Oseledets' Theorem

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These seminar notes closely follow Simion Filip's published notes on the multiplicative ergodic theorem, and contain ambiguities and inaccuracies.

1 Statement of the Theorem - intuition and motivation (1h session)

1.1 Reminder

Definition 1.1. A *probability preserving system* (p.p.s) $(\Omega, \mathcal{B}, \mu, T)$ is a separable, second-countable metric space Ω with $\mathcal{B}$ it's Borel σ-algebra, μ a probability measure on Ω and $T : \Omega \to \Omega$ measurable such that μ is T invariant ($T_*\mu = \mu$ or $\forall A \in \mathcal{B}; \mu(T^{-1}(A)) = \mu(A)$)

Definition 1.2. A p.p.s $(\Omega, \mathcal{B}, \mu, T)$ is *ergodic* if measurable T-invariant sets are either full or null : $T^{-1}(A) = A \Rightarrow \mu(A) \in \{0, 1\}$.

Theorem 1.3 (Birkhoff Pointwise Ergodic Theorem). *Let $(\Omega, \mathcal{B}, \mu, T)$ an ergodic p.p.s, $f : \Omega \to \mathbb{R}$ with $f^+ \in L_1(\Omega, \mu)$ ($f = f^+ + f^-$ with $f^+ = \max(f, 0), f^- = \min(f, 0)$) then for μ-a.e $\omega \in \Omega$:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(T^i(\omega)) = \int_\Omega f d\mu$$

Where the limit is allowed to be $-\infty$.

Remark 1.4 (Kind of multiplicative theorem). In the same setting, take $g = \exp(f)$ then for a.e $\omega \in \Omega$ we have:

$$\lim_{N\to\infty}(g(\omega)g(T\omega)\dots g(T^{N-1}(\omega))^{1/N} = \exp\left(\int_\Omega f d\mu\right)$$

1.2 Stating the theorem

Example 1.5. Let M a smooth manifold of $\dim = n$, $F : M \to M$ a smooth diffeomorphism preserving a measure μ on M. For every point in $p \in M$, the smooth structure of M defines a vector space called the tangent space T_pM (directions of all paths on M through p). The idea of tangent spaces attached to every point in M is given the form of the *tangent bundle*

$$TM = \coprod_{p\in M}\{p\} \times T_pM$$

so an element of TM can be thought of as a pair (p, v) with v a direction on T_pM, and a natural projection $\pi : TM \to M$ by $\pi(x, v) = x$. TM has the form of a $2n$-dimensional manifold: if $\{(U_\alpha, \phi_\alpha)\}$ is an atlas of charts ($\phi_\alpha : U_\alpha \to R^n$ are diffeomorphisms) we take:

$$\tilde{\phi}_\alpha : \pi^{-1}(U_\alpha) \to R^{2n} \text{ by } \tilde{\phi}_\alpha(x, v) = (\phi_\alpha(x), v)$$

These maps provide the topology and the smooth structure on TM - A subset $A \subset TM$ is open if and only if $\tilde{\phi}_\alpha(A \cap \pi^{-1}(U_\alpha))$ is open for every α.
F induces a map on the tangent bundle. For $p \in M$:

$$D_pF : T_pM \to T_{F(p)}M \text{ is a linear map}$$

If we take a ball at the tangent space T_pM We can apply DF iteratively and see in what directions it has various growth rates. At each iteration the set moves to a point $F^N(p)$ and it's vector space shape is changed by the m-fold derivative DF^N (drawing). The chain rule implemented by the N-fold composition:

$$D_{F^{N-1}(p)}F \circ \cdots \circ D_pF : T_pM \to T_{F^N(p)}M$$

This is an example of a problem answered by the Oseledets Multiplicative Ergodic theorem. Unlike the Birkhoff multiplicative variant, here our multiplicative action lies in the extension of our space M, μ to the tangent bundle of M.

To formalize the idea of vector spaces parametrized by some manifold or probability space we introduce vector bundles, and consider TM as a prototypical example.

Definition 1.6 (vector bundle)**.** A vector bundle $V \to (\Omega, \mathcal{B}, \mu, T)$ of dimension d is a space V with a continuous surjection $\pi : V \to \Omega$ such that for every $\omega \in \Omega$:

1. the fiber $V_\omega := \pi^{-1}(\{\omega\})$ is a d-dimensional vector space.
2. $\exists \omega \in U \subset \Omega$ open neighborhood and a homeomorphism $\varphi : U \times \mathbb{R}^d \to \pi^{-1}(U)$ such that for all $x \in U$:
   - $\forall v \in \mathbb{R}^d; (\pi \circ \varphi)(x, v) = x$
   - $v \mapsto \varphi(x, v)$ is a linear isomorphism $\mathbb{R}^d \to \pi^{-1}(\{x\})$.

V is called the *total space*, Ω the *base space*, (U, φ) is a *local trivialization* of the vector bundle.

The action of T on the base space Ω can be lifted to an action on V provided that on each fiber the action would be a linear map such that a composition compatibility criterion is met. We introduce a general concept of a system of compatible linear maps and use it to define the lift of T.

