Today's talk

* reminders & Presenting main Thm
* long sketch of proof (3 steps proof)
* proving step 1

Previously on the seminar

Yiftach

Measure theoretic Entropy

$(X, \mathcal{B}, \mu, T)$ – Borel probability Measure-preserving System

α = measurable partition

$h_\mu(X, T, \alpha)$ = entropy of T w.r.t α

$\Downarrow$

$h_\mu(X, T)$ = entropy of T (sup over partitions)

Florent

Topological Entropy

(X, d, T) – metric space & continuous function

$\mathcal{U}$ = open cover of X

$h_{top}(X, T, \mathcal{U})$ = topological entropy of T w.r.t to $\mathcal{U}$

$\Downarrow$

$h_{top}(X, T)$ = topological entropy of T (sup over covers)

$\Downarrow$

Variational principle: if X is compact, so

$$h_{top}(X,T) = \sup_{\mu} \{ h_\mu(X,T) \}$$

where μ is radon, Probality, T-inv

Erez

Littlewood Conjecture (1930's)

$\forall \alpha, \beta \in \mathbb{R}$ $\quad \inf_{n \in \mathbb{N}} n \cdot \langle n\alpha \rangle \cdot \langle n\beta \rangle = 0$

$\iff \quad \liminf n \langle n\alpha \rangle \langle n\beta \rangle = 0$

[where $\langle x \rangle$ denotes the fractional part of x, i.e
$$\langle x \rangle = \min \{ x - \lfloor x \rfloor, \lceil x \rceil - x \} \text{]}$$

Observation: $\forall \alpha, \beta \in \mathbb{R}^2$ $\quad \inf_{n>0} n \cdot \langle n\alpha \rangle \langle n\beta \rangle < \frac{1}{2\sqrt{5}}$

following a survey on EKL and other results by Einsiedler & Lindenstrauss,

Today & next week will prove the following thm:

Thm For $\delta>0$ define $\mathbb{E}_\delta := \{(\alpha,\beta) \mid \liminf_{n\to\infty} n\langle n\alpha\rangle\langle n\beta\rangle \geq \delta\}$

$\mathbb{E}'_\delta$ inf

so $\mathbb{E}_\delta$ has upper box dimension equal to 0.

[i.e, $\exists c$ s.t $\forall \varepsilon>0$ if $r\in(0,1)$ is small enough, then one can cover $\mathbb{E}_\delta$ by at most $c\cdot\frac{1}{r^\varepsilon}$ boxes of size $r\times r$]

remark 1: may assume $(\alpha,\beta)\in[0,1]^2$.

remark 2: can we take inf instead liminf?

by the thm, the set $\mathbb{E} := \{(\alpha,\beta) \mid \liminf n\langle n\alpha\rangle\langle n\beta\rangle > 0\}$ is a countable union of sets with upper box dimension 0, so particularly - $\mathbb{E}$ has hausdorff dimension 0.

indeed, for $Y\subseteq\mathbb{R}^3$

* $$\dim_H(Y) := \inf\Big\{d>0 \;\Big|\; \liminf_{\substack{r\to 0\\ C_r=(C_1,C_2,C_3\dots)}} \Big(\sum_i (\operatorname{diam} C_i)^d\Big)=0\Big\}$$

where C_r is a countable cover of Y with $\operatorname{diam}(C_i)<r$ $\forall i$

* $$\dim_{\substack{\text{upper}\\ \text{box}}}(Y) := \limsup_{r\to 0}\frac{\log(b_r(Y))}{|\log r|}$$

where $b_r(Y)$ is the biggest cardinality of $F\subseteq Y$ s.t $\forall y,y'\in F$ $\|y-y'\|\geq r$ (r-seperated set)

$$\Rightarrow \dim_H(\mathbb{E}) = \dim_H\Big(\bigcup_{\delta=1}^{\infty}\mathbb{E}_{\frac{1}{\delta}}\Big) = \sup_\delta\{\dim_H(\mathbb{E}_{\frac{1}{\delta}})\}$$

Observation

sub-additivity of Hausdorff outer measure

$$\leq \sup_\delta\{\dim_{\substack{\text{upper}\\ \text{box}}}(\mathbb{E}_{\frac{1}{\delta}})\} = 0.$$

Detailed sketch of proof

$G = SL_3(\mathbb{R})$, $\Gamma = SL_3(\mathbb{Z})$

$$X_3 := \left\{ x \subset \mathbb{R}^3 \;\middle|\; \begin{array}{l} x \text{ is a lattice} \\ covol(x) = 1 \end{array} \right\}$$

Prop $G/\Gamma \simeq X_3$ by the bijection $[g\Gamma] \mapsto g\mathbb{Z}^n$

Prop X_3 is non-compact metric space equiped with the chabuty-Fell metric. Furthermore, the topology induced from the metric coincide with the qoutent topology.

