GEOMETRIC AND ARITHMETIC ASPECTS OF APPROXIMATION VECTORS – introduction

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$\|\cdot\|$ arbitrary norm on $\mathbb{R}^d$.

$B_r = B_r(\|\cdot\|)$ the ball around the origin with radius r w.r.t $(\|\cdot\|)$.

For $\vec{\theta} \in \mathbb{R}^d$, define **the best approximation sequence ("bap") of $\vec{\theta}$** w.r.t $\|\cdot\|$ be the sequence

$$\vec{v}_k = \left(\vec{p}_k, q_k\right)_{k=0} \subset \mathbb{Z}^d \times \mathbb{N}$$

such that:

- $q_0 = 1$
- $q_k = min\ \{q \in \mathbb{N} \mid \exists \vec{p} \in \mathbb{Z}^d \text{ such that } \|q\vec{v} - \vec{p}\| < \|q_{k-1}\vec{\theta} - \vec{p}_{k-1}\|$
- $\forall \vec{p} \in \mathbb{Z}^d \quad \|q_k\vec{\theta} - \vec{p}_k\| \leq \|q_k\vec{\theta} - \vec{p}\,\|$

  Informally, it is the "best" sequence which satisfies

$$1 = q_0 < q_1 < \ldots.. < q_{k-1} < q_k < \ \ldots..$$

$$\left\|q_0\vec{\theta} - \vec{p}_0\right\| > \left\|q_1\vec{\theta} - \vec{p}_1\right\| > \ \ldots.. > \left\|q_{k-1}\vec{\theta} - \vec{p}_{k-1}\right\| > \left\|q_k\vec{\theta} - \vec{p}_k\right\| > \ldots..$$

Remark: $\vec{p}_k$ not uniquely defined for a finite number of indices.

Remark: $d = 1$, can "easily" calculate bap via continued fractions. More difficult once $d \geq 2$.

Theorem . ***Corollary from generalized Dirichlet theorem***

1. $(q_k, \vec{p}_k)_{k=0}^{\infty}$ *infinite sequence* $\iff \vec{\theta} \in \mathbb{R}^d \setminus \mathbb{Q}^d$.

   Furthermore, $\|q_k\vec{\theta} - \vec{p}_k\| \longrightarrow 0$.

2. $\exists\, \gamma = \gamma_{\|\cdot\|}$ *(MINIMAL) such that for all* $\vec{\theta} \in \mathbb{R}^d$ *and for all sufficiently large* k

$$q_k^{\frac{1}{d}}\left(q_k\vec{\theta} - \vec{p}_k\right) \in B_\gamma.$$

For $d = 1$, $\gamma = 1$ and this is the usual "corollary from Dirichlet theorem".

More generally, given $\vec{v} = (\vec{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$ define **the displacement** (w.r.t $\vec{\theta}, \vec{v}$) to be

$$\boldsymbol{disp}\,(\vec{\boldsymbol{\theta}}, \vec{\boldsymbol{v}}) := q^{\frac{1}{d}}\,(q\vec{\theta} - \vec{p}\,).$$

For $\varepsilon > 0$, we say that $\vec{w} = (\vec{P}, Q) \in \mathbb{Z}^d \times \mathbb{N}$ is a **ε-approximation of $\vec{\theta}$** w.r.t $\|\cdot\|$ if

- $disp\,(\vec{\theta},\vec{w}\,) \in B_\varepsilon$
- $gcd(P_1,...,P_d,Q)=1$

Let $(\vec{w}_k)_{k=0}$ be the sequence of ε-app of $\vec{\theta}$ ordered by $Q_0 \le Q_1 \le Q_2 ....$

Claim: For Lebesgue a.e $\vec{\theta}$ and all $\varepsilon > 0$ we have that $(\vec{w}_k)_{k=0}^{\infty}$ is infinite
$\iff$ $\liminf_{k\to\infty} \|disp(\vec{\theta},\vec{v}_k)\| = 0$ where $\vec{v}_k$ are bap of $\vec{\theta}$.
$\iff$ $\vec{\theta}$ is not **badly approximable**.