Definition 1.7 (cocycle)**.** A *cocycle* of (Ω, T) is a map $\alpha : \mathbb{Z}_{\geq 0} \times \Omega \to GL(d, \mathbb{R})$ for which the following identity holds for all $m, n \in \mathbb{Z}_{\geq 0}$:

$$\alpha(m + n, x) = \alpha(m, T^n x) \circ \alpha(n, x) \tag{1}$$

With $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$. For invertible T, we define the cocycle as a map $\alpha : \mathbb{Z} \times \Omega \to GL(d, \mathbb{R})$ with the condition that identity (1) holds for all $m, n \in \mathbb{Z}$.

Definition 1.8. A vector bundle $V \to (\Omega, \mathcal{B}, \mu, T)$ is *equipped with a cocycle* α if the action of T lifts to V by the linear maps of α. That is, if there is a map $\overline{T} : V \to V$ such that for a.e ω, every $v \in V_\omega$ and every $m \in \mathbb{Z}_{\geq 0}$:

$$\overline{T}^m(\omega, v) \;=\; (T\omega, \alpha(m, \omega)v)$$

So fixing bases for the local trivializations, we obtain linear maps:

$$(T^m)_\omega : V_\omega \to V_{T^m\omega}$$

such that the cocycle identity holds:

$$\forall v \in V_\omega \; \forall m, n \in \mathbb{Z}_{\geq 0};\; (T^{m+n})_\omega v = T^m_{T^n\omega} \circ T^n_\omega v$$

Which by iterative applications can also be written (similarly to the differential chain rule) as the condition:

$$T^N_\omega = T_{T^{N-1}\omega} \circ \cdots \circ T_\omega$$

Remark 1.9.

1. When the context is clear we remove the ω subscript.

2. When T is invertible, we use a cocycle defined over $\mathbb{Z}$ and the respective conditions.

Definition 1.10. A set $W \subset V$ is a subbundle of dimension $d' \leq d$ if the same vector bundle definition holds for W with d' as the dimension of the model real vector space. We denote W_ω the fiber over ω in W. W is T-*invariant* if for all ω, T_ω maps W_ω to $W_{T\omega}$ ($T_\omega W_\omega = W_{T\omega}$)

Example 1.11 (To motivate the result a bit more and shed some light on the statement's notation)**.** Consider a single matrix A acting on $\mathbb{R}^n$. Assume that A has positive eigenvalues $e^{\lambda_1} > \cdots > e^{\lambda_k}$ (multiplicities allowed). For $v \in \mathbb{R}^n$ we consider the behavior of $\|A^N v\|$ as N gets large. Let E_{λ_i} the eigenspace corresponding to eigenvalue e^{λ_i}, and take: $E^{\leq \lambda_i} := E_{\lambda_k} \oplus \cdots \oplus E_{\lambda_i}$. Taking $v \in E^{\leq \lambda_i} \setminus E^{\leq \lambda_{i+1}}$ and assume $\lambda_i \geq 0$, our term would have growth rate of $e^{N\lambda_i}$ and so:

$$\frac{1}{N} \log \|A^N v\| \to \lambda_i$$

Analyzing growth rates we could also recover the filtration

$$E^{\leq \lambda_k} \subset \cdots \subset E^{\leq \lambda_1} = \mathbb{R}^n$$

Theorem 1.12 (Oseledets Multiplicative Ergodic Theorem)**.** *Suppose* $(\Omega, \mathcal{B}, \mu, T)$ *is a ergodic p.p.s,* $V \to (\Omega, \mathcal{B}, \mu, T)$ *is a vector bundle equipped with a cocycle* α*. Assume that every fiber in* V *is a normed vector space (has a symmetric, positive-definite bilinear form) such that:*

$$\int_\Omega \log^+ \|T_\omega\|_{op} \; d\mu(\omega) < \infty$$

Here $\log^+(x) := \max(0, \log x)$ *and* $\| - \|_{op}$ *denotes the operator norm of a linear map between normed vector spaces. Then there exist real numbers* $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ *(allowing* $\lambda_k = -\infty$*) and* T*-invariant subbundles of* V *defined on a full measure set:*

$$0 \subsetneq V^{\leq \lambda_k} \subsetneq \cdots \subsetneq V^{\leq \lambda_1} = V$$

such that for every i *and all* $v \in V_\omega^{\leq \lambda_i} \setminus V_\omega^{\leq \lambda_{i+1}}$ *we have:*

$$\frac{1}{N} \log \|T^N v\| \to \lambda_i$$

and the limit is uniform for fixed ω *and* i *over* $v \in V_\omega^{\leq \lambda_i} \setminus V_\omega^{\leq \lambda_{i+1}}$ *with* $\|v\| = 1$.

Note 1.13.

1. The filtration $V^{\leq \lambda_\star}$ is called the *forward Oseledets filtration.* The numbers $\{\lambda_i\}$ are called Lyapunov exponents.

2. The *multiplicity* of exponent λ_i is defined to be $\dim V^{\leq \lambda_i} - \dim V^{\leq \lambda_{i+1}}$.

3. The above result is a defining property for the Lyapunov exponents and the Oseledets Filteration, which in this sense are canonical.

