Chapter 2

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Motivation 1: passing from a discrete measure-preserving dynamical system to a continuous measure-preserving one – suspension

Motivation 2: understanding the typical behaviour within this new constructed flow

First recall the conditional expectation Theorem: if $(X, \mathcal{B}, \mu)$ is a probability space $\mathcal{A} \subset \mathcal{B}$ is a Borell sub-σ-algebra, then there exists a map $\mathbb{E}(\cdot\,|\mathcal{A}) : L^1(X, \mathcal{B}, \mu) \longrightarrow L^1(X, \mathcal{A}, \mu)$ such that for any φ which $\mathcal{B}$-measurable and any $A \in \mathcal{A}$ we have that

$$\int_A \varphi \, d\mu = \int_A \mathbb{E}(\varphi|A) d\mu$$

and $\mathbb{E}(\varphi|\mathcal{A})$ is unique up to null-set.

Definition of suspension

Let $(B, [\mathcal{B}], \beta, T)$ be a probability ergodic measure-preserving system such that $(B, [\mathcal{B}], \beta)$ is a Lebesgue probability space – isomorphic (up to null sets) to an interval with Lebesgue measure, a finite or countable set of atoms, or a combination (disjoint union) of both.

roof function – let $\tau : B \to \mathbb{R}_{>0}$ be an integrable function.

General case – there is also a compact metrizable group M, and a function $\tau_M : M \to \mathbb{R}_{>0}$.

For $p \in \mathbb{N}$ and $\vec{b} \in B$ define a map

$$\tau_p(\vec{b}) := \tau(\vec{b}) + \tau\big(T^1(\vec{b})\big) + \ ... \ + \tau\big(T^{p-1}(\vec{b})\big) \tag{1}$$

By Birkhoff ergodic Thoerem, for β a.e $\vec{b} \in B$ we have that

$$\lim_{p\to\infty} \frac{1}{p}\tau_p(\vec{b}) = \int_B \tau \, d\beta > 0 \tag{2}$$

I.e.

$$\lim_{p\to\infty} \tau_p(\vec{b}) = \infty.$$

For $l \in \mathbb{R}_{\geq 0}$ we define a map

$$p_l : B \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{N} \tag{3}$$

$$p_l(b, k) := \max\{p \in \mathbb{N} \,|\, k + l - \tau_p(\vec{b}) \geq 0\,\} \tag{4}$$

which is well defined for a.e $b \in B$ by (2).

Define the **suspension of the dynamical system $(B, [\mathcal{B}], \beta, T)$ of the roof function τ** to be the flow $\{T_l^\tau \,|\, l \in \mathbb{R}_{\geq 0}\,\}$ of the dynamical system $(B^\tau, [\mathcal{B}]^\tau, \beta^\tau)$ defined by the followings:

- $B^\tau := \{(\vec{b}, k) \,|\, 0 \leq k \leq \tau(\vec{b})\,\}$
- $[\mathcal{B}]^\tau = [\mathcal{B}] \otimes LEB_{\mathbb{R}}$ i.e. the product σ-algebra of $[\mathcal{B}]$ and the Lebesgue σ-algebra of $\mathbb{R}$. (note that we do not need τ for this notation)

- $\beta^\tau = \left(\beta \times Leb_{\mathbb{R}>0}\right)_{|B^\tau}$ **normalized**.
-
$$T_l^\tau(\vec{b},k) := \left(\,T^{\,p_l(\vec{b},k)}(\vec{b})\,,\; k+l-\tau_{p_l(\vec{b},k)}(\vec{b})\,\right)$$
- In order to show T_l^τ is map from B^τ to B^τ we need to know that
$$k+l-\tau_{p_l(b,k)}(\vec{b}) \leq \tau\big(T^{\,p_l(\vec{b},k)}(\vec{b})\big)$$
which holds exactly by the definition of the index $p_l(b,k)$.
- Example: $\tau(\vec{b}) = c$ for all $\vec{b}$ for some $c>0$. So
$$p_l(\vec{b},k) = \max\left\{p\in\mathbb{N} \,|\, k+l-cp \geq 0\,\right\} = \left\lfloor \frac{k+l}{c} \right\rfloor \tag{5}$$
$\longrightarrow$ not dependent on $\vec{b}$.

claim: This is a flow. I.e. for all l_1 and l_2 we have $T_{l_1}^\tau \circ T_{l_2}^\tau = T_{l_1+l_2}^\tau$.