Example: $d=2$, $\|\cdot\| = \|\cdot\|_2$, $\vec{\theta} = (\pi, e)$, $\gamma = \sqrt{\frac{2}{\sqrt{3}}} \approx 1.07$

	q	p	$\|q\vec{\theta}-\vec{p}\|_2$	$disp(\vec{\theta},\vec{v}) = \sqrt{q}\,(q\vec{\theta}-\vec{p})$	$\|disp(\vec{\theta},\vec{v})\|_2$
	1	(3,3)	0.31	(0.14 , -0.28)	0.31
	2	(6,5)	0.51	(0.28 , 0.39)	0.73
	3	(9,8)	0.45	(0.42 , 0.73)	0.77
	4	(13,11)	0.45	(-0.43 , -0.87)	0.9
	5	(16,14)	0.5	(-0.29 , -0.65)	1.12
	6	(19,16)	0.34	(-0.15 , -0.37)	0.84
	7	(22,19)	0.02	(-0.013 , -0.03)	0.07
	8	(25,22)	0.2	(0.128 , 0.36)	0.8
	9	(28,24)	0.53	(0.26 , 0.8)	1.6

PICTURE

Theorem: (Rogers 1951) $\|\cdot\| = \|\cdot\|_{\max}$, $d \ge 1$, $\vec{v}_k$ bap of $\vec{\theta}$ for some $\vec{\theta}$.
Then any two consecutive displacements $disp(\vec{\theta},\vec{v}_k)$ cannot lie in the same quadrant.

Question: Does this hold for arbitrary norm?

Theorem: (Moschevitin 2000) – No. There exists a norm $\|\cdot\|$ on $\mathbb{R}^2$ (explicit) and $\vec{\theta}$'s with bap such that $disp(\vec{\theta},\vec{v}_k)$ always lie on the first quadrant.

Observation: If $\vec{p},q$ is bap for some θ, then $gcd\,(p_1,...,p_d,q)=1$.

Claim: If $\vec{v}_k = (\vec{p}_k,q_k)$ and $\vec{v}_{k+1} = (\vec{p}_{k+1},q_{k+1})$ are two consecutive bap for some $\vec{\theta}$, then they are **primitve w.r.t the lattice $\mathbb{Z}^d$** – there exists $\vec{u}_3,...,\vec{u}_d$ such that $\vec{v}_k,\vec{v}_{k+1},\vec{u}_3,...,\vec{u}_d$ is a basis of $\mathbb{Z}^d$.

In the example above: there exists $\vec{u}$ such that $\{\,(1,3,3),(7,22,19),\vec{u}\,\}$ is a basis of $\mathbb{Z}^3$.

Basic question for the paper: What is the typical behavior of the bap $(\vec{v}_k)$ and the ε-app $(\vec{w}_k)$ with respect to their length, direction, arithmetical properties?

Fix $m \in \mathbb{N}$. For $\vec{a} \in \mathbb{Z}_m^{d+1}$ we say that **$\vec{a}$ is primitive (mod m)** if there are no $c, d_1, ..., d_n$ with $c \ne 1$ such that $a_i = cd_i$. Define

$$N_{m,d+1} := \#\{\vec{a} \in \mathbb{Z}_m^{d+1} \,|\, \vec{a}\ is\ primitve\,\}.$$

Theorem 1.1.a. – ***arithmetic properties of bap***

$\|\cdot\|$ norm on $\mathbb{R}^d$. Let $m \in \mathbb{N}$ and let $a_1, a_2, ..., a_d, b \in \mathbb{Z}_m$ such that $(a_1, a_2, ..., a_d, b)$ is primitive (mod m). Then for Lebesgue a.e $\vec{\theta} \in \mathbb{R}^d$

$$\frac{1}{k}\#\left\{1 \le i \le k \,|\, \begin{cases} p_{i,j} = a_j \,(mod\, m) \\ q_i = b\,(mod\, m) \end{cases}\right\} \longrightarrow \frac{1}{N_{m,d+1}}$$

where $(\vec{p}_k, q_k)$ is bap of $\vec{\theta}$ w.r.t $\|\cdot\|$.