2 Proof of the Theorem (2h)

2.1 Reminder

Setting: Ω a compact metric space (may be relaxed with separable, second-countable metric space). $\mathcal{B}$ it's Borel σ-algebra, $T : \Omega \to \Omega$ a μ-invariant continuous function, $(\Omega, \mathcal{B}, \mu, T)$ an ergodic probability preserving system ($T(A) = A \Rightarrow \mu(A) \in \{0, 1\}$). Note that we stated the Birkhoff ergodic theorem for general probability spaces but for the multiplicative ergodic theorem we take more specific conditions on Ω.

Note 2.1. Suppose that T is invertible and that the cocycle on V for T^{-1} satisfies the same assumptions in the Oseledets theorem. Applying the result to the inverse operator gives a set of k' exponents η_j and the *backwards* Oseledets filtration $V^{\leq \eta_{k'}} \subsetneq \cdots \subsetneq V^{\leq \eta_1}$ such that for $v \in V^{\leq \eta_j} \setminus V^{\leq \eta_{j+1}}$:

$$\lim_{N \to \infty} \log \|T^{-N} v\| = \eta_j$$

Now assume that this is compatible with the forward behavior, namely that: $\eta_j = -\lambda_{k+1-j}$ for all j and $k = k'$. Taking $V^{\lambda_j} := V^{\leq \lambda_j} \cap V^{\leq \eta_{k+1-j}}$ gives a a T-invariant direct sum decomposition:

$$V \ = V^{\lambda_1} \oplus \cdots \oplus V^{\lambda_k}$$

with the defining property for the decomposition:

$$0 \neq v \in V^{\lambda_i} \iff \lim_{N \to \pm\infty} \frac{1}{N} \log \|T^N v\| = \lambda_i$$

2.2 Preliminaries

Definitions for vector bundles equipped with cocycles:

Definition 2.2 (vector bundle morphism)**.** Let V, W vector bundles equipped with a cocycle. A map $f : V \to W$ is a *vector bundle morphism* if first, the following diagram commutes:

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "\pi_1"] & & W \arrow[dl, "\pi_2"] \\ & \Omega & \end{tikzcd}$$

That is, if $\pi_1 = \pi_2 \circ f$, which guarantees that fibers are mapped to fibers $f(V_\omega) = W_\omega$ and second, if for every $\omega \in \Omega$, $f_\omega := f|_{V_\omega}$ is a linear map. Composition of bundle morphisms are bundle morphisms so we have a category of vector bundles over Ω. The *kernel* of a vector bundle morphism can be considered as the preimage of the zero vector bundle.

Definition 2.3 (quotient)**.** Let $E \subset V$ vector bundles over Ω equipped with a cocycle. The space V/E for which every fiber $(V/E)_\omega = V_\omega/E_\omega$ is the quotient fiber bundle of V by E. Note that this is just (or barely) a description of an object we do not carefully define and exhibit that it is a vector bundle.

Definition 2.4 (direct sum)**.** $V \oplus W$ is the space in which every fiber is $(V \oplus W)_\omega = V_\omega \oplus W_\omega$

Definition 2.5 (exact sequence)**.** A sequence of vector bundles and morphisms

$$\dots \stackrel{f_{i-1}}{\to} V_i \stackrel{f_i}{\to} V_{i+1} \stackrel{f_{i+1}}{\to} V_{i+2} \to \dots$$

is an *exact sequence* if for all i, Im $(f_i) = \ker(f_{i+1})$, equivalently, if for all ω, the induced sequence of $(V_\omega)_i$'s is exact. A *short exact sequence* is an exact sequence of the form

$$0 \to E \to V \to F \to 0$$

Note 2.6.

1. In an exact sequence $W \to V \to V/W$ we have that the Lyapunov exponents of V are the union of those in W and those in V/W.

2. Let $E \subset V$ and E is T-invariant, then the same boundedness condition of the Oseledec theorem holds for E and V/E with the natural norms.

Definition 2.7 (projective space bundle)**.** For a vector bundle V, the projective space bundle $\mathbb{P}(V)$ is a fiber bundle over Ω ($\mathbb{P}(V) \xrightarrow{\pi} \Omega$), derived from V in a way that each fiber is the projective space of the corresponding V-fiber: $(\mathbb{P}(V))_\omega := \pi^{-1}(\{\omega\}) = \mathbb{P}(V_\omega)$ (for some vector space H, $\mathbb{P}(H)$ is the quotient space $(H \setminus \{0\})/\sim$ with $v \sim v' \iff \exists \lambda \in \mathbb{R}; v = \lambda v'$).
The action of T on V lifts to $\mathbb{P}(V)$ by projective linear maps:

$$\widetilde{T}(\omega, [v]) = (T_\omega, [T_\omega v])$$

2.3 Equivalence of the Birkhoff and the 1-dimensional Oseledec Theorems

Proposition 2.8. *For a 1-dimensional vector bundle, the Oseledets theorem is equivalent to the Birkhoff theorem.*

Proof. (B $\Rightarrow$ O): Observe that $f(\omega) := \log \frac{\|T_\omega v\|}{\|v\|}$ is independent of a choice of $0 \neq v \in V_\omega$ so $f(\omega) = \log \|T_\omega\|_{op}$. Assumptions of integrability is equivalent in both statements as $f^+ = \log^+ \ \|T_\omega\|_{op}$ and the Birkhoff average converges:

$$\frac{1}{N}(f(\omega) + \cdots + f(T^{N-1}\omega)) \overset{N\to\infty}{\longrightarrow} \int_\Omega f d\mu$$

