Rudolph's x2 x3 Theorem

Plan of the talk

- On the theorem

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- The invertible extension

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- Certain conditional measures as translates of a measure on a group.

Theorem

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1. It actually holds that
$h_\mu(S_2) > 0 \iff h_\mu(S_3) > 0 \iff h_\mu(S_2^m S_3^n) > 0$ for some $m, n \in \mathbb{N}$.
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Open question (Furstenberg). Is it true that the Haar measure of $\mathbb{T}$ is the unique non-atomic measure invariant under S_2 and S_3?

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Theorem. Assume that $A \subseteq \mathbb{T}$ is a forward invariant under S_2 and S_3 (namely $\forall x \in A$, $S_i x \in A$, for $i \in \{2, 3\}$). Then either A is finite or A is dense.

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By Rudolph's theorem we obtain the following result which can give some insight (in some cases) that Furstenberg's result can't.

Corollary from Rudolph's theorem (Exercise 9.3.2. ELW book). Let μ be an S_3 invariant and ergodic probability measure with positive entropy. Then μ almost every $x \in \mathbb{R}/\mathbb{Z}$ has a dense orbit under S_2.

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Example. Consider the middle third cantor set

$$C = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{3^i} \mid a_i \in \{0,2\} \right\},$$

which is clearly S_3 invariant. The Bernoulli shift on two symbols gives C an S_3 invariant ergodic measure μ_C with positive entropy.

The invertible extension

We will change the setting to the space

$$X \stackrel{\text{def}}{=} \left\{ x \in \mathbb{T}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} = 2x_n,\ x_{\mathbf{n}+\mathbf{e}_2} = 3x_n,\ \forall \mathbf{n} \in \mathbb{Z}^2 \right\},$$

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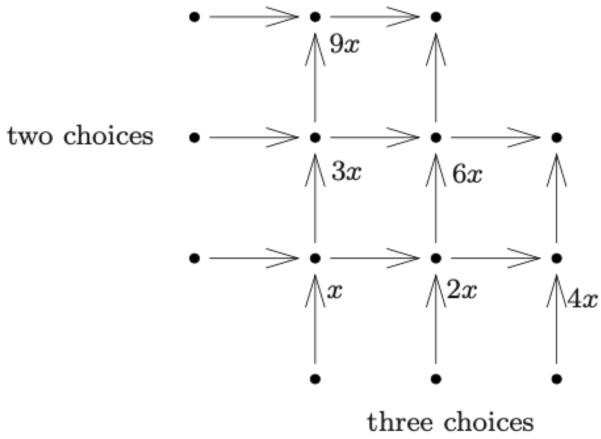

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X is a closed subgroup of the compact group $\mathbb{T}^{\mathbb{Z}^2}$ hence X is a compact abelian group with respect to the induced topology τ_X.
Let $I \subseteq \mathbb{Z}^2$ be a finite set and for each $\mathbf{n} \in I$ let $E_{\mathbf{n}} \subseteq \mathbb{T}$ be an open set, and define

$$[E_{\mathbf{n}}]_{\mathbf{n} \subseteq I} \stackrel{\text{def}}{=} \{x \in X \mid x_{\mathbf{n}} \in E_{\mathbf{n}},\ \mathbf{n} \in I\}.$$

Then the sets $[E_{\mathbf{n}}]_{\mathbf{n} \subseteq I}$ form a basis for τ_X.

Cylindrical sets in view of coordinate projections

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$$[E_{\mathbf{n}}]_{\mathbf{n}\subseteq I} = \left\{ x \in X \mid x_{m_0,n_0} \in \bigcap_{(m,n)\in I} S_2^{-(m-m_0)} S_3^{-(n-n_0)} E_{m,n} \right\} =$$

$$\pi_{m_0,n_0}^{-1}\left(\bigcap_{(m,n)\in I} S_2^{-(m-m_0)} S_3^{-(n-n_0)} E_{m,n} \right).$$

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Hence $\tau_{m,n} \overset{\text{def}}{=} \pi_{m,n}^{-1}\tau_{\mathbb{T}}$ generate the topology, and moreover $\tau_{m-1,n} \supseteq \tau_{m,n}$, $\tau_{m,n-1} \supseteq \tau_{m,n}$. We conclude

$$\tau_{m,n} \nearrow \tau_X.$$

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Then we conclude that $\mathcal{B}_{m-1,n} \supseteq \mathcal{B}_{m,n}$, $\mathcal{B}_{m,n-1} \supseteq \mathcal{B}_{m,n}$, and $\bigvee_{n=0}^{\infty} \bigvee_{m=0}^{\infty} \mathcal{B}_{-m,-n} = \mathcal{B}_X$, where $\mathcal{B}_X$ is the Borel σ-algebra on X.