* $G > A$:= diagonal matrices in G
* $A \supset \underbrace{A^+}_{\text{semi-group}} := \{a(s,t) \mid s,t \geq 0\}$ where $a(s,t) = \begin{pmatrix} e^s & & \\ & e^t & \\ & & e^{-s-t} \end{pmatrix}$
* for $\sigma, \tau \geq 0$ define $a_{\sigma,\tau}(t) = a(\sigma t, \tau t)$
* for $(\alpha,\beta) \in [0,1]^2$ define $x_{\alpha,\beta} := \begin{pmatrix} 1 & & \alpha \\ & 1 & \beta \\ & & 1 \end{pmatrix} \mathbb{Z}^n \in X_3$
* let G act on X_3 in the natural way, i.e by left multiplication.

main thm from EKL (Measure classifaction for $A \curvearrowright X_3$)

Let μ be an A-invariant, Ergodic, Probality Measure on X_3. then one of following holds

1) $h_\mu(X,a) = 0 \quad \forall a \in A$

2) μ is algebraic.

i.e, $\exists\, H \leq G$ st $A \leq H \leq G$, H is closed & connected and $\exists\, x_0 \in X_3$ s.t μ is the H-inv measure on $H \cdot x_0$.

Corollary: if 1 doesn't hold, then μ is not compactly-supported.

[by the classification of algebraig measures by Lindenstrauss & weiss]

Detailed sketch of proof

step 1: (α,β) satisfies Littlewood iff the orbit $A^+ \cdot x_{\alpha,\beta}$ is unbounded.

[Moreover, $\forall \delta > 0$ $\exists K_\delta$ compact in X_3 s.t

We prove this for $\mathbb{E}_\delta$ $(\alpha,\beta) \in \mathbb{E}_\delta \Longrightarrow A^+ \cdot x_{\alpha,\beta} \subset K_\delta$]

step 2: (i) if $A^+ \cdot x_{\alpha,\beta}$ is bounded then $\forall \sigma, \tau \geq 0$

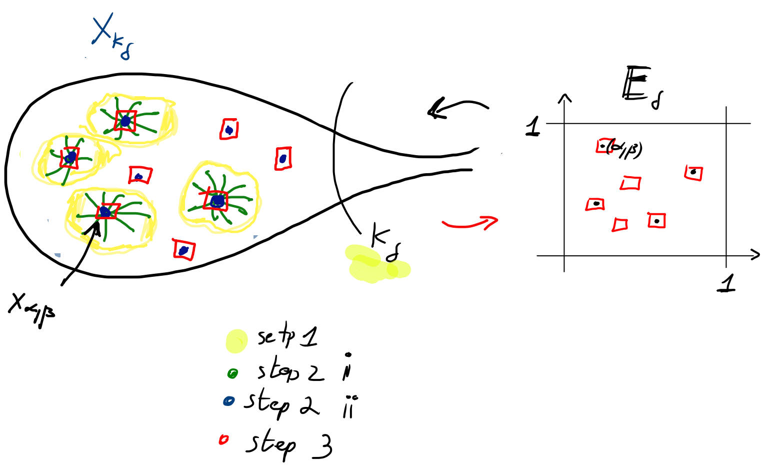
$$h_{top}\left(\overline{\{a_{\sigma,\tau}(t)\}_{t\geq 0} \cdot x_{\alpha\beta}}\,,\ a(\sigma,\tau)\right) = 0$$

* assume $h_{top}(\text{''},\text{''}) > 0$
* use variational principle to get μ with $h_\mu(\text{''},\text{''}) > 0$
* using μ, finds μ_∞ which satisfies EKL conditions but has positive entropy & compactly supported

(ii) for all $K \subset X_3$ compact, define $X_K := \{x \in X_3 \mid A^+ \cdot x \subset K\}$

then $\forall \sigma, \tau \geq 0$ $h_{top}(X_K, a(\sigma,\tau)) = 0$

step 3: deduce thm using ii and applying cover size arguments.



Proving step 1

(α,β) satisfies Littlewood iff the orbit $A^+ \cdot x_{\alpha,\beta}$ is unbounded.