We use the cocycle identity ($T_{T^m\omega} T^m_\omega v = T^{m+1}_\omega v$) and manipulate the Birkhoff averages term, each time choosing $f(T^m\omega) = \log \frac{\|T_{T^m\omega} T^m v\|}{\|T^m v\|}$:

$$\frac{1}{N}(f(\omega) + \cdots + f(T^{N-1}\omega)) = \frac{1}{N}\log\left(\frac{\|Tv\|}{\|v\|}\frac{\|T^2 v\|}{\|Tv\|}\cdots\frac{\|T^N v\|}{\|T^{N-1}v\|}\right) = \frac{1}{N}\log\left(\frac{\|T^N v\|}{\|v\|}\right)$$

But $\frac{1}{N}\log(\frac{\|T^N v\|}{\|v\|}) = \frac{1}{N}\log\|T^N v\| - \frac{\log\|v\|}{N}$ It follows that

$$\frac{1}{N}\log\|T^N v\| \to \int_\Omega f d\mu = \lambda_1$$

(O $\Rightarrow$ B): Assume $f^+ \in L^1$, define $T_\omega v := \exp(f(\omega))$. again $f^+ \in L^1 \iff \log^+ \|T_\omega\|_{op} \in L^1$ and applying the above in reverse order grants us the convergence of Birkhoff averages to Lyapunov exponent λ_1. Using T-invariance ($\int f \circ T^m = \int f$) with dominated convergence we conclude that $\lambda_1 = \int f$. $\square$

2.4 The two Lemmas

Lemma 2.9. *At least one but perhaps both of the following holds:*

1. *There is $\lambda \in \mathbb{R}$ such that for a.e $\omega \in \Omega$ and for all nonzero $v \in V_\omega$*
$$\lim_{N\to\infty} \frac{1}{N} \log \|T^N v\| = \lambda$$
and for fixed ω, the limit is uniform over $\{v \in V_\omega : \|v\| = 1\}$

2. *There exists a non-trivial proper T-invariant subbundle $E \subsetneq V$ which is defined at μ- a.e ω.*

Remark 2.10. In our setting, when $E \subset V$ is an a.e defined T-invariant subbundle then the boundedness condition in Oseledec theorem holds for E and V/E with the natural norms.

Lemma 2.11. *Consider a short exact sequence of cocycles over Ω*

$$0 \to E \to V \to F \to 0$$

Assume that there exists λ_E, λ_F such that for a.e $\omega \in \Omega$

$$\forall 0 \neq e \in E_\omega : \ \frac{1}{N} \log \|T^N e\| \to \lambda_E$$
$$\forall 0 \neq f \in F_\omega : \ \frac{1}{N} \log \|T^N f\| \to \lambda_F$$

and that the limits are uniform over $\|e\| = \|f\| = 1$ and fixed ω. If $\lambda_E > \lambda_F$ then the sequence is split: there exists a linear (splitting) map $\sigma : F \to E$ such that $V = E \oplus \sigma(F)$ and $p \circ \sigma = 1_F$ and this decomposition of V is T-invariant. λ_F is allowed to be $-\infty$.

Remark 2.12. The lyapunov exponent of F and $\sigma(F)$ are the same and V has the two exponents λ_E, λ_F. Having an Oseledec filtration, the maximal exponent is getting smaller as we go down the sets in the filtration. When maximal exponent is not reduced when moving to a subbundle, we have that subbundle as a direct summand of the entire cocycle.

Proof. (of Oseledec MET) Proceed by induction on dimension $d := \dim V$. For $d = 1$ the claim follows from the Birkhoff equivalence. It also follows by Lemma 2.9 as a line bundle has no proper non-trivial subbundles so case 1 of the lemma must hold and this is the case of a single Lyapunov exponent and trivial filtration.
Now assume the theorem holds for cocycles of dimension $\leq d-1$. Let V a cocycle of $\dim V \ = d$. We observe two cases:

case 1 There exists λ such that for a.e $\omega \in \Omega$ and all $v \in V_\omega$ we have

$$\lim_{N\to\infty} \frac{1}{N} \log \|T^N v\| = \lambda$$

In this case the result holds for V with a single exponent and trivial filtration. If it does not occur, by Lemma 2.9 we are in the second case:

case 2 There exists a non-trivial proper T-invariant subbundle. Take such subbundle E that has maximal dimension. The theorem holds for E with exponents $\lambda_1, \ldots, \lambda_k$ and a filtration $E^{\leq\lambda_k} \subsetneq \cdots \subsetneq E^{\leq\lambda_1} = E$. The quotient V/E has no proper T-invariant subbundles so by Lemma 2.9 it has one Lyapunov exponent λ'. Note that $\lambda', \lambda_1, \ldots, \lambda_k$ are all of V's Lyapunov exponents. If $\lambda' \geq \lambda_1$ we show that the forward Oseledec filtration is either $E^{\leq\lambda_k} \subsetneq \cdots \subsetneq E^{\leq\lambda_1} \subsetneq V$ in case $\lambda' > \lambda_1$, with $V \setminus E^{\leq\lambda_1}$ having growth rate λ', or $E^{\leq\lambda_k} \subsetneq \cdots \subsetneq E^{\leq\lambda_2} \subsetneq V$ in case $\lambda_1 = \lambda'$. For an appropriate change of basis $V = E \oplus \sigma_0(V/E)$ (not necessarily a T-invariant decomposition), we can use the notation:

$$T_\omega = \begin{bmatrix} T_{E,\omega} & U_\omega \\ 0 & H_\omega \end{bmatrix} \quad , \; T^N = \begin{bmatrix} T^N_{E,\omega} & U_{N,\omega} \\ 0 & H^N_\omega \end{bmatrix}$$