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$$[x]_{\mathcal{B}_{m,n}} = \{y \in X \mid x_{a,b} = y_{a,b},\ \forall a \geq m,\ b \geq n\}$$

.

Shift maps

We consider the left shift map $T_2(x)_{(m,n)} \overset{\text{def}}{=} x_{(m+1,n)}$ and the down shift map $T_3(x)_{(m,n)} \overset{\text{def}}{=} x_{(m,n+1)}$ which are invertible and keep X invariant.

$$\begin{CD} X @>{T_2^m T_3^n}>> X \\ @V{\pi_{m,n}}VV @VV{\pi_{m,n}}V \\ \mathbb{T} @>{S_2^m S_3^n}>> \mathbb{T} \end{CD}$$

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Then there exists a borel probability measure μ_X on X which is T_2,T_3 invariant and $\left(\pi_{m,n}\right)_* \mu_X = \mu$ for all $(m,n) \in \mathbb{Z}^2$.

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Lemma (without proof). Assume that μ is S_2 and S_3 invariant borel probability measure on $\mathbb{T}$.
Then there exists a borel probability measure μ_X on X which is T_2,T_3 invariant and $\left(\pi_{m,n}\right)_* \mu_X = \mu$ for all $(m,n) \in \mathbb{Z}^2$.
Moreover, if μ is ergodic for the joint S_2,S_3 action, then μ_X is ergodic for the joint T_2, T_3 action.

Consider the partition $\xi_{\mathbb{T}} \stackrel{\text{def}}{=} \{[0, \frac{1}{6}), [\frac{1}{6}, \frac{2}{6}), ..., [\frac{5}{6}, 1)\}$, and $\xi_X \stackrel{\text{def}}{=} \pi_{0,0}^{-1}(\xi_{\mathbb{T}})$.

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In fact:

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By the same argument $h_{\mu_X}(T_3, \xi_X) = h_\mu(S_3, \xi_{\mathbb{T}})$.
Now ξ_X and $\xi_{\mathbb{T}}$ are generators for both S_2 and S_3, thus we get
Corollary. $h_{\mu_X}(T_l, \xi_X) = h_\mu(S_l, \xi_{\mathbb{T}}) = h_\mu(S_l)$, for $l \in \{2, 3\}$.

The reduction of the problem

Assuming that μ_X is T_2,T_3 invariant and ergodic, such that $h_{\mu_X}(T_2, \xi_X) > 0$, our goal will be to show

$$h_{\mu_X}(T_2, \xi_X) = \log(2).$$

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Proof. Consider the generator $\xi_0 \overset{\text{def}}{=} \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ for S_2.

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Lemma. $h_\mu(S_2) = \log(2) \iff \mu$ is the Haar measure.

Proof. Consider the generator $\xi_0 \stackrel{\text{def}}{=} \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ for S_2.
The partition $\bigvee_{i=0}^{N-1} S_2^{-i}\xi_{\mathbb{T}}$ consists of 2^N dyadic intervals $I_{j,N} \stackrel{\text{def}}{=} [\frac{j}{2^N}, \frac{j+1}{2^N})$ of length $\frac{1}{2^N}$. Once we will show that $\mu(I_{j,N}) = \frac{1}{2^N}$ for all $j \leq N$ and $N \in \mathbb{N}$, it will follow that $\mu = m_{\mathbb{T}}$.

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Hence

$$\frac{1}{N} H_\mu\left(\bigvee_{i=0}^{N-1} S_2^{-i} \xi_{\mathbb{T}}\right) < \frac{1}{N} \log(2^N) = \log(2).$$

Assume for contradiction that there exists $I_{j,N}$ such that $|I_{j,N}| \neq \frac{1}{2^N}$. Now recall that in general, if ξ is a partition of N elements then $H_\nu(\xi) \leq \log N$ and

$$H_\nu(\xi) = \log N \iff \nu(P) = \frac{1}{N}, \ \forall P \in \xi.$$

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and since $h_\mu(S_2) = h_\mu(S_2, \xi_0) = \inf_{n \geq 1} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} S_2^{-i} \xi_{\mathbb{T}} \right)$, we have a contradiction.