Theorem 1.2.a. – ***arithmetic properties of ε-app***

Let $\|\cdot\|$ norm on $\mathbb{R}^d$. Let $\varepsilon > 0$. Let $m \in \mathbb{N}$ and let $a_1, a_2, ..., a_d, b \in \mathbb{Z}_m$ such that $(a_1, a_2, ..., a_d, b)$ is primitive (mod m). Then for Lebesgue a.e $\vec{\theta} \in \mathbb{R}^d$

$$\frac{1}{k}\#\left\{1 \le i \le k \,|\, \begin{cases} P_{i,j} = a_j \,(mod\, m) \\ Q_i = b\,(mod\, m) \end{cases}\right\} \longrightarrow \frac{1}{N_{m,d+1}}$$

where $(\vec{P}_k, Q_k)$ is ε-approximations of $\vec{\theta}$ w.r.t $\|\cdot\|$.

Notice that both 1.1.a and 1.2.a doesn't depend on the norm $\|\cdot\|$.

1.1.a extends Moeckel 1982 for $d = 1$. 1.1.b extends Szusz for $d = 1$.

Theorem 1.1.b. – ***equidistribution of bap displacements***

$\|\cdot\|$ norm on $\mathbb{R}^d$. Then there exists $\mu_{\|\cdot\|}$ a probability measure on $\mathbb{R}^d$ (boundedly supported, absolutely continuous w.r.t Lebesgue, not restriction of Lebesgue), such that for Lebesgue a.e $\vec{\theta}$ we have

$$\left(disp(\vec{\theta}, \vec{v}_k)\right)_{k=0}^{\infty}$$

equidistribute w.r.t $\mu_{\|\cdot\|}$, $(\vec{v}_k)$ is bap of $\vec{\theta}$ w.r.t $\|\cdot\|$

Theorem 1.2.b. – ***equidistribution of ε-displacements***

$\|\cdot\|$ norm on $\mathbb{R}^d$, $\varepsilon > 0$. Then for Lebesgue a.e $\vec{\theta}$ we have that

$$\left(disp(\vec{\theta}, \vec{w}_k)\right)_{k=0}^{\infty}$$

equidistribute w.r.t $Vol_{|B_\varepsilon}$ normelised, where $\vec{w}_k$ is the ε-app of θ w.r.t $\|\cdot\|$.

1.2.a extends Bosma Jager and Wiedijk from 1983. The measure μ on $\mathbb{R}$ in their work is the measure given by the density function

$$F(t) = \begin{cases} \frac{1}{2log(2)} & t \in [-0.5, 0.5] \\ \\ \frac{1}{2log(2)}(1/t - 1) & t \in [-1, -0.5) \cup (0.5, 1] \end{cases}$$

1.2.b new, even for $d = 1$. There had been a little work about direction only.

Short introduction to Lattices:

- $\boldsymbol{L \subseteq \mathbb{R}^n}$ **is a lattice** if there exists linearly-independent $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ such that $L = span_{\mathbb{Z}}(\vec{v}_1, \vec{v}_2 \dots \vec{v}_n) \iff L = h\mathbb{Z}^n$, where $h = (\vec{v}_1, \dots \vec{v}_n)$. In this case, we say that $\boldsymbol{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n}$ **(respectively $\boldsymbol{h}$) is a basis of $\boldsymbol{L}$**.
- claim: If $L = h_1\mathbb{Z}^n = h_2\mathbb{Z}^n$ then $|\det(h_1)| = |\det(h_2)|$.
- Define the $covol(L) := |\det h|$, where h is any matrix satisfying $L = h\mathbb{Z}^n$
- We have the following one-to-one correspondences

$$X_n := \{Lattices\ in\ \mathbb{R}^n\ covol = 1\} \longleftrightarrow SL_n(\mathbb{R})/SL_n(\mathbb{Z})$$

$$g\mathbb{Z}^n \longleftrightarrow g\,SL_n(\mathbb{Z}).$$

- endow X_n with the quotient topology.
- Denote by m_{X_n} the **Haar measure on $\boldsymbol{X_n}$** – the unique measure which is
  - Borel
  - Probability
  - Regular – for any A borel we have that

    $$m_{X_n}(A) = \inf\{m(B)\,|\,B\ open\ with\ A \subset B\} = \sup\{m(K)\,|\,K\ compacy\ with\ K \subset A\}.$$

  - invariant for left $Sl_n(\mathbb{R})$ multiplication. I.e. for all $A \subset X_n$ Borel and all $g \in Sl_n(\mathbb{R})$ we have that $m_{X_n}(A) = m_{X_n}(gA)$.