Moreover, $\forall \delta > 0$ $\exists K_\delta$ compact in X_3 s.t

$$(\alpha,\beta) \in \mathbb{E}_\delta \implies A^+ \cdot x_{\alpha,\beta} \subset K_\delta$$

we prove the "moreover" for $\mathbb{E}_\delta^+$ instead of $\mathbb{E}_\delta$

in order to prove thm recall **Mahler compactness criterion**

$\mathcal{U}$ is bounded $[\overline{\mathcal{U}}$ is compact$]$ $\iff$ $\exists \varepsilon > 0$ s.t $\forall x \in \mathcal{U}$ $\varepsilon \leq \lambda_1(x)$

where $\lambda_1(x)$ is length of shortest non-zero vector in x.
(sup-norm)

<u>sketch of proof for Mahler</u>

<u>lemma</u>: the map $x \mapsto \lambda_1(x)$ is continous

$\Rightarrow$ assume $\forall j \in \mathbb{N}$ $\exists x_j \in \mathcal{U}$ s.t $\lambda_1(x_j) < \frac{1}{j}$. passing to sub-sequence and using compactness of $\mathcal{U}$,

$\exists L \in \mathcal{U}$ s.t $x_{j_k} \longrightarrow L$. but then by lemma 1

$\frac{1}{j_k} > \lambda_1(x_{n_k}) \xrightarrow{k\to\infty} \lambda_1(L)$, hence $\lambda_1(L) = 0$, contradiction.

$\Leftarrow$ Exercise (more difficult)

$$\boxed{A^+ \cdot x_{\alpha,\beta} \text{ unbounded} \implies (\alpha,\beta) \text{ satisfies Littlewood}}$$

want to show $\inf_{n\in\mathbb{N}} n \cdot \langle n\alpha \rangle \langle n\beta \rangle = 0$

let $\varepsilon > 0$ arbitrary and assume $\varepsilon < \frac{1}{2}$.

$A^+ \cdot x_{\alpha,\beta}$ is unbounded, and so by Mahler $\exists a \in A^+$ s.t

$$\lambda_1(a \cdot x_{\alpha,\beta}) < \varepsilon$$

$\Rightarrow$ $\exists s,t \geq 0$ s.t $\lambda_1(a(s,t) \cdot x_{\alpha,\beta}) < \varepsilon$

$$a(s,t) \cdot x_{\alpha,\beta} = \begin{pmatrix} e^s & & \\ & e^t & \\ & & e^{-s-t} \end{pmatrix} \begin{pmatrix} 1 & & \alpha \\ & 1 & \beta \\ & & 1 \end{pmatrix} \cdot \mathbb{Z}^3,$$

and so if

$\lambda_1(a(s,t) \cdot x_{\alpha,\beta}) < \varepsilon$, $\exists m,k,n \in \mathbb{Z}$ s.t

$$\left\| \begin{pmatrix} e^s & & \\ & e^t & \\ & & e^{-s-t} \end{pmatrix} \begin{pmatrix} 1 & & \alpha \\ & 1 & \beta \\ & & 1 \end{pmatrix} \begin{pmatrix} m \\ k \\ -n \end{pmatrix} \right\|_\infty < \varepsilon$$

$$\Updownarrow$$

$$\max\left\{ |e^s(m-\alpha n)|, |e^t(k-\beta n)|, |e^{-s-t}(-n)| \right\} < \varepsilon < \frac{1}{2}$$

$$\Rightarrow \quad |m-\alpha n| < \tfrac{1}{2}\cdot \tfrac{1}{e^s} < \tfrac{1}{2}$$
$$|k-\beta n| < \tfrac{1}{2}\cdot\tfrac{1}{e^t} < \tfrac{1}{2}$$

$\Rightarrow$ and so by the properties of fractional part function

$$\langle \alpha n\rangle = |m-\alpha n|$$
$$\langle \beta n\rangle = |k-\beta n|$$

$$\Rightarrow n\cdot\langle n\alpha\rangle\langle n\beta\rangle = e^t\cdot e^s\cdot e^{-s-t}\cdot n\cdot\langle n\alpha\rangle\langle n\beta\rangle$$
$$= e^s|m-n\alpha|\cdot e^t|k-\beta n|\cdot e^{-s-t}\cdot n$$
$$< \varepsilon^3 < \varepsilon$$

<u>remarks:</u>

* $n\neq 0$, otherwise get $e^t|k|$ & $e^s|m| < \frac{1}{2}$, contradiction.
* for $\varepsilon > \frac{1}{2}$, choose $\frac{1}{2}$ for Mahler and repeat.

