Some implications of effective results in homogeneous dynamics for Diophantine approximation

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Dirichlet Theorem: (1842) For all $y \in \mathbb{R}$ and for all $T > 1$ there exists $q \in \mathbb{N}, p \in \mathbb{Z}$ such that $|qy - p| < 1/T$ and $q \leq T$.

Higher dimensions
Let $m, n \in \mathbb{N}$. We say $\vec{t} \in \mathbb{R}^{m+n}$ is a weight (for $m+n$) if $t_i \geq 0$ for all coordinates and $\sum_{i=1}^{m} t_i = \sum_{j=1}^{n} t_{m+j}$.

Weighted Dirichlet theorem for matrices: For all $Y \in M_{m,n}(\mathbb{R})$ and for every weight vector $\vec{t}$ there exists $0 \neq \vec{q} \in \mathbb{Z}^n, \vec{p} \in \mathbb{Z}^m$ such that

- $|Y_i\vec{q} - p_i| \leq e^{-t_i} \qquad i \in \{1, ..., m\}$
- $|q_j| \leq e^{t_{m+j}} \qquad j \in \{1, ..., n\}$.

Proof: Define

$$\boldsymbol{u_Y} = \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix} \in SL_{m+n}(\mathbb{R})$$

and define the unimodular lattice

$$\boldsymbol{\Lambda_Y} := u_Y \mathbb{Z}^{m+n}.$$

In particular, $\Lambda_Y := \left\{ \begin{pmatrix} Y\vec{q} - \vec{p} \\ \vec{q} \end{pmatrix} \middle| \vec{q} \in \mathbb{Z}^n \ \vec{p} \in \mathbb{Z}^m \right\}$.

Define a centrally symmetric convex body

$$\Pi_{\vec{t}} := \left\{ (\vec{a}, \vec{b}) \in \mathbb{R}^m \times \mathbb{R}^n \mid |a_i| \leq e^{-t_i} \ |b_i| \leq e^{t_{m+j}} \right\}.$$

So $vol(\Pi_{\vec{t}}) = 2^m e^{-t_1 - \ldots - t_m} 2^n e^{t_{m+1} + \ldots + t_{m+n}} = 2^{m+n}$. So by Minkowski's convex body theorem $\Lambda_Y \cap \Pi_{\vec{t}} \neq \{\vec{0}\}$. Any point in this intersection cannot have $\vec{q} = \vec{0}$ hence satisfies the theorem.

Can we improve Dirichlet theorem?

Given $\vec{t} \in \mathbb{R}^{m+n}$ a weight define the floor of $\vec{t}$ to be

$$\lceil \vec{t} \rceil := min_{i=1,\ldots,m+n}\ t_i.$$

Given τ a collection of weights, we say τ is unbounded if for any $T_0 \in \mathbb{R}$ there exists $\vec{t} \in \tau$ such that $\lceil \vec{t} \rceil \geq T_0$.

For $Y \in M_{m,n}(\mathbb{R})$ and τ a collection of unbounded weights, we say Y is weighted Dirichlet improvable along τ if there exists $\varepsilon < 1$ and there exists T_0 such that for any weight $\vec{t} \in \tau$ with $\lceil \vec{t} \rceil \geq T_0$ there exists $0 \neq \vec{q} \in \mathbb{Z}^n, \vec{p} \in \mathbb{Z}^m$ such that

- $|Y_i\vec{q} - p_i| \le \varepsilon e^{-t_i} \quad i \in \{1, ..., m\}$
- $|q_j| \le \varepsilon e^{t_{m+j}} \quad j \in \{1, ..., n\}$.

We will proof the following:

Theorem 1 (KW 2006, KM 2009). *Let τ be an unbounded collection of weights. Then Lebesgue almost every $Y \in M_{m,n}(\mathbb{R})$ is not weighted Dirichlet improvable.*

- Was first proven by a non-effective theorem by Kleinbock-Weiss on 2006.
- We will prove it using an effective extension of KW.
- For the private case of the standard weights $\tau := \{\, (\frac{t}{m}, ..., \frac{t}{m}, \frac{t}{n}, ..., \frac{t}{n}) \,|\, t > 0\}$ – proved by Dvenport and Schmidt on 1970. No dynamics involved.

In order to state the effective theorem by KM we need some defintions.

