GEOMETRIC AND ARITHMETIC ASPECTS OF APPROXIMATION VECTORS – Temperedness and lack thereof in Case I

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Goal 1: explaining why $\mathcal{B} \subset S_{r_0}$ is tempered.

Goal 2: explaining why $\mathcal{S}_\varepsilon \subset S_{r_0}$ is not tempered.

Goal 3: explaining how to use (and overcome) the above to get the required equidistribution.

Reminders:

"case 1" – Lebesgue typical case (both for best and ε app).

"case 2" – specific θ's.

Today's talk: we only deal with case 1.

$$S_{r_0} := \{\Lambda \in \mathcal{X}_n \,|\, \#(\Lambda_{prim} \cap D_{r_0}) \geq 1\}$$
$$S_{r_0}^{\#} := \{\Lambda \in \mathcal{X}_n \,|\, \#(\Lambda_{prim} \cap D_{r_0}) = 1\}$$

Recall: S_{r_0} is $m_{\mathcal{X}_n}$-reasonable (Theorem 8.6, Alon's talk).

Define $\mathcal{B} \subset S_{r_0}$ in the following way:

For $\Lambda \in S_{r_0}^{\#}$ define
$$v(\Lambda) := \Lambda \cap D_{r_0} \in \mathbb{R}^n$$
(well defined by definition of $S_{r_0}^{\#}$), and then define (a bit confusing notation)
$$v_\Lambda := \pi_{\mathbb{R}^d}(v(\Lambda)) \in \mathbb{R}^d$$

I.e., v_Λ is the horizontal component of $\Lambda \cap D_{r_0}$.

Define for $\Lambda \in S_{r_0}^{\#}$
$$r(\Lambda) := \|\vec{v}_\Lambda\|\,,$$

I.e. the distance of $\Lambda \cap D_{r_0}$ from the vertical axis. Now define

$$\mathcal{B} := \{\Lambda \in S_{r_0}^{\#} \,|\, C_{r(\Lambda)} \cap \Lambda_{prim} = \{\, \pm\vec{v}(\Lambda)\}\,\}.$$

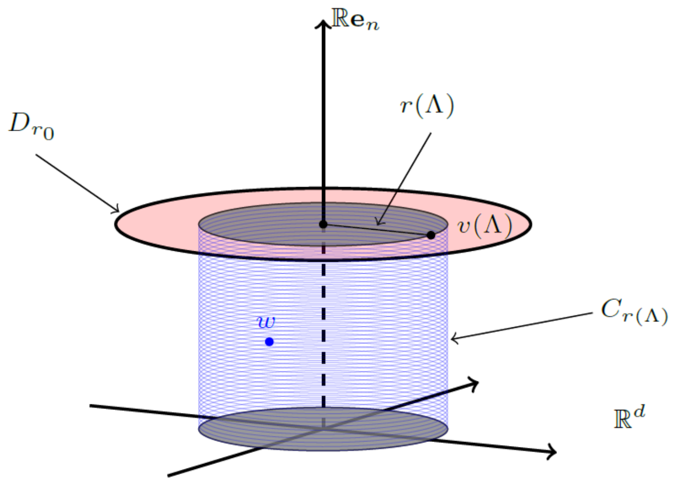

FIGURE 2. If the cylinder $C_{r(\Lambda)}$ defined by the unique vector $v(\Lambda) \in \Lambda \cap D_{r_0}$ contains another lattice point w, then $\Lambda \notin \mathcal{B}$.

- S_{r_0} – at least one primitive vector in D_{r_0}
- $S_{r_0}^{\#}$ – exactly one primitive vector in D_{r_0}
- $\mathcal{B}$ – exactly one primitive vector in D_{r_0} + no primitive vectors in the cylinder below him

The set $\mathcal{B}$ will detect best approximations (v_k). Note that $\mathcal{B}$ is norm dependent.

Theorem (Lemma 9.1). *The set $\mathcal{B}$ is open in S_{r_0}.*

We use this Lemma to prove the following:

Theorem (Lemma 9.2). *For any norm, the set $\mathcal{B}$ is $\mu_{S_{r_0}} - JM$ and $\mu_{S_{r_0}}(\mathcal{B}) > 0$.*

Recall from chapter 5 (Rotem's talk) – if $\mathcal{B} \subset S_{r_0}$ is $\mu_{S_{r_0}}$-JM with positive measure which is tempered, then if $\Lambda \in \mathcal{X}_n$ is (a_t, μ)-generic then Λ is also $(a_t, \mu_{s_{r_0}}|_{\mathcal{B}})$-generic. So we want to prove the following:

Theorem (Proposition 9.8). *The set $\mathcal{B}$ is tempered.*

Recall that $\mathcal{B}$ is tempered if there exists $M \in \mathbb{N}$ such that for $\mu_{S_{r_0}}$ a.e Λ we have that

$$\#\{\, t \in [0,1] \,|\, a_t\Lambda \in \mathcal{B}\,\} < M.$$

Proof:

- For $r > 0$ define $C_r(e) := \{(\vec{x}, c) \in \mathbb{R}^n \mid \|\vec{x}\| \le r, c \in [-e, e]\,\}$.