(α,β) satisfies Littlewood $\Rightarrow$ $A^+\cdot x_{\alpha,\beta}$ is unbounded.

in order to proof this direction, recall Dirichlet thm:

<u>Dirichlet thm:</u> $\forall y\in\mathbb{R}\ \forall Q\geq 1\ \exists q\neq 0\in\mathbb{N},\ p\in\mathbb{Z}$ s.t $\begin{cases} |qy-p| < \frac{1}{Q} \\ 1\leq q\leq Q\end{cases}$

assume $\varepsilon<1$. want to show that $\exists \vec{v}\in A^+\cdot x_{\alpha,\beta}$ s.t $\|\vec{v}\|<\varepsilon$.
by the assumption, $\exists n\in\mathbb{N}$ s.t $n\langle n\alpha\rangle\langle n\beta\rangle < \varepsilon^5$

$\Rightarrow \exists k,m\in\mathbb{N}$ s.t $n|n\alpha-m||n\beta-k| < \varepsilon^5 < \varepsilon^3 < \varepsilon$

if $\max\{|n\alpha-m|, |n\beta-k|\} < \varepsilon$ then we are done.

** <u>indeed</u> choose $s,t\geq 0$ $e^s|n\alpha-m| = e^t|n\beta-k| = \varepsilon$.
so $\vec{v} = (e^s(n\alpha-m), e^t(n\beta-k), -n\cdot e^{-t-s}) \in a_{s,t}\cdot x_{\alpha,\beta} \subset A^+\cdot x_{\alpha,\beta}$
and also $\|\vec{v}\|<\varepsilon$: two coordinates are clear, and for the third $|-n\cdot e^{-t-s}| = n\cdot e^{-t}\cdot e^{-s} = \dfrac{n\cdot|n\alpha-m|\cdot|n\beta-k|}{\varepsilon^2} < \dfrac{\varepsilon^5}{\varepsilon^2} < \varepsilon$

[notice that in this case we only used $n|n\alpha-m||n\beta-k| < \varepsilon^3$]

For the second case, assume WLOG that $|n\alpha - m| > \varepsilon$

hence
$$\begin{cases} |n\alpha - m| > \varepsilon \\ n \cdot |n\beta - k| < \varepsilon^4 \end{cases}$$

now use Dirichlet thm for $y = n\alpha$, $Q = \frac{1}{\varepsilon}$.

so $\exists p \in \mathbb{Z}$ $\exists q \neq 0$ s.t $1 \leq q < \frac{1}{\varepsilon}$ s.t $|qn\alpha - p| < \varepsilon$

Furthermore,

$$qn \cdot |qn\beta - qk| = q^2 \cdot n |n\beta - k| < \frac{1}{\varepsilon^2} \cdot \varepsilon^4 = \varepsilon^2$$

so in total
$$\begin{cases} qn |qn\beta - qk| |qn\alpha - p| < \varepsilon^3 \\ \max\{|qn\beta - qk|, |qn\alpha - p|\} < \varepsilon \end{cases}$$

write $qn = \tilde{n}$, $qk = \tilde{k}$, $p = \tilde{m}$

and repeat (*)(*) to find corresponding $\vec{v}$.

lastly, Proving the moreover comment

[proof for $\mathbb{E}_\delta$]

Weaker version: $\forall \delta > 0$ $\exists K_\delta \subset X_3$ compact s.t

$$\inf_{n \in \mathbb{N}} n \cdot \langle n\alpha \rangle \langle n\beta \rangle \geq \delta \implies A^+ \cdot x_{\alpha,\beta} \subset K_\delta$$

for $\delta > 0$, choose $K_\delta := \{ L \in X_3 \mid \delta \leq \lambda_1(L) \}$

by mahler, K_δ is compact (equivalent statement of Mahler)

now assume $A^+ \cdot x_{\alpha,\beta} \not\subset K_\delta$.

so $\exists s,t \geq 0$ s.t $\lambda_1(a(s,t) \cdot x_{\alpha,\beta}) < \delta$

in the first direction we showed that

$$\lambda_1(a(s,t) \cdot x_{\alpha,\beta}) < \delta \implies \exists n \neq 0 \;\; n \cdot \langle n\alpha \rangle \langle n\beta \rangle < \delta$$

contradicting the fact that $\inf_{n \in \mathbb{N}} n \langle n\alpha \rangle \langle n\beta \rangle \geq \delta$.