(similarly done in the proof of Lemma 2.11) with

$$\begin{aligned} T_{E,\omega} : E_\omega &\to E_{T\omega} \\ U_\omega : \sigma_0(V/E)_\omega &\to E_{T\omega} \\ H_\omega : \sigma_0(V/E)_\omega &\to \sigma_0(V/E)_{T\omega} \end{aligned}$$

and

$$U_{N,\omega} : \sigma_0(V/E)_\omega \to E_{T^N\omega}$$

In this basis, $v \in \sigma_0(V/E)_\omega$ grows under H in rate $\frac{1}{N}\log\|H^N v\| \to \lambda'$, uniformly on unit vectors and similarly $v \in E_\omega$ grows under $T_{E,\omega}$ in rate at most λ_1 with uniformity. The issue is with vectors that project non-trivially to V/E, we need to make sure that they grow in rate λ'. As such vector has growth rate at least λ', we are left with showing that

$$\limsup \; \frac{1}{N}\log\|U_N\|_{op} \; \leq \lambda'$$

that would grant us with uniform convergence for unit vectors in $V_\omega \setminus E_\omega$. For that, observe that by multiplying the above two matrices we get:

$$U_{N+1,\omega} = T_{E,T^N\omega}U_{N,\omega} + U_{T^N\omega}H^N_\omega$$

this inductively expand to:

$$U_{N+1,\omega} \; = T^N_{E,T\omega}U_\omega + T^{N-1}_{E,T^2\omega}U_{T\omega}H_\omega + \cdots + T_{E,T^N\omega}U_{T^{N-1}\omega}H^{N-1}_\omega + U_{T^N\omega}H^N_\omega = \sum_{m=0}^{N} T^{N-m}_{E,T^{m+1}\omega}U_{T^m\omega}H^m_\omega$$

Note the resemblance to the formal sum constructed in the proof of Lemma 2.11. For each N pick $0 \leq k \leq N$ such that the respective term in the sum is maximal in operator norm. We have:

$$\|U_{N+1,\omega}\|_{op} \; \leq (N+1)\|T^{N-k}_{E,T^{k+1}\omega}U_{T^k\omega}H^k_\omega\|_{op}$$

Note that

$$\frac{1}{N+1}\log \|U_N+1\|_{op}$$

If $\lambda' < \lambda_1$ we apply Lemma 2.11 to the exact sequence:

$$0 \to E/E^{\leq \lambda_2} \hookrightarrow V/E^{\leq \lambda_2} \twoheadrightarrow V/E \to 0$$

($E/E^{\leq \lambda_2}$ is a cocycle with single Lyapunov exponent λ_1) that so splits with a splitting map $\sigma : V/E \to V/E^{\leq \lambda_2}$

$$V/E^{\leq \lambda_2} = E/E^{\leq \lambda_2} \oplus \sigma(V/E)$$

Let $\tau : V \to V/E^{\leq \lambda_2}$ the natural projection and take $V^{\leq \lambda_2} := \tau^{-1}(\sigma(V/E))$, which is a subbundle ($\tau$ is a bundle homomorphism so the inverse of every fiber in $\sigma(V/E)$ is a linear space. τ pulls back local trivializations of $\sigma(V/E)$ to local trivializations of $\tau^{-1}(\sigma(V/E))$). Observe that for $E^{\leq \lambda_2} \subset V^{\leq \lambda_2}$ (inverse projection takes the zero of any bundle to the space by which we quotient). For $v \in V \setminus V^{\leq \lambda_2}$, $\tau(v) \in E/E^{\leq \lambda_2}$ so $v \in E \setminus E^{\leq \lambda_2}$, with Lyapunov exponent λ_1. If $v \in V^{\leq \lambda_2}$, then either $v \in E^{\leq \lambda_2}$ or $\tau(v)$ is a nonzero vector in $\sigma(V/E)$ so it has Lyapunov exponent λ' (using our remark on exponents and quotient vector bundles). So $V^{\leq \lambda_2}$ is a subbundle with maximal exponent $\max\{\lambda_2, \lambda'\}$. Now $\dim V^{\leq \lambda_2} < d$ so by the inductive assumption it has a filtration and Lyapunov exponents (that we already named) so putting $V^{\leq \lambda_1} := V$ above this filtration we are done. Alternatively, we can proceed the above construction to produce $V^{\leq \lambda_i}$'s until $\lambda_{i+1} < \lambda'$, each time use

$$0 \to E^{\leq \lambda_i}/E^{\leq \lambda_{i+1}} \hookrightarrow V^{\leq \lambda_i}/E^{\leq \lambda_{i+1}} \twoheadrightarrow V^{\leq \lambda_i}/E^{\leq \lambda_i} \to 0$$