- $k = m + n$
- $G = SL_k(\mathbb{R}), \Gamma = SL_k(\mathbb{Z})$
- $X = G/\Gamma$. Particularely, for $Y \in M_{m,n}(\mathbb{R})$ we have that $u_Y \in G$ and $\Lambda_Y \in X$.
- Define $H := \{u_Y \,|\, Y \in M_{m,n}(\mathbb{R})\} \subset G$.
- Given $\vec{t} \in \mathbb{R}^k$ a weight, define

  $$g_{\vec{t}} := diag(e^{t_1}, ..., e^{t_m}, e^{-t_{m+1}}, ..., e^{-t_{m+n}}).$$

  Particularly $g_{\vec{t}} \in G$ for any weight.
- Given $t > 0$ define

  $$g_t := diag(e^{t/m}, ..., e^{t/m}, e^{-t/n}, ..., e^{-t/n}) \in G.$$
- m_X Haar on X.
- vol haar on H. I.e vol is Lebesgue.
- For $\varepsilon > 0$ define
  $$K_\varepsilon := \left\{ x \in X \;\middle|\; \|\vec{x}\|_\infty \ge \varepsilon \;\forall \vec{x} \in X \setminus \{\vec{0}\} \right\}.$$

Claim: Let τ be a collection of weights. Then the following are equivalent:

1. $Y \in M_{m,n}(\mathbb{R})$ is weighted Dirichlet improvable along τ.
2. There exists $\varepsilon < 1$ such that $g_{\vec{t}}\Lambda_Y \notin K_\varepsilon$ for all weight $\vec{t} \in \tau$ with $[\vec{t}]$ large enough.

Theorem (KM 2009 Theorem 1.3). *There exists $\widetilde{\gamma} > 0$ such that for any $f \in C^\infty_{comp}(H)$, $\psi \in C^\infty_{comp}(X)$ and any compact $L \subset X$ there exists $\widetilde{C} = \widetilde{C}(f, \psi, L)$ such that for all $x_0 \in L$ and all $\vec{t}$ weight we have that*

$$\left| \int_H f(h)\, \psi(g_{\vec{t}} h x_0)\, dvol(h) - \int_H f dvol \int_X \psi \, dm_X \right| \le \widetilde{C} e^{-\widetilde{\gamma}\,[\vec{t}]} \,.$$

Proof of Theorem 1:

Let $\varepsilon < 1$. For $i \in \mathbb{N}$ define

$$B_i := \bigcap_{[\vec{t}]>i \ \ \vec{t}\in\tau} \{u_Y \in H \,|\, g_{\vec{t}}\Lambda_Y \notin K_\varepsilon\}.$$

Notice that $\cup_{i\in\mathbb{N}} B_i = \{Y \in M_{m,n} \mid \exists i \ s.t \ g_{\vec{t}}\Lambda_Y \notin K_\varepsilon \ \forall \, \vec{t} \in \tau \ with \ [\vec{t}] > i\,\}$.

Assume by contradiction that $vol(B_i) > 0$.

Then we can choose $B \subset B_i$ compact with positive measure as well.

Take $f \in C^\infty_{comp}(H)$ such that $f(h) = 1$ in B and vanishes outside of it (up to).

Take a non-negative function $\psi \in C^\infty_{comp}(X)$ which is supported on K_ε but vanishes outside of it (up to).

Choose $x_0 = \mathbb{Z}^k$.

Apply the theorem above and get there exists $\widetilde{C}, \delta_1, \delta_2$ such that for any weight $\vec{t} \in \tau$ we have

$$\left| \int_B \psi(g_{\vec{t}}\Lambda_Y)\, dvol(Y)\Big(1+\delta_1\Big) - vol(B)(1+\delta_2)\int_X \psi\, dm_X \right| \le \widetilde{C}e^{-\widetilde{\gamma}\,[\vec{t}]} \ .$$

Take $\vec{t}$ so that $[\vec{t}]$ is large enough ($[\vec{t}] > i$ and RHS is small).

We get that $\int_B \psi\,(g_{\vec{t}}\Lambda_Y)\, dvol(Y) > 0$, contradicting the definition of B_i and ψ.

So $vol\,(\bigcup_{i\in\mathbb{N}} B_i) = 0$, i.e. for Lebesgue almost every Y there is an unbounded positive sequence $(t_k) \subset \mathbb{R}$ and a sequence of weights $(\vec{t}_k) \subset \tau$ such that $g_{\vec{t}_k}\Lambda_Y \in K_\varepsilon$ with $[\vec{t}_k] = t_k$.

As this hold for any $\varepsilon < 1$, Lebesgue almost every Y is not Dirichlet improvable along τ.

Proof of KM 2009

More definitions:

- '$dist_G$' is a right invariant metric on G, giving rise to '$dist_X$' similarly on X.
- For $l \in \mathbb{N}$ define $\|\cdot\|_l$ is the $(2,l)$-Sobolev norm. I.e for $\psi : H \to \mathbb{R}$

$$\|\psi\|_l := \Big(\sum_{|\alpha|\le l} (\|D^\alpha\psi\|_{L^2})^2\Big)^{1/2}$$

  where $D^\alpha\psi$ is a shorthand for the collection of all α-order partial derivatives of ψ.
- Define $W^{2,\infty}(X) := \{\psi \in C^\infty(X) \,|\, \|\psi\|_l < \infty \ \forall\, l \in \mathbb{N}\,\}$.
- For $l \in \mathbb{N}$ define $\|\cdot\|_{C^l}$ the C^l norm. I.e. for $\psi : H \to \mathbb{R}$

$$\|\psi\|_{C^l} := \sum_{|\alpha|\le l} \sup_{h\in H} \left|\partial^\alpha\psi(h)\right|$$

  where $\partial^\alpha\psi$ is a shorthand for the collection of all α-order partial derivatives of ψ.

