar{w}_2 \} \) not primitive.

(Picture 5)

For \( L \subset \mathbb{R}^n \) lattice define

\[ \lambda_j(L) := \inf \left\{ \text{covol}(L_0) \mid L_0 \text{ is sub-lattice of rank } j \right\} \]

claim \( \lambda_j(L) = \min \left\{ \right\} \)

\[ \text{"Proof": assume } \{ L_k \} \subset \mathbb{Z}^\infty \text{ sub-lattices of rank } j \]

set \( \text{covol}(L_k) \xrightarrow{k \to \infty} \lambda_j(L) \).

\( \forall k, \text{ Pick } \mathbb{Z} \text{-basis of } L_k, \text{ i.e. } \| \bar{v}_i \| \times \lambda_j(L_k) \)

\( \forall i \in \{ 1, \ldots, j \} \). Using Minkowski's 2\text{nd}, get \( \exists C > 0 \)

\( \forall k \quad \lambda_j(L) \leq \| \bar{v}_i \| \leq C \)

Using properties of \( L \) get that \( \exists \) sub-sequence s.t. \( \forall \bar{v}_i^\infty \xrightarrow{m_0} \bar{v}_i^\infty \in L \) and that \( \bar{v}_1^\infty, \ldots, \bar{v}_n^\infty \) are lin-ind. Define \( L_0 = \text{span}_Z \{ \bar{v}_1^\infty, \ldots, \bar{v}_n^\infty \} \)

and get \( \text{covol}(L_0) = \lim_{m_0} \text{covol}(L_k) = \lambda_j(L) \).
Claim: if \( \lambda_g(l) = \text{covol}(l_0) \) for some \( l_0 \leq l \) of rank \( g \), then \( l_0 \) is primitive.

Proof: assume \( l_0 \leq l \) not primitive

Define \( l_1 = \text{span}_{l_0} \cap l \) (Primitive)

\( l_0 \subseteq l_1 \Rightarrow \text{covol}(l_1) \leq \text{covol}(l_0) \)

So can look only on primitives.

What do we know about the size of \( \lambda_g(l) \)?

Claim: \( \exists C > 0 \) s.t. \( \forall l \) \( \forall g \in \{1, \ldots, g\} \)

\[
C \frac{\prod_{i=1}^{g} \lambda_i(l)}{n} \leq \lambda_g(l) \leq \frac{\prod_{i=1}^{g} \lambda_i(l)}{n}
\]

Proof: if \( \vec{u}_i, \vec{u}_2, \ldots, \vec{u}_g \) realises \( \lambda_1, \ldots, \lambda_g \)

and \( l_0 = \text{span}_2 \{\vec{u}_1, \ldots, \vec{u}_g\} \) so

\[
\lambda_g \leq \text{covol}(l_0) \leq \prod_{i=1}^{g} ||\vec{u}_i|| = \prod_{i=1}^{g} \lambda_i
\]

def of \( \lambda_g(l) \) \quad \text{parallelipiped vol. get equality if angles are right, otherwise get } \leq

on the other side, suppose \( l_0 = \text{span}_2 \{\vec{u}_1, \ldots, \vec{u}_g\} \)

\( \text{st } \frac{1}{2} \text{covol}(l_0) < \lambda_g(l) \)

(exist such by definition of \( \lambda_g(l) \))

\[
\lambda_g(l) > \frac{1}{2} \text{covol}(l_0) \geq C \prod_{i=1}^{g} \lambda_i(l_0) \geq C \prod_{i=1}^{g} \lambda_i(l)
\]

get such \( C \) from minkowski 2nd

for \( g \)-dim.

From minkowski 2nd

for \( g \)-dim.
Define $L = \Lambda \oplus \mathbb{Z} \hat{e}_3 \subset \mathbb{R}^3$

$(Picture 6)$

Always

$$
\begin{cases}
\lambda_1 = 1 \text{ realised by } \hat{e}_3 \\
\lambda_4 = 1 \text{ realised by } \text{span}_\mathbb{Z}(\hat{e}_3)
\end{cases}
$$

$\lambda_2 = \mathcal{C}$ realised by $\{c\hat{e}_1, c\hat{w}\}$

$\lambda_2 = \Lambda$ realised by $\Lambda$

("Proof" by geometric intuition)

Problems with $\lambda_2$:

1) not easy to find
2) can have many realisers, for example if $L = \mathbb{Z}^n$, $\lambda_2(L)$ have at least $\binom{n}{2}$ realisers.

Partial solution for 2 - Harish Narasimhan Filtration.
Harder-Narasimhan Filtration

Assume we know $L_J = \text{covol}(L_0)$ for some sub-lattice of rank $J$, Picture 7.

For $0 \leq J \leq n$, [convention] $a_0(L) = 1$

Step 1: Draw in plain the figure $A^3(L) = \left\{ (J, y) \mid 0 \leq J \leq n, y \geq \log(L_J) \right\}$ join

= union of vertical lines $L_J = \text{covol}(L_J)$

Step 2: Take convex hull of $A^3(L)$

\[ \text{Conv } A^3 = \text{minimal convex set containing } A^3(L) \]

Step 3: Define Profile of $L$

to be the bottom polygonal line of $\text{Conv}(A^3)$, and mark all vertices which form a vertex $!!$

on Profile $(L)$.