□

Proof. (Of Lemma 2.9)

Define the space of probability measures on $\mathbb{P}(V)$ which project (or pushed forward by $\pi : \mathbb{P}(V) \to \Omega$) to μ:

$$\mathcal{M}^1(\mathbb{P}(V), \mu) := \{\eta \text{ prob. measure on } \mathbb{P}(V) \text{ with } \pi_*\eta = \mu\}$$

The action of T on $\mathbb{P}(V)$ can be extended to $\mathcal{M}^1(\mathbb{P}(V), \mu)$ naturally ($\widetilde{T}\eta = \widetilde{T}_*\eta$) as for all $A \in \mathcal{B}$, $\widetilde{T}\eta$ projects to μ:

$$\pi_*(\widetilde{T}_*\eta)(A) = \widetilde{T}_*\eta(\pi^{-1}(A)) = \eta(\widetilde{T}^{-1}\pi^{-1}(A)) \stackrel{\star}{=} \eta(\pi^{-1}(T^{-1}A)) = \mu(T^{-1}A) = \mu(A)$$

$\star$ follows from the set equality $\widetilde{T}^{-1}\pi^{-1}(A) = \pi^{-1}(T^{-1}A)$ which indeed holds as the linear maps of the lifted T (to V and $\mathbb{P}(V)$) that acts on the fibers are assumed to be invertible, and so each fiber $P_\omega := \pi^{-1}(\{\omega\}) \subset \mathbb{P}(V)$ of $\omega \in A$ has $\widetilde{T}^{-1}\{P_\omega\} = \bigcup_{T\delta=\omega}\{P_\delta\}$. On the RHS, the inverse projection takes each $\delta \in T^{-1}\ \{\omega\}$ to it's fiber P_δ so $\pi^{-1}(T^{-1}\{\omega\}) = \bigcup_{T\delta=\omega}\{P_\delta\}$ and each $\omega \in A$ has the same contribution to both sides.
Equip $\mathcal{M}^1(\mathbb{P}(V), \mu)$ with the weak-* topology, here it is the smallest topology such that for every $\phi : \mathbb{P}(V) \to \mathbb{R}$ that is measurable on $\mathbb{P}(V)$ and continuous on a.e fiber, the operator

$$\mathcal{M}^1(\mathbb{P}(V), \mu) \to \mathbb{R}, \quad \eta \mapsto \int_{\mathbb{P}(V)} \phi d\eta \ \text{ is continuous}$$

Theorem 2.13 (Krylov-Bogoliubov)**.** *$\mathcal{M}^1(\mathbb{P}(V), \mu)$ is weak-* compact and sequentially compact. It has at least one T-invariant measure.*

Define $f : \mathbb{P}(V) \to \mathbb{R}$ by:

$$f([v]) := \log \frac{\|Tv\|}{\|v\|}$$

(formally, in local trivialization $f(\omega, [v]) = \log \frac{\|T_\omega v\|}{\|v\|}$) and observe the continuous operator

$$\int f : \mathcal{M}^1(\mathbb{P}(V), \mu) \to \mathbb{R}, \quad \eta \mapsto \int_{\mathbb{P}(V)} f d\eta$$

Let $\mathcal{M}^1(\mathbb{P}(V), \mu)^T$ be the space of T-invariant measures. By Krylov-Bogolyubov, it's non-empty, weak-* closed (take a $\eta_n \to \eta$, and take the limit on $\int \phi d\eta_n = \int \phi \circ T d\eta_n$) so it's weak-* compact. Thus f achieves a minimum on $\mathcal{M}^1(\mathbb{P}(V), \mu)^T$.

Theorem 2.14 (Krein-Milman)**.** *A compact convex set in a topological vector space is the convex hull of its extreme points.*

The set of measures on which the minimum is achieved is convex (for $\eta_1, \eta_2 \in \mathcal{M}^1(\mathbb{P}(V), \mu)^T$, $s \in [0,1]$, write $s\eta_1 + (1-s)\eta_2$ and observe that this is a T-invariant measure that projects to μ). That set is also closed (for a weak-* convergent $\eta_n \to \eta$, use the above idea for T-invariance and $\int f d\eta_n$ is a constant series converging to $\int f d\eta$) so by Krein-Milman it has an extreme point, denoted η. This measure is ergodic for the T action on $\mathbb{P}(V)$ (otherwise, for a set $A \subset \mathbb{P}(V)$ with $0 < \eta(A) < 1$ and $T^{-1}(A) = A$ define $\eta_1 = \frac{\eta(A \cap \,\cdot\,)}{\eta(A)}, \eta_2 = \frac{\eta(A^C \cap \,\cdot\,)}{\eta(A^C)}$ and observe that those are T-invariant measures that minimize the operator, project to μ and exhibit η as a nontrivial convex combination, with the same properties.)
Apply the Birkhoff ergodic theorem to f and the measure η, for η-a.e $[v] \in \mathbb{P}(V)$:

$$\frac{1}{N}\sum_{m=0}^{N-1} f(T^m[v]) = \frac{1}{N}\log\left(\frac{\|T^N v\|}{\|v\|}\right) \to \int_{\mathbb{P}(V)} f d\eta =: \lambda$$

Let $M \subset \mathbb{P}(V)$ the full measure set of vectors for which the above limit holds. For $\omega \in \pi(M)$, let $M_\omega = M \cap \mathbb{P}(V_\omega)$.