- $\|\psi\|_{Lip} := sup_{h_1,h_2\in H}\,\dfrac{|\psi(h_1)-\psi(h_2)|}{dist_H(h_1,h_2)}$ is the Lipschitz constant of ψ.
- $Lip(H) := \{\psi \mid \|\psi\|_{Lip} < \infty\}$.
- For $\vec{t}$ a fixed weight, define an inner automorphism on H defined by

$$\Phi_{\vec{t}}(h) := g_{\vec{t}}\, h\, (g_{\vec{t}})^{-1}.$$

  So we have $(\Phi_{\vec{t}})^{-1} = \Phi_{\vec{t}^{-1}}$

Proof of KM 2009 Theorem 1.3 Given $\psi \in C^{\infty}_{comp}(X)$ define $\psi' := \psi - \int_X \psi$. Then $\int_X \psi' = 0$ and $\psi' \in W^{2,\infty}(X) \cap Lip(X)$.

So enough to prove that there exists $\widetilde{\gamma} > 0$ such that for any $f \in C^{\infty}_{comp}(H)$, $\psi \in W^{2,\infty}(X) \cap Lip(X)$ with $\int_X \psi = 0$ and any compact $L \subset X$ there exists $\widetilde{C}$ such that for all $x_0 \in L$ and all $\vec{t}$ weight we have that

$$\left| \int_H f(h) \cdot \psi(g_{\vec{t}} h x_0)\, dvol(h) \right| \leq \widetilde{C} e^{-\widetilde{\gamma}\,[\vec{t}]}\ . \tag{1}$$

As $\widetilde{C} = \widetilde{C}(f, \psi, L)$ we can also show the above holds for $\vec{t}$ with $[\vec{t}]$ large enough.

Given $\vec{t}$ define

$$t := [\vec{t}]/2$$
$$\vec{u} = \vec{u}(\vec{t}) := \vec{t} - (t/m, ..., t/m, t/n, ...t/n).$$

Note that we get $[\vec{u}] \geq [\vec{t}]/2$ and that $g_{\vec{t}} = g_t g_{\vec{u}}$.

We introduce a new function in the following way. Let $\theta \in C^{\infty}_{comp}(H)$ which satisfies the followings:

- $supp(\theta), supp(f) \subset B := B_H(r)$ for $r = e^{-\beta t}$ for β to be specified later.
- $\int_H \theta = 1\, dvol$
- $\|\theta\|_l \ll r^{-(l+mn/2)}$

By Lemma 2.2(a) such a function θ exists. So we have the following:

$$\begin{aligned}
\int_H f(h)\, \psi(g_{\vec{t}} h x_0)\, dvol(h) &= \int_H f(h)\, \psi(g_{\vec{t}} h z)\, dvol(h) \int_H \theta(y) dvol(y) \\
&= \int_H \int_H f(h)\, \psi(g_{\vec{t}} h x_0)\, \theta(y)\ dvol(y) dvol(h) \\
&\overset{Left-inv}{=} \int_H \int_H f\big(\Phi_{\vec{u}}^{-1}(y) h\big)\, \psi\big(g_t y g_{\vec{u}}\, h x_0\big)\, \theta(y)\ dvol(y) dvol(h) = (**).
\end{aligned} \tag{2}$$

Note that $\Phi_{\vec{u}}^{-1}$ is a **contracting** automorphism of H – for any $y \in H$ we have $\left(\Phi_{\vec{u}}^{-1}\right)^n(y) \longrightarrow I$.

Furthermore, a direct calculation shows that in fact we have

$$dist_G(I, \Phi_{\vec{u}}^{-1}(y)) \overset{(*)}{\ll} e^{-2[\vec{u}]} dist_G(I, y) \leq e^{-2t} dist_G(I, y)$$

Then by right invariance for the metric $dist_G$, for any $\widetilde{f}$ defined by $\widetilde{f}(h) = f(\Phi_{\vec{u}}^{-1}(y)h)$ we have that

$$supp(\widetilde{f}\,) \subset \widetilde{B} \; = B_H(r(1+e^{-2t})) = B_H(r + e^{-(2+\beta)t}).$$

Theorem (Corollary 3.4). *Let $L \subset X$ be compact and let B any ball in H. Then there exists $T = T(B, L)$ such that for every $0 < \varepsilon < 1$, any $x_0 \in L$ and any weight $\vec{t}$ with $[\vec{t}\,] \geq T$ one has*

$$vol\Big(\Big\{h \in B\,|g_{\vec{t}}hx_0 \notin K_\varepsilon\Big\}\Big) \ll \varepsilon^{(mn(k-1))^{-1}} vol(B)$$

Now take t large enough so that

- $vol(\widetilde{B}) \leq 2vol(B)$ (as t is larger we have $\widetilde{B} \to B$).
- $t > T(\widetilde{B}, L)$ as in corollary 3.4.