Pictures 7 + 8
Remark: For all $\text{loc} \ L$ of rank $j$, the point $(j, \log \text{covol} (L_j))$ is above the profile of $L$. 

By definition of $f_j$!
Thm – HN filtration (1,2,3,4,5)

1) If \((J, \log \text{conv} (L_j)) \in \text{Vertices}\),
   then \(L_j\) is unique. i.e. - if \(L_j \subset L\) is sub-lattice of rank \(J\), so
   \(\text{conv} (L_j) \subset \text{conv} (L_j')\)

2) The sub-lattices for which \((J, \text{conv}(L_j))\) is a vertex are nested. i.e.

\[
\bigcap_{0 \leq k \leq n-1} L_{J_0} \subset L_{J_1} \subset \ldots \subset L_{J_k} \subset L_{J_{k+1}}
\]

\(k\) intersecting ones
\(0 \leq k \leq n-1\)
\(1 \leq J_i \leq n-1\)

(*) is called then Hunder-Narasimhan filtration of \(L\) and we say it has length \(k\), say it is stable if \(k = 0\).
For each $i \in \{4, 000, k\}$

$$\mathcal{I}_i = L_0 \subset L_{g_1} \subset \cdots \subset L_j$$

is the $\mathcal{H}_{11}$-filtration of $L_{J^p}$
If proof uses heavily the following claim:

if \( L_1, L_2, C, L \) are primitive sub-lattices

(a) \( \text{rank}(L_1 \cap L_2) + \text{rank}(L_1 + L_2) = \text{rank}(L_1) + \text{rank}(L_2) \)

(b) \( \text{covol}(L_1 \cap L_2) \cdot \text{covol}(L_1 + L_2) \leq \text{covol}(L_1) \cdot \text{covol}(L_2) \)

\[ \Rightarrow \]

\[ \log(C_0(a)) + \log(C_0(\ell^2)) \leq \log(C(L_1)) + \log(C(L_2)) \]

Assume we have two primitive sub-lattices \( L_1, L_2 \) s.t. \( \text{covol}(L_1) = \text{covol}(L_2) = \chi \)

and \( \chi \) is a vertex of the profile.

\[ \ell = \frac{c-b}{2m \cdot \chi} (t-m) + c \]
\[ f(j) = \frac{C-b}{2(m-j)} j \cdot (j-m) + c \]

\[ = \frac{b-C}{2} + c = \frac{b+c}{2} \leq 2^j \]

Violating the fact that \( 2^j < f(j) \)

This is for the case \( m \geq j \).

Assume \( m=j \)

\[ \Rightarrow \text{rank}(L_1 \cap L_2) = \text{rank}(L_1). \]

\( L_1, L_2 \) are primitive, so

cannot have \( L_1 \cap L_2 \subset L_1 \)

If have same rank

\[ \Rightarrow L_1 \cap L_2 = L_1. \]

\[ \Rightarrow L_1 \cap L_2 = L_2 \quad (\text{same argument}) \]

\[ \Rightarrow L_1 = L_2. \]
**Pf 2:**

\[ (m-a_2-a_1, b) \rightarrow \mathbb{L}(H) \]

Assume by contradiction \( L_1 \not\parallel L_2 \).

\[ (L_1 \cap L_2) \neq L_1 \]

\[ \Rightarrow \quad \text{rank } (L_1 \cap L_2) < \text{rank } L_2 \]

\[ \text{volume } L_2 < \text{rank } (L_1 + L_2) \]

Writing the equation \( f(t) \) and using again that \( b+c < a_1 + a_2 \)

We get that at least one of point lies beneath \( f(t) \), contradicting the fact it is in the profile of \( L \).
$L_J < L$ primitive subgroup of rank $J$.

$\vec{V}_1, \ldots, \vec{V}_J, \vec{V}_{j+1}, \ldots, \vec{V}_n$

basis of $L_J$

basis of $L$

$\pi_J : \mathbb{R}^n \xrightarrow{proj_J} (\text{span}_{\mathbb{R}}(L_J))^\perp \sim \mathbb{R}^{n-J}$

$\star \pi_J(L) = \text{span}_{\mathbb{R}}(\pi_J(\vec{V}_J), \ldots, \pi_J(\vec{V}_n))$

claim 2  $\pi_J(L)$ is discrete in $(\text{span}_{\mathbb{R}}(L_J))^\perp$

$\Rightarrow \pi_J(L)$ is an $n-J$ lattice in $(\text{span}_{\mathbb{R}}(L_J))^\perp$

[Short discussion
Prior to results 45]
\{0\} \subseteq L_{\bar{f}_1} \subseteq L_{\bar{f}_2} \subseteq \cdots \subseteq L_{\bar{f}_k} \subseteq L \quad (\text{reminder})

4) \forall i \in \{1, \ldots, k\} \text{ define } \pi_{\bar{f}_i} \text{ as before so that}

\{0\} \subseteq \pi_{\bar{f}_i}(L_{\bar{f}_{i+1}}) \subseteq \pi_{\bar{f}_i}(L) \subseteq \pi_{\bar{f}_i}(L_k) \subseteq \pi_{\bar{f}_i}(L)

is the $HN$-filtration of $\pi_{\bar{f}_i}(L)$.

5) for all $i \in \{1, \ldots, k\}$

$\pi_{\bar{f}_i}(L_{\bar{f}_{i+1}})$ is stable.