- $\vec{u} \in \mathbb{Z}^n$ primitive if the coordinates of $\vec{u}$ are co-prime. Denote $\boldsymbol{\mathbb{Z}^n_{prim}}$ the set of primitive vectors on $\mathbb{Z}^n$.
- Define $n = d + 1$.
- Define $\boldsymbol{\pi^{\vec{u}}_{\vec{u}^\perp}} : \mathbb{R}^n \longrightarrow \vec{u}^\perp$ be the orthogonal projection.
  In case $\mathbb{R}^n = \mathbb{R}^d \bigoplus span_{\mathbb{R}}(\vec{u})$ (where $\mathbb{R}^d = \{\vec{x} \in \mathbb{R}^n\,|\,x_n = 0\}$), define $\boldsymbol{\pi^{\vec{u}}_{\mathbb{R}^d}} : \mathbb{R}^n \longrightarrow \mathbb{R}^d$ be the projection to $\mathbb{R}^d$ with kernel $span_{\mathbb{R}}(\vec{u})$.
- Claim: $\pi^{\vec{u}}_{\vec{u}^\perp}(\mathbb{Z}^n)$ and $\pi^{\vec{u}}_{\mathbb{R}^d}(\mathbb{Z}^n)$ are d-dim lattices.
- Question: Fix a measure on primitive vectors in $\mathbb{Z}^n_{prim}$. For example, for $T > 0$, take uniform distribution on $\{\vec{u} \in \mathbb{Z}^n_{prim}\,|\,\|\vec{u}\| \le T\}$. What is the typical behavior of the d-dim lattices of the form $\pi^{\vec{u}}_{\vec{u}^\perp}(\mathbb{Z}^d)$ or $\pi^{\vec{u}}_{\mathbb{R}^d}(\mathbb{Z}^d)$ as $T \to \infty$?
- No really good answers so far, even in this mild version (can also take primitive sets instead of primitive vectors, can also look at projections of more general lattices than $\mathbb{Z}^n$).
- Two ways to simplify:
  1) "simplify the space" (Schmidt) 2) look at more specific collections (WS).

We say two **lattices $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ are homothetic** if there exists $t \neq 0$ such that $L_1 = tL_2$. The homothaty class of each lattice L contains a unique representative in X_n, which we denote by $[L]$.

We say **two lattices L_1 and L_2 are similar** if there exists $t \neq 0$ and $P \in SO_d$ such that $L_1 = P\,t(L_2)$.
We denote by $< L > = \Big[SO_d(\mathbb{R})[L]\Big]$ the similarity class of a lattice L.

Define $\mathcal{S}X_n := SO_n(\mathbb{R})\backslash\Big(SL_n(\mathbb{R})/SL_n(\mathbb{Z})\Big)$ as **the space of Shapes of lattices**.
Define $m_{\mathcal{S}X_n}$ as the push-forward of Haar measure from X_n to $\mathcal{S}X_n$. It is the unique X_n-right-inv Haar probability measure on $\mathcal{S}X_n$.

Specifically, for $\vec{u}$ primitive, $< \pi_{\mathbb{R}^d}^{\vec{u}}(\mathbb{Z}^n) >$ is the class of d-dim lattices which are similar to $\pi_{\mathbb{R}^d}^{\vec{u}}(\mathbb{Z}^n)$.