By the future formula for entropy $h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1)$, where

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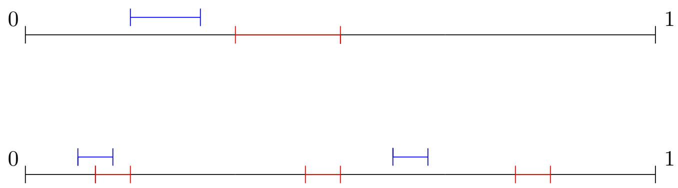
Proof by picture



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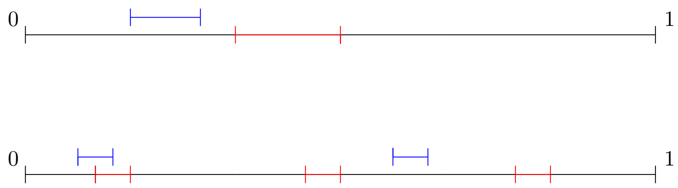

Proof of the picture: Note that if $|x - a| < \frac{1}{2 \cdot 3^{n+1}}$ and $x \in (a, b)$ such that $b - a = \frac{1}{2 \cdot 3^{n+1}}$ then its impossible that $x + \frac{1}{2} \in (a + \frac{j}{3^n}, b + \frac{j}{3^n})$. In fact, if we assume the contrary, then

$$\frac{1}{2 \cdot 3^n} - \frac{1}{2 \cdot 3^{n+1}} \leq \left| \frac{1}{2} - \frac{j}{3^n} \right| - |x - a| \leq \left| \left(x + \frac{1}{2} \right) - \left(a + \frac{j}{3^n} \right) \right| < b - a = \frac{1}{2 \cdot 3^{n+1}},$$

which is a contradiction.

$$(T_3^{-n}\xi_X) \vee \mathcal{A}_1 = \xi_X \vee \mathcal{A}_1 \implies \xi_X \vee T_3^n \mathcal{A}_1 = T_3^n(\xi_X \vee \mathcal{A}_1)$$

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Let $\mathcal{A} \stackrel{\text{def}}{=} \bigvee_{n=0}^{\infty} T_3^n \mathcal{A}_1$, which is the σ-algebra generated by the coordinates in the right-half plane $\left\{(m, n) \in \mathbb{Z}^2 \mid m > 0\right\}$.

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Then $T_3^n\mathcal{A}_1\nearrow\mathcal{A}$ and

$$h_{\mu_X}(T_2,\xi_X)=H_{\mu_X}(\xi_X\mid\mathcal{A}_1)=\lim_{n\to\infty}H_{\mu_X}(\xi_X\mid T_3^n\mathcal{A}_1)=H_{\mu_X}(\xi_X\mid\mathcal{A}).$$

—-End of first talk—

Second talk - Reminder

We consider the space
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$$\mathcal{A} \overset{\text{def}}{=} \bigvee_{n=0}^{\infty} T_3^n T_2^{-1}\left(\pi_{0,0}^{-1}(\mathcal{B}_{\mathbb{T}})\right) = \bigvee_{n=0}^{\infty} \pi_{1,-n}^{-1}(\mathcal{B}_{\mathbb{T}}),$$

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Once we show $H_{\mu_X}(\xi_X \mid \mathcal{A}) = \log(2)$ we are done.

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Next, since $\xi_X \vee \mathcal{A} = T_2\mathcal{A}$,

$$[x]_{\xi_X \vee \mathcal{A}} = [x]_{T_2\mathcal{A}} = T_2[T_2^{-1}x]_{\mathcal{A}} = x + T_2 G.$$

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$G \overset{\text{def}}{=} [0]_{\mathcal{A}} = \left\{x \in X \mid x_{m,n} = 0,\ \forall\,(m,n) \in \mathbb{N} \times \mathbb{Z}\right\}$ i s a closed subgroup of X and $[0]_{\xi_X \vee \mathcal{A}} = T_2 G \leq G$ of index 2.
Moreover, $[x]_{\mathcal{A}} = x + G$ and $[x]_{\xi_X \vee \mathcal{A}} = x + T_2 G$.