Lemma 2.2. *The flow* $\{\,T_l^\tau \,|\, l\in\mathbb{R}_{\geq 0}\,\}$ *preserves the measure* β^τ.

Motivation: use suspension in a more specific settings to get deeper results.

Let $(A,[\mathcal{A}],\alpha)$ be a Lebesgue probability space, and define

- $B := A^{\mathbb{N}}$
- $[\mathcal{B}] := [\mathcal{A}]^{\otimes\mathbb{N}}$
- $\beta := \alpha^{\mathbb{N}}$

and let $T: B\to B$ be the left shift – if $\vec{b} = (b_n)_{n=0}^\infty \in B$ then $T(\vec{b}) = (b_{n+1})_{n=0}^\infty$.

Notice that in this case $(B,[\mathcal{B}],\beta)$ is also a Lebesgue probability space, and that the left shift T is measure preserving and ergodic.

Let $\tau: B\to\mathbb{R}_{>0}$ be an integrable function.

Let $(B^\tau,[\mathcal{B}]^\tau,\beta^\tau,\{T_l^\tau\}_{l\geq 0})$ be the suspension as before.

$(T_l^\tau)^{-1}([\mathcal{B}]^\tau)$ is a σ-algebra which satisfies $(T_l^\tau)^{-1}([\mathcal{B}]^\tau)^\tau \subseteq [\mathcal{B}]^\tau$.

Motivation: compute explicitly $\mathbb{E}\left(\,\cdot\,\,|\,(T_l^\tau)^{-1}([\mathcal{B}]^\tau)\right)$.

Recall the conditional expectation Theorem: There exists a map
$\mathbb{E}\left(\,\cdot\,\,|\,(T_l^\tau)^{-1}([\mathcal{B}]^\tau)\right) \,:\, L^1(B^\tau,[\mathcal{B}]^\tau,\beta^\tau) \longrightarrow L^1(B^\tau,(T_l^\tau)^{-1}([\mathcal{B}]^\tau),\beta^\tau)$ such that for any φ which is $[\mathcal{B}]^\tau$-measurable and any $A\in(T_l^\tau)^{-1}([\mathcal{B}]^\tau)$ we have that
$$\int_A \varphi\, d\beta^\tau = \int_A \mathbb{E}(\varphi\,|\,(T_l^\tau)^{-1}([\mathcal{B}]^\tau))d\beta^\tau$$
and $\mathbb{E}(\varphi|(T_l^\tau)^{-1}([\mathcal{B}]^\tau))$ is unique up to null-set.

Assume from now on l is fixed.

The following definitions appear in the paper with an l index, which we omit for convenience.

For $q\in\mathbb{N}$ and $\vec{a},\vec{b}\in B = A^{\mathbb{N}}$ we denote $\vec{a}[q] := (a_{q-1},a_{q-2},...,a_1,a_0)$, and by $\vec{a}\,[q]\,\vec{b}\in B$ the concatenated word
$$\vec{a}\,[q]\,\vec{b} := (a_{q-1},a_{q-2},...,a_1,a_0,b_0,b_1,b_2...)$$

For $(\vec{b}, k)$ define maps in the following way

$$\widetilde{q}_{\vec{b},k} : B \longrightarrow \mathbb{N} \tag{6}$$

$$\widetilde{q}_{\vec{b},k}(\vec{a}) := \min\{q \in \mathbb{N} \mid k - l + \tau_q(\vec{a}\,[q]\,\vec{b}) \geq 0\,\} \tag{7}$$