Theorem (Schmidt 98). *The uniform measures on the finite sets*

$$\{ < \pi_{\mathbb{R}^d}^{\vec{u}}(\mathbb{Z}^n) > \mid \vec{u} \in \mathbb{Z}^n_{prim} \textit{ with } \|\vec{u}\| \leq T \}$$

are equidistributed in $\mathcal{S}X_d$ w.r.t $m_{\mathcal{S}X_d}$ as $T \to \infty$.

Weiss-Shapira: Less simplification of the space, more specific collections.

Theorem 1.1.c equidistribution of bap-projected lattices.

$\|\cdot\|$ norm on $\mathbb{R}^d$. Then there exists $\mu(X_d) = \mu(X_d)_{\|\cdot\|}$ a probability measure on X_d (equivalent to m_{X_d} but different [i.e same null sets]), such that for Lebesgue a.e $\vec{\theta}$ we have

$$\left([\pi_{\mathbb{R}^d}^{\vec{v}_k}(\mathbb{Z}^n)] \right)_{k=0}^{\infty}$$

equidistribute w.r.t $\mu(X_d)$, where $\vec{v}_k$ is bap of $\vec{\theta}$ w.r.t $\|\cdot\|$.

Theorem 1.2.c equidistribution of ε-projected lattices.

$\|\cdot\|$ norm on $\mathbb{R}^d$, $\varepsilon > 0$. Then for Lebesgue a.e $\vec{\theta}$ we have

$$\left([\pi_{\mathbb{R}^d}^{\vec{w}_k}(\mathbb{Z}^n)] \right)_{k=0}^{\infty}$$

equidistribute w.r.t m_{X_d}, where $\vec{w}_k$ is ε-app of $\vec{\theta}$ w.r.t $\|\cdot\|$. Note that this results does not depend on ε or on the norm.

To summarize, we have in addition the following:

Theorem Theorem 1.1. *Theorems 1.1.a 1.1.b and 1.1.c hold jointly.*

Theorem Theorem 1.2. *Theorems 1.2.a 1.2.b and 1.2.c hold jointly.*

Question: Given $\vec{\theta}$ with $(\vec{p}_k, q_k)$ bap, or with $(\vec{P}_k, Q_k)$ ε-app, what can we know about the rate of growth of the denominators (q_k) and (Q_k)?

Theorem: (Khinchine, 1935) $d = 1$ there exists $C > 0$ such that for Lebesgue almost every θ with (q_k, p_k) bap we have

$$\lim_{k\to\infty} q_k^{1/k} = C$$

(Levy, 1936) $C = e^{\pi^2/(12ln(2))}$. This constant is called the Levy-Khinchine constant.

What happens if $d > 1$? – there had been some results in recent years.

Theorem (corollary 3.4). *$\|\cdot\|$ on $\mathbb{R}^d$ and $\varepsilon > 0$. Then there exists $\delta_{\|\cdot\|}$ and $\delta' = \delta'_{\|\cdot\|,\varepsilon}$ such that for Lebesgue a.e $\vec{\theta}$ with $(q_k, \vec{p}_k)$ bap and $(Q_k, \vec{P}_k)$ ε-app we have that*

$$\lim_{k\to\infty} q_k^{1/k} = \delta$$

$$\lim_{k\to\infty} Q_k^{1/k} = \delta'$$

- For bap – improves recent results.
- For ε-app – completely new.

Open question: calculate δ and δ' for specific cases.

What about non-typical cases?

A Field $\mathbb{K}$ is called a **totally real number field** if $\mathbb{K} = \mathbb{Q}[\beta]$ and all roots of the minimal polynomial of β are real.

Example $\mathbb{Q}[\beta]$ where β is one of the 3 real roots of the polynomial $f(x) = x^3 + x^2 - 2x - 1$. So $\mathbb{Q}[\beta] = span_{\mathbb{Q}}(1, \beta, \beta^2)$ if of degree 3.

Claim: If $\vec{\alpha} = (\alpha_1, \dots \alpha_d) \in \mathbb{R}^d$ such that $span_{\mathbb{Q}}(1, \alpha_1, \dots, \alpha_d)$ is a totally real number field, then $\vec{\alpha}$ is Badly-approximable.