Proof. Note that $x \in [0]_{\mathcal{A}}$ if and only if $x_{m,n} = 0$, for all $m > 0$ and $n \in \mathbb{N}$.
Hence if we consider the continuous projection

$$(x)_{(m,n) \in \mathbb{Z}^2} \mapsto (x)_{(m,n) \in \mathbb{N} \times \mathbb{Z}},$$

then $G = [0]_{\mathcal{A}}$ is the kernel, which is a closed subgroup, hence also $[x]_{\mathcal{A}} = x + G$.
Next, since $\xi_X \vee \mathcal{A} = T_2\mathcal{A}$,

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and $x \in T_2 G \leq G \iff x_{0,n} = 0$ for all $n \in \mathbb{Z}$. Hence $T_2 G$ is of index 2.

Intro to leafwise measures

Consider the probability measure supported on G defined by

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Finally

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The plan (roughly) to show that $\nu_x = m_G$ is to first prove that ν_x are the Haar measures on a certain subgroup G_x, and then to use ergodicity and entropy assumption to prove that $\nu_x = m_G$ for a.e. x.

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Since $x \mapsto \nu_x$ is $\mathcal{A}$ measurable, and since $x \mapsto \mu_x^{\mathcal{A}}$ is $\mathcal{A}$ measurable (by the theorem about conditional measures), we get that for all $y \in [x]_{\mathcal{A}} \backslash N$

$$\nu_x = \nu_y, \quad \mu_x^{\mathcal{A}} = \mu_y^{\mathcal{A}}.$$

(since then for all $f \in C(X)$, $\phi_f(\cdot) \stackrel{\text{def}}{=} \nu_{\cdot}(f)$ is a real $\mathcal{A}$ measurable function and $[x]_{\mathcal{A}} \subseteq \phi_f^{-1}(\nu_x(f))$ which implies $\nu_x(f) = \nu_y(f)$ for all $y \in [x]_{\mathcal{A}}$).

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It remains to verify that

$$\underbrace{\text{Stab}(\nu_x)}_{\text{Closed subgroup}} = \text{Support}(\nu_x),$$

where the support of a measure is the smallest set of points of which any nbhd has a positive measure.

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Then

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By a theorem which we will not prove (its a long detour, see Theorem 2.29 in ELW book), it follows that

$$\mathcal{P}(T_2) \underset{\mu_X}{=} \bigvee_{k \geq 0} T_3^k \bigcap_{n=0}^{\infty} \bigvee_{i=n}^{\infty} T_2^{-i}(\xi_X)$$

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and the right hand side is contained in $\mathcal{A}$.
We will first prove that $x \mapsto \nu_x$ is measurable with respect to the Pinsker σ-algebra of T_3 and then we will show that the Pinsker algebras of T_2 and T_3 are the same!

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Recall $G = [0]_{\mathcal{A}} = \{x \in X \mid x_{m,n} = 0, \ \forall (m,n) \in \mathbb{N} \times \mathbb{Z}\}$. We will show that $G \cong \mathbb{Z}_2$ where $\mathbb{Z}_2$ are the 2-adic integers.

$$\mathbb{Z}_2 = \varprojlim_{k \in \mathbb{N}} \mathbb{Z}/2^k\mathbb{Z} = \{(x_1, x_2, \ldots.) \mid x_j \in \mathbb{Z}/2^j\mathbb{Z}, \ x_{j+1} = x_j \mod 2^j\}$$

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We note that a convenient way to indentifty $\mathbb{Z}_2$ elements is by formal sums $\sum_{j=0}^{\infty} a_j 2^j$ where $a_j \in \{0,1\}$ (the identification is $\sum_{j=0}^{\infty} a_j 2^j \mapsto (a_0 + 2\mathbb{Z}, a_0 + 2a_1 + 2^2\mathbb{Z}, \dots))$.