Define $\varepsilon := s_0 e^{-\beta t/k}$ for $s > 0$ which we explain later on, and denote

$$A := \{h \in \widetilde{B}\,|g_{\vec{u}}hx_0 \notin K_\varepsilon\}.$$

Then for any weight $\vec{u}$ with $[\vec{u}\,] \geq T$ and any $x_0 \in L$ we have by corollary 3.4 that

$$vol(A) \ll \varepsilon^{mn(k-1)^{-1}} vol(\widetilde{B}).$$

Hence we have

$$\begin{aligned} |\int_A \int_H f\big(\Phi_{\vec{u}}^{-1}(y)h\big)\, \psi\big(g_t y g_{\vec{u}}\, hx_0\big)\, \theta(y)\; dvol(y)dvol(h)| &\ll \big(\varepsilon^{mn(k-1)^{-1}} vol(\widetilde{B})\big)\, sup|f|\, sup|\psi| \int_H \theta(y)dy \\ &\overset{\varepsilon = s_0 e^{-\beta t/k}}{\ll} vol(B)\, e^{\frac{-\beta t}{mnk(k-1)}}\, sup|f|\, sup|\psi| \end{aligned} \tag{3}$$

So this is the estimate we have in A.

Now let $h \in \widetilde{B} \setminus A$ and write for convenience

$$F_h(y) := f\big(\Phi_{\vec{u}}^{-1}(y)h\big)\, \theta(y).$$

We start with the following Thm:

Theorem (Theorem 2.3). *There exists r_0 such that the followings hold. Let $F \in C^\infty_{comp}(H)$, let $0 < r < r_0$ and $x_0 \in X$ such that*

- *$supp(F) \subset B_H(r)$*
- *the map $G \to X$ defined by $g \mapsto gx_0$ is injective on $B_G(2r)$.*

Then for any $\psi \in W^{2,\infty}(X) \cap Lip(X)$ with $\int_X \psi = 0$ there exists $\gamma, E > 0$ and $N, l \in \mathbb{N}$ such that for any $t \geq 0$ we have

$$|\int_H F(h)\psi(g_t hx_0)dvol(h)| \leq E\Big(r\int_H |F| + r^{-(2l+N/2)}\, \|F\|_l\; e^{-\gamma t}\Big).$$

So we from Lemma 2.2 we have the following:

$$\|F_h\|_l \ll \|f\|_{C^l} \|\theta\|_l \ll r^{-(l+mn/2)} \|f\|_{C^l}. \tag{4}$$

So by Thm 2.3 (for r small enough and s_0 we chose) there exists $E, \gamma > 0$ and $N, l \in \mathbb{N}$ such that

$$\begin{aligned}
&|\int_{\widetilde{B}\setminus A}\int_H f\big(\Phi_{\vec{u}}^{-1}(y)h\big)\,\psi\big(g_t y g_{\vec{u}}\,hx_0\big)\,\theta(y)\ dvol(y)dvol(h)| \\
&\quad\le \int_{\widetilde{B}\setminus A}\left|\int_H F_h(y)\,\psi\big(g_t y g_{\vec{u}}\,hx_0\big)\ dvol(y)\right| dvol(h) \\
&\quad\overset{Thm\,2.3}{\le} E\bigg(r\int_H |F_h| dvol(y)\ + r^{-(2l+N/2)}\ \|F_h\|_l\ e^{-\gamma t}\bigg) vol(\widetilde{B}) \\
&\quad\overset{equation\,4}{\ll} E\bigg(sup|f|e^{-\beta t}\ + \|f\|_{C^l}\, e^{-(\gamma-(2l+N/2)\beta)t}\bigg) vol(B)
\end{aligned} \tag{5}$$

Now lets go back to $(**)$ on equation 2. So we have that

$$|(**)| \le C_1 e^{\frac{-\beta t}{mnk(k-1)}} + C_2 e^{-\beta t} + C_3 e^{-(\gamma-(2l+N/2)\beta)t} \le max(C_1, C_2) e^{\frac{-\beta t}{mnk(k-1)}} + C_3 e^{-(\gamma-(2l+N/2)\beta)t}.$$

Choose β so that both exponents are equal and we are done.