Example: $\tau(\vec{b}) = c$ for all $\vec{b}$. So we have

$$\widetilde{q}_{\vec{b},k}(\vec{a}) := \min\{q \in \mathbb{N} \mid k - l + cq \geq 0\,\} = \left\lceil \frac{l-k}{c} \right\rceil$$

$\longrightarrow$ independent on $\vec{a}$ and $\vec{b}$.

$$\widetilde{h}_{\vec{b},k} : B \longrightarrow B^{\tau} \tag{8}$$

$$\widetilde{h}_{\vec{b},k}(\vec{a}) := \big(\,\vec{a}\,[\widetilde{q}_{\vec{b},k}(\vec{a})\,]\,\vec{b}\;,\; k - l + \tau_{\widetilde{q}_{\vec{b},k}(\vec{a})}(\vec{a}\,[\widetilde{q}_{\vec{b},k}(\vec{a})\,]\,\vec{b})\,\big) \tag{9}$$

By Birkhoff ergodic Thoerem, for β a.e $\vec{a}, \vec{b} \in B$ we have that

$$\lim_{q\to\infty} \frac{1}{q}\tau_q(\vec{a}\,[q]\,\vec{b}) = \int_B \tau\, d\beta > 0 \tag{10}$$

hence $\widetilde{q}_{\vec{b},k}(\vec{a})$ and $\widetilde{h}_{\vec{b},k}(\vec{a})$ is well defined for a.e $\vec{a}, \vec{b} \in B$.

Claim:

$$\{\,\widetilde{h}_{\vec{b},k}(\vec{a}) \,|\, \vec{a} \in B\,\} = (T_l^{\tau})^{-1}(\vec{b}, k) \tag{11}$$

$$\{\,\widetilde{h}_{T_l^{\tau}(\vec{b},k)}(\vec{a}) \,|\, \vec{a} \in B\,\} = \{\,(b', k') \,|\, T_l^{\tau}(b', k') = T_l^{\tau}(b, k)\,\} = [(\vec{b}, k)]_{(T_l^{\tau})^{-1}([\mathcal{B}]^{\tau})} \tag{12}$$

Proposition 2.3. ***The law of last jump:*** *For any positive $[\mathcal{B}]^{\tau}$-measurable function φ on and for β^{τ}-a.e. $(\vec{b}, k) \in B^{\tau}$ we have*

$$\mathbb{E}\big(\varphi \,|\, (T_l^{\tau})^{-1}([\mathcal{B}]^{\tau})\big)(\vec{b}, k) = \int_B \varphi\big(\widetilde{h}_{T_l^{\tau}(\vec{b},k)}(\vec{a})\big)\, d\beta(\vec{a})$$

I.e. if $(\vec{b}, k) = T_l^{\tau}\big(\,(a_{N-1}, a_{N-2}, \ldots, a_1, a_0, \vec{b})\,,\, x\,\big)$ then $N = \widetilde{q}_{T_l^{\tau}(\vec{b},k)}(\vec{a})$ and statistically each digit of a_i is chosen randomly with respect to the measure α.

proof of law of last jump:

Need to show that for any $A \in (T_l^{\tau})^{-1}([\mathcal{B}]^{\tau})$ we have

$$\int_A \varphi(\vec{b}, k)\, d\beta^{\tau}(\vec{b}, k) = \int_A \int_B \varphi\big(\widetilde{h}_{T_l^{\tau}(\vec{b},k)}(\vec{a})\big)\, d\beta(\vec{a})\, d\beta^{\tau}(\vec{b}, k) \tag{13}$$

I.e. enough to show that for any ψ positive $[\mathcal{B}]^{\tau}$-measurable we have that

$$\int_{B^{\tau}} \psi\big(T_l^{\tau}(\vec{b}, k)\big)\varphi(\vec{b}, k)\, d\beta^{\tau}(\vec{b}, k) = \int_{B^{\tau}} \psi\big(T_l^{\tau}(\vec{b}, k)\big)\Big(\int_B \varphi\big(\widetilde{h}_{T_l^{\tau}(\vec{b},k)}(\vec{a})\big)\, d\beta(\vec{a})\Big)\, d\beta^{\tau}(\vec{b}, k). \tag{14}$$