Recall: $\vec{\alpha}$ is Badly approximable if there exists a norm $\|\cdot\|$ such that

$$\inf\{\ \|disp(\vec{\alpha}, \vec{v})\| \mid \vec{v} \in \mathbb{Z}^d \times \mathbb{N}\,\} > 0.$$

Theorem 1.5. *Let $\vec{\alpha} = (\alpha_1, \dots \alpha_d) \in \mathbb{R}^d$ such that $span_{\mathbb{Q}}(1, \alpha_1, \dots, \alpha_d)$ is a totally real number field of degree $d+1$ with $d \geq 2$.*

(For example $d = 2$ and $\vec{\alpha} = (\beta, \beta^2)$) as above).

Define

$$\begin{aligned} \varepsilon_0 :&= \inf\{\varepsilon' > 0 \,|\, (\vec{w}_k)_{k=0}\ \varepsilon' - app\ are\ infinite\} \\ &= \liminf_{k\to\infty} \|\, disp\,(\vec{\alpha}, \vec{v}_k)\,\|. \end{aligned}$$

Let $\varepsilon > \varepsilon_0$. Let $(\vec{v}_k)_{k=0}^{\infty} = (\vec{p}_k, q_k)$ be the bap of $\vec{\alpha}$ and $(\vec{w}_k)_{k=0}^{\infty} = (\vec{P}_k, Q_k)$ be the ε-app of $\vec{\alpha}$.

Let $m \in \mathbb{N}$ and let $b_1, \dots, b_d, B \in \mathbb{Z}_m$ such that $(b_1, b_2, \dots, b_d, B)$ is primitive $(\bmod\, m)$.

Then the following holds:

- *(a.1) Assume that* $\|\cdot\|$ *is Euclidean or max. . Then there exists* $c = c(b_1, .., b_d, B) > 0$ *such that*

$$\frac{1}{k}\#\left\{1 \leq i \leq k \,|\, \begin{cases} p_{i,j} = b_j \,(mod\, m) \\ q_i = B \,(mod\, m) \end{cases}\right\} \longrightarrow c$$

- *(a.2) There exists* $c' = c'(b_1, ..., b_d, B) > 0$ *such that*

$$\frac{1}{k}\#\left\{1 \leq i \leq k \,|\, \begin{cases} P_{i,j} = a_j \,(mod\, m) \\ Q_i = b \,(mod\, m) \end{cases}\right\} \longrightarrow c'$$

- *(b.1) Assume that* $\|\cdot\|$ *is Euclidean or max. Then there exists a probability measure* $\mu = \mu_{\|\cdot\|}$ *on* $\mathbb{R}^d$ *such that*

$$\left(disp(\vec{\alpha}, \vec{v}_k)\right)_{k=0}^{\infty}$$

equidistribute w.r.t $\mu_{\|\cdot\|}$.

- *(b.2) There exists a probability measure* $\nu = \nu_{\|\cdot\|}$ *on* $\mathbb{R}^d$ *such that*

$$\left(disp(\vec{\alpha}, \vec{w}_k)\right)_{k=0}^{\infty}$$

equidistribute w.r.t $\nu_{\|\cdot\|}$.

- *(c.1) Assume that* $\|\cdot\|$ *is Euclidean or max. Then there exists a probability measure* $\mu(X_d) = \mu(X_d)_{\|\|}$ *on* X_d *such that*

$$\left(\left[\pi_{\mathbb{R}^d}^{\vec{v}_k}(\mathbb{Z}^n)\right]\right)_{k=0}^{\infty}$$

equidistribute w.r.t $\mu(X_d)$.

- *(c.2) There exists a probability measure* $\nu(X_d) = \nu(X_d)_{\|\|}$ *on* X_d *such that*

$$\left(\left[\pi_{\mathbb{R}^d}^{\vec{w}_k}(\mathbb{Z}^n)\right]\right)_{k=0}^{\infty}$$

equidistribute w.r.t $\nu(X_d)$.

Furthermore, the support of the measures above are null sets w.r.t Vol *and* m_{X_d} *and we have that* $a.1, b.1, c.1$ *and* $a.2, b.2, c.2$ *hold jointly.*