Observation 2.15. The following properties hold:

1. T-invariance: $TM_\omega = M_{T\omega}$ for a.e ω
2. M_ω is nonempty for a.e ω
3. $E = \bigcup_{\omega \in \pi(M)} \operatorname{span}_V\{M_\omega\}$ Is a T-invariant subbundle of V defined μ-a.e.

Proof. 1. Notice that as T_ω is invertible:

$$M_{T\omega} = \{[v'] \in \mathbb{P}(V_{T\omega}) : \frac{1}{N}\sum_{m=0}^{N-1} f(T^m[v']) \to \lambda\} =$$

$$\{T[v] : [v] \in \mathbb{P}(V_\omega), \frac{1}{N}\sum_{m=0}^{N-1} f \circ T(T^m[v]) \to \lambda\}$$

Birkhoff averages of $f \circ T$ converge to $\int_{\mathbb{P}(V)} f \circ T$ and as η is T-invariant, this integral equals $\int f = \lambda$ so $M_{T\omega} = TM_\omega$.

2. $\mu(\pi(M)) = \eta(\pi^{-1}(\pi(M))) \geq \eta(M) = 1$

3. We can use an ergodicity argument to show that the dimension of $\text{span}_V\{M_\omega\}$ is some constant d', a.e. Take a μ-positive measure set $K \subset \Omega$ in which this dimension is constant. Denote $\tau : E \to \Omega$ the projection. Observe that for $m \geq 0$, all the fibers in $T^{-m}\tau^{-1}(K)$ has the same constant dimension as the cocycle maps on the fibers are invertible. By ergodicity, $\mu \bigcup_{m\geq 0} T^{-m}K$ is a full measure set. For local trivializations, reduce to an appropriate full measure $\Omega' \subset \Omega$. For ω in Ω', take (U, φ) the local trivialization as an element of V, take $U' = U \cap \Omega'$ and φ' such that $\varphi'^{-1} = \varphi^{-1}|_{\tau^{-1}(U)}$. Up to a composition with a change of basis, φ' is a homeomorphism $U' \times \mathbb{R}^{k'} \to \tau^{-1}(U')$. T-invariance follows from (1) extended to linear combinations.

□

If E is a proper subbundle of V, this is case 2 of the lemma. Suppose $E = V$, in this case for a.e ω and every $v \in V_\omega = \text{span}_V\{M_\omega\}$ is a linear combination of vectors with growth rate λ. By triangle inequality, we get that v grows at rate at most λ. To see that the growth rate is exactly λ, uniformly on $\|v\| = 1$ on a fixed fiber, assume by contradiction that there exists a μ-generic ω (ω for which the pointwise ergodic theorem's result holds for continuous functions), $\varepsilon > 0$ and a sequence $[v_i] \in \mathbb{P}(V_\omega)$ such that:

$$\limsup_{N_i \to \infty} \frac{1}{N_i} \log\left(\|T^{N_i} v_i\|\right) \leq \lambda - \varepsilon$$

Where $\|v_i\| = 1$ and N_i is a sequence tending to ∞. Define the following series of measures on $\mathbb{P}(V)$:

$$\eta_i = \frac{1}{N_i} \sum_{m=0}^{N_i - 1} \delta_{T^m[v_i]}$$

Let η_ε be one of their weak-* limits (there is a limit due to the fact that the space of Borel measures on the projective space, denoted $\mathcal{M}(\mathbb{P}(V))$, is a weak-* compact metrizable space, hence sequentially compact). There is a standard method to show that η_ε is T-invariant. Let i_k such that $\eta_{i_k} \to \eta_\varepsilon$ in weak-* and observe that for ϕ measurable and continuous on every fiber:

$$|\int_{\mathbb{P}(V)} \phi \circ T d\eta_{i_k} - \int_{\mathbb{P}(V)} \phi d\eta_{i_k}| = \frac{1}{N_{i_k}} |\int \sum_{m=1}^{N_{i_k}-1} (\phi \circ T^{m+1} - \phi \circ T^m)\delta_{[v_i]}| =$$
$$\frac{1}{N_{i_k}} |\int (\phi \circ T^{N_{i_k}} - \phi) d\delta_{[v_i]}| \leq \frac{2}{N_{i_k}} \|f\|_\infty \to 0$$

so their limits $\int \phi d\eta_\varepsilon = \int \phi \circ T d\eta_\varepsilon$. Moreover, η_ε is projected to μ. Let $\varphi : \Omega \to \mathbb{R}$ continuous:

$$\int_{\mathbb{P}(V)} \varphi \circ \pi d\eta_\varepsilon =$$
$$\lim_{k\to\infty} \frac{1}{N_{i_k}} \sum_{m=0}^{N_{i_k}-1} \int_{\mathbb{P}(V)} \varphi \circ \pi d\delta_{T^m[v_i]} =$$
$$\lim_{k\to\infty} \frac{1}{N_{i_k}} \sum_{m=0}^{N_{i_k}-1} \varphi(T^m\omega) \overset{\star}{=} \int_\Omega \varphi d\mu$$