Define the LHS of (14) as G and Notice that

$$
\begin{aligned}
G &= \int_{B^\tau} \psi\big(T^{\,p_l(\vec{b},k)}(\vec{b})\,,\, k+l-\tau_{p_l(\vec{b},k)}(\vec{b})\,\big)\ \varphi(b,k)\, d\beta^\tau(\vec{b},k) \\
&= \sum_{p=0}^{\infty} \int_{B^\tau} \mathbb{1}_{\{\,p_l(\vec{b},k)=p\,\}}(\vec{b},k)\ \psi\big(T^{\,p}(\vec{b})\,,\, k+l-\tau_p(\vec{b})\,\big)\ \varphi(b,k)\, d\beta(\vec{b})\, dk
\end{aligned} \tag{15}
$$

Write $(b',k') := (\,T^{\,p}(\vec{b})\,,\, k+l-\tau_p(\vec{b})\,) = f(\vec{b},k)$, and define

$$
D(b',k',p) : = \{\vec{a} \in B \,|\, \widetilde{q}_{\,b',k'}(\vec{a}) = p\} \tag{16}
$$

By this,

$$
\{\, f^{-1}(b',k') \mid p_l(f^{-1}(b',k')) = p\,\} = \big\{\, (\vec{a}\,[p]\,b'\,,\ k'-l+\tau_p(\vec{a}\,[p]\,b')) \,|\, \vec{a} \in D(b',k',p)\,\big\}.
$$

So using a change of variables we have that

$$
\begin{aligned}
G &= \sum_{p=0}^{\infty} \int_{B^\tau} \psi\big(T^{\,p}(\vec{b})\,,\, k+l-\tau_p(\vec{b})\big)\Big(\mathbb{1}_{\{\,p(\vec{b},k)=p\,\}}(\vec{b},k)\ \ \varphi(b,k)\Big)\, d\beta(\vec{b})\, dk \\
&= \sum_{p=0}^{\infty} \int_{B^\tau} \psi(b',k')\,\Big(\int_{D(b',k',p)} \varphi\big(\vec{a}\,[p]\,b'\,,\ k'-l+\tau_p(\vec{a}\,[p]\,b')\big)\, d\beta(\vec{a})\Big)\, d\beta(b')dk' \\
&= \int_{B^\tau} \psi(b',k')\,\Big(\sum_{p=0}^{\infty} \int_{D(b',k',p)} \varphi\big(\vec{a}\,[p]\,b'\,,\ k'-l+\tau_p(\vec{a}\,[p]\,b')\big)\, d\beta(\vec{a})\Big)\, d\beta(b')dk' \\
&= \int_{B^\tau} \psi(b',k')\,\Big(\int_B \varphi\big(\widetilde{h}_{\,b',k'}(\vec{a})\big)\, d\beta(\vec{a})\Big)\, d\beta^\tau(b',k') \\
&= \int_{B^\tau} \psi(T_l^\tau(b',k'))\,\Big(\int_B \varphi\big(\widetilde{h}_{\,T_l^\tau(\vec{b},k)}(\vec{a})\big)\, d\beta(\vec{a})\Big)\, d\beta^\tau(b',k')
\end{aligned} \tag{17}
$$

where the last equality holds since T_l^τ preserves the measure β^τ.

$$
\widetilde{h}_{\,\vec{b},k}(\vec{a}) := \big(\,(\vec{a}\,[\widetilde{q}_{\vec{b},k}(\vec{a})\,]\,\vec{b})\,\ ,\, k-l+\tau_{\widetilde{q}_{\vec{b},k}(\vec{a})}(\vec{a}\,[\widetilde{q}_{\vec{b},k}(\vec{a})\,]\,\vec{b})\,\big) \tag{18}
$$