- Choose M large enough such that any $r > 0$ any $M+1$ points in $C_r(e^d)$ contains two points $(\vec{x}, c)$ and $(\vec{x}', c')$ with
  - $\|\vec{x} - \vec{x}'\| < r/3$
  - $|c - c'| < 1$

  Why there exists such M for $r = 1$ is obvious by compactness. Then apply a linear transformation which dilates the horizontal subspace.
- We claim that $\mathcal{B}$ is M-tempered (in a stronger sense, where the definition holds for all $\Lambda \in \mathcal{B}$).
- Indeed, suppose by contradiction that there exists $\Lambda \in \mathcal{B}$ and $0 = t_0 < t_1 ... < t_M \leq 1$ such that $a_{t_j} \in \mathcal{B}$ for all j.
- Define $(w_j, 1) := v\,(a_{t_j}\Lambda) \in \mathbb{R}^n$. I.e. we have that
  - $(w_j, 1) \in a_{t_j}\Lambda_{prim} \cap D_{r_0}$
  - $a_{t_j}\Lambda \cap C_{\|w_j\|}(1)^0 = \{\vec{0}\}$.
- For each j, apply a_{-t_j} on the above. So for all j there exists $x_j \in \mathbb{R}^d$
  - $(x_j, e^{dt_j}) \in \Lambda_{prim} \cap a_{t_j}\, D_{r_0}$
  - $\Lambda_{prim} \cap C^o_{\|\vec{x}_j\|}(e^{t_j}) = \{\vec{0}\}$, for some $\vec{x}_j \in \mathbb{R}^d$.
- So we have $\|x_0\| \geq \|x_1\| \geq .... \geq \|x_m\|$ hence $(x_j, e^{dt_j}) \in C_{\|x_0\|}(e^d)$ for all j.
- So by the properties previously discussed, there exists $j < i$ such that
  - $\|x_i - x_j\| < \frac{\|x_0\|}{3}$
  - $|e^{dt_j} - e^{dt_i}| < 1$.
- So the difference vector satisfies

  $$(x_j, e^{dt_j}) - (x_i, e^{dt_i}) \in \Lambda_{prim} \cap C_{\|x_0\|/3}(1) \subseteq \Lambda_{prim} \cap C_{r(\Lambda)}(1),$$

  contradicting the assumption that $\Lambda \in \mathcal{B}$ (if the difference is not primitive then just shorten it).

remark: Notice that the proof shows that there exists $M \in \mathbb{N}$ such that for *any* $x \in S_{r_0}$ we have that

$$\#\,\{t \in [0,1] \,|\, a_t x \in \mathcal{B}\} < M.$$

remark: so the reason why we need to differ between best approximations and ε approximations is that $\mathcal{B}$ is useful for best, but not useful for ε (Chen's talk).

Theorem (Lemma 9.9). *For any $0 < r \leq r_0$, the sets S_r are $\mu_{S_{r_0}} - JM$.*

(weird that this lemma is with S_r but next one is with S_ε).

Theorem (Proposition 9.10). *For any $d > 1$, for any $\varepsilon > 0$ and any norm, S_ε is not tempered. For $d = 1$, and any $\varepsilon > 0$, S_ε is tempered.*

Proof for $d > 1$:

- Let $\varepsilon > 0$ and let $M \in \mathbb{N}$. Let $\Lambda \in \mathcal{X}_n$ such that
  - $(2M)^{-1}\,\vec{e}_n \in \Lambda$
  - $(x, 1) \in \Lambda$ such that $\|x\| < \varepsilon/2$.

- Then for every $j \in \{0, 1, ..., M\}$ we have that $y_j := (x, 1 + \frac{j}{2M}) \in \Lambda$. (NEEDS TO BE PRIMITIVE. THROUGH THE TALK WE DISCUSSED HOW TO OVERCOME THIS OBSTACLE).
- Let $t_j := \dfrac{1}{d}\, log\,(1 + \frac{j}{2M}) \in [0, 1)$.
- So we have that $a_{t_j}(y_j) = ((1 + \frac{j}{2m})^{1/d} x, 1) \in a_{t_j}\Lambda$ for all j. (NEEDS TO BE PRIMITVE)
- So $a_{t_j}\Lambda \in S_\varepsilon$ for all j.
- As we started with M arbitrary, we get that S_ε is not tempered (ONLY IF WE ASSUME THE STRONG DEFINITION OF TEMPERATENESS).