And $\star$ results from ω being a μ-generic point. So $\eta_\varepsilon \in \mathcal{M}^1(\mathbb{P}(V), \mu)^T$ Applying the assumption to f we get $\limsup \int_{\mathbb{P}(V)} f d\eta_{i_k} \leq \lambda - \varepsilon$ therefor $\int_{\mathbb{P}(V)} f d\eta_\varepsilon \leq \lambda - \varepsilon$ which contradicts the definition of λ being the minimal value achieved on $\mathcal{M}^1(\mathbb{P}(V), \mu)^T$. □

Proof. (of lemma 2.11) Let $0 \to E \hookrightarrow V \xrightarrow{p} F \to 0$ a short exact sequence of vector bundles. Note that by definition E can be identified with a T-invariant subbundle of V. Choose a lift $\sigma_0 : F \to V$ (A vector bundle morphism with $p \circ \sigma_0 = Id_F$, since we have a metric on V a possible choice is the identification $F \simeq E^\perp$) such that $V = E \oplus \sigma_0(F)$. Under this identification, the cocycle map takes the form of the block matrix:

$$T = \begin{bmatrix} T_E & U \\ 0 & T_F \end{bmatrix}$$

With linear maps:

$$T_{E,\omega} : E_\omega \to E_{T\omega}$$
$$T_{F,\omega} : F_\omega \to F_{T\omega}$$
$$U_\omega : F_\omega \to E_{T\omega}$$

Any other lift $\sigma : F \to V$ differs from σ_0 by a map $\tau : F \to E$. To justify that, let $\tau = \sigma - \sigma_0 : F \to V$, so τ has image in $E = \ker p$:

$$p \circ \tau = p \circ (\sigma - \sigma_0) = 0$$

We look for a $\sigma = \tau + \sigma_0$ for which the decomposition $V = E \ \oplus \sigma(F)$ is T-invariant ($T_\omega \sigma(F) = \sigma(F)_{T\omega}$ a.e). For $\begin{bmatrix} e \\ f \end{bmatrix} \in \sigma(F) \subset E \oplus \sigma_0(F)$ (working in the σ_0 decomposition) we present a condition for $T \begin{bmatrix} e \\ f \end{bmatrix}$ to be in $\sigma_{T\omega}(F)$ and show that there is a lift σ that fulfills this condition. For $\begin{bmatrix} e \\ f \end{bmatrix} \in \sigma(F)$ we can write $e = \tau(f)$ and apply T:

$$T \begin{bmatrix} \tau(f) \\ f \end{bmatrix} = \begin{bmatrix} T_{E,\omega} \circ \tau_\omega(f) + U_\omega(f) \\ T_{F,\omega}(f) \end{bmatrix} \in V_{T\omega}$$

The condition for this vector to be in $\sigma_{T\omega}(F)$ is:

$$T_{E,\omega} \circ \tau_\omega(f) + U_\omega(f) = \tau_{T\omega} \circ T_{F,\omega}(f)$$

This condition is an equality of linear maps $F_\omega \to E_{T\omega}$, equivalently:

$$\tau_\omega = T_{E.\omega}^{-1} \circ \tau_{T\omega} \circ T_{F,\omega} - T_{E,\omega}^{-1} \circ \ U_\omega$$

And it is enough to show it for a normal basis of F_ω. A formal solution of this equation is given by:

$$\tau_\omega = -\sum_{n=0}^{\infty} (T_{E,\omega}^{n+1})^{-1} \circ U_{T^n\omega} \circ T_{F,\omega}^n$$

The assumption that $\lambda_F - \lambda_E < 0$ is used to show that the sum converges like a sum of an exponentially convergent sequence. For a bound on the contribution of the middle term, apply Birkhoff theorem on $\log \|U_\omega\|_{op}$ ($\log^+ \|U_\omega\|_{op} \in L^1$) to achieve:

$$\|U_{T^n\omega}\|_{op} = e^{o(n)}$$

Further, Observe that:

$$\forall v_1 \in E_\omega \ ; \ \|T^{n+1}_{E,\omega} v_1\| = e^{n\lambda_E + o(n)} \|v_1\|$$

So going backwards we have:

$$\forall v_2 \in E_{T^{n+1}\omega} \ ; \quad \|(T^{n+1}_{E,\omega})^{-1} v_2\| = e^{-n\lambda_E + o(n)} \|v_2\|$$

Let $v \in F_\omega ; \|v\| = 1$:

$$\|T^n_{F,\omega} v\| = e^{n\lambda_f + o(n)}$$

Recalling that the growth rates are uniform over unit circles in the respective vector bundle fibers, the $o(n)$ terms obey the uniform convergence. The term in the formal sum is bounded by an exponentially convergent series:

$$\|(T^{n+1}_{E,\omega})^{-1} \circ U_{T^n\omega} \circ T^n_{F,\omega} v\| \leq e^{n(\lambda_F - \lambda_E) + o(n)}$$

And the sum converge uniformly. Hence choosing a basis for F_ω we have that τ_ω such that $\sigma(F)$ is T-invariant is well defined. $\square$