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$$\begin{array}{ccc} G & \longrightarrow & \mathbb{Z}_2 \\ {\scriptstyle T_3}\downarrow & & \downarrow{\scriptstyle \times 3} \\ G & \longrightarrow & \mathbb{Z}_2 \end{array}$$

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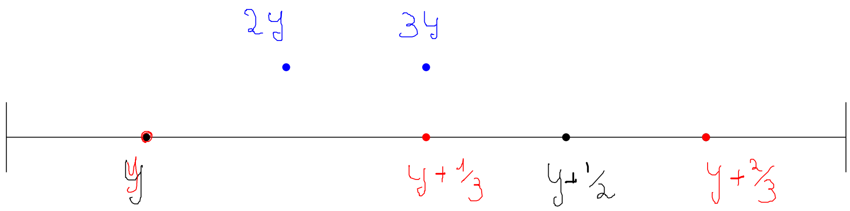
Figure: Knowledge of $2y$ and $3y$ modulo 1 determines a unique y modulo 1



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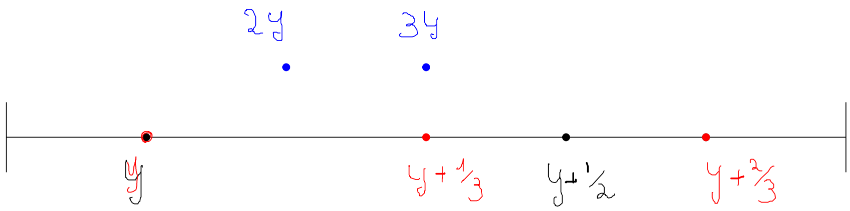

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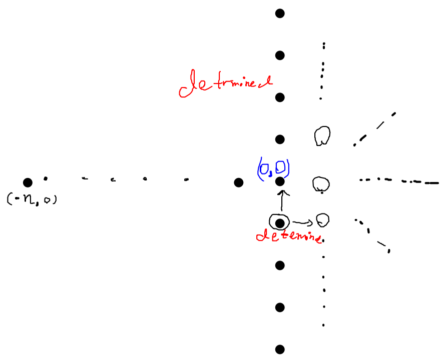

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If $x \in G$ then $x_{-m,0} = \sum_{j=0}^{m} \frac{a_{m-j}}{2^{j+1}} + \mathbb{Z}$, $a_i \in \{0,1\}$ and
$2^{m+1}x_{-m,0} = a_0 + a_1 2 + ... + a_m 2^m + 2^{m+1}\mathbb{Z}$. It is now easy to check that ϕ has a trivial kernel and any 2-adic integer $\sum_{k=0}^{\infty} a_k 2^k$ is attained.

Since $x_{-m,0} = \frac{q}{2^{m+1}}$ and $2x_{-m-1,0} + \mathbb{Z} = x_{-m,0} + \mathbb{Z}$ the map

$$(..., x_{-1,0} + \mathbb{Z}, x_{0,0} + \mathbb{Z}) \mapsto \left(2x_{0,0} + 2\mathbb{Z},\ 2^2 x_{-1,0} + 2^2\mathbb{Z}, ...\right)$$

defines a homomorphism $\phi : G \to \mathbb{Z}_2$. To show that ϕ is an isomorphism we note a more explicit form.
If $x \in G$ then $x_{-m,0} = \sum_{j=0}^{m} \frac{a_{m-j}}{2^{j+1}} + \mathbb{Z}$, $a_i \in \{0,1\}$ and $2^{m+1}x_{-m,0} = a_0 + a_1 2 + ... + a_m 2^m + 2^{m+1}\mathbb{Z}$. It is now easy to check that ϕ has a trivial kernel and any 2-adic integer $\sum_{k=0}^{\infty} a_k 2^k$ is attained.
Finally we note that $(T_3 x)_{-m,0} = x_{-m,1} = 3x_{-m,0}$ and since the map ϕ is a group homomorphism, we get $\phi(T_3 x) = 3\phi(x)$.

Measurability w.r.t. Pinsker σ-algebra of T_3.

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1. Recall that a map $x \mapsto \nu_x$ is measurable with respect a σ-algebra if for a dense subset $\{f_n\}_{n=1}^{\infty} \subseteq C(X)$, each of the real maps $\Psi_{f_n}(x) \stackrel{\text{def}}{=} \nu_x(f_n)$ is measurable.

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$$\bigvee_{i=0