Let $\mathcal{X}_n^{\mathbb{A}}$ be the adelic space, let $\pi : \mathcal{X}_n^{\mathbb{A}} \longrightarrow \mathcal{X}_n$ be the projection, and let $m_{\mathcal{X}_n^{\mathbb{A}}}$ be the Haar measure on the adelic space discussed in chapter 7.

For $\theta \in \mathbb{R}^d$ define $\widetilde{\Lambda}_\theta := \big(u(-\theta), e_f\big)SL_n(\mathbb{Q}) \in \mathcal{X}_n^{\mathbb{A}}$, and define further:

$$\widetilde{S}_{r_0} := \pi^{-1}\big(S_{r_0}\big) \quad \widetilde{S}_\varepsilon := \pi^{-1}\big(S_\varepsilon\big) \quad \widetilde{\mathcal{B}} := \pi^{-1}\big(\mathcal{B}\big).$$

In chapter 13 we will conclude (using results obtained in chapter 5) that

- $\widetilde{\Lambda}_\theta$ is $(a_t, \mu_{\widetilde{S}_{r_0}}|_{\widetilde{\mathcal{B}}})$-generic $\longrightarrow$ equidist best approximations (triples)
- $\widetilde{\Lambda}_\theta$ is $(a_t, \mu_{\widetilde{S}_{r_0}}|_{\widetilde{S}_\varepsilon})$-generic $\longrightarrow$ equidist of ε approximations (triples)

So in order to get there, we prove the following:

Theorem (Proposition 12.1). *For Lebesgue a.e. θ we have that $\widetilde{\Lambda}_\theta$ is $(a_t, \mu_{\widetilde{S}_{r_0}})$-generic. Moreover, it is $(a_t, \mu_{\widetilde{S}_{r_0}}|_{\widetilde{\mathcal{B}}})$-generic, as well as $(a_t, \mu_{\widetilde{S}_{r_0}}|_{\widetilde{S}_\varepsilon})$-generic for any $\varepsilon \in (0, r_0)$.*

Recall Theorem 5.11 from Chen's talk: If S_{r_0} is a μ-reasonable cross section (Thm 8.6) then we have the following:

- if $\mathcal{B} \subset S_{r_0}$ is $\mu_{S_{r_0}}$-JM with positive measure (Lemma 9.2) which is tempered (proposition 9.8), then if $\Lambda \in \mathcal{X}_n$ is (a_t, μ)-generic then Λ is also $(a_t, \mu_{s_{r_0}}|_{\mathcal{B}})$-generic.
- If $\Lambda \notin \Delta^{\mathbb{R}}_{S_{r_0}}$ then if $\Lambda \in \mathcal{X}_n$ is (a_t, μ)-generic then Λ is also $(a_t, \mu_{s_{r_0}}|_{S_\varepsilon})$-generic for any $\varepsilon < r_0$.
- Recall:

$$\begin{aligned}\Delta^{\mathbb{R}}_{S_{r_0}} :=& \{a_t\Lambda \,|\, t \in \mathbb{R} \;\&\; \exists\, \delta \;s.t\; \Lambda \in \Delta_{S_{r_0},\delta}\,\} \\ =& \{a_t\Lambda \,|\, t \in \mathbb{R} \;\&\; \exists\, \delta \;s.t\; \forall \varepsilon > 0 \;\limsup_{T\to\infty} \frac{1}{T} N(\Lambda, T, S_{r_0,<\varepsilon}) > \delta\,\}\end{aligned}$$

  where $S_{r_0,<\varepsilon} := \{\Lambda \in S_{r_0} \,|\, min(t > 0 \,|\, a_t\Lambda \in S_{r_0}) < \varepsilon\}$.

With a little work and some famous results about lattices, we have that for Lebesgue a.e Θ we have that Λ_Θ is $(a_t, m_{\mathcal{X}_n})$-generic.

So by lifting properties + proposition 9.8 the $(a_t, \mu_{\widetilde{S}_{r_0}}|_{\widetilde{\mathcal{B}}})$-generic case is done.

By proposition 9.10, the difficulty is to prove the $(a_t, \mu_{\widetilde{S}_{r_0}}|_{\widetilde{S}_\varepsilon})$-generic part. I.e, to prove that for $m_{\mathcal{X}_n}$ a.e Λ satisfies that $\Lambda \notin \Delta^{\mathbb{R}}_{S_{r_0}}$. This is done by propositions 12.3, 12.4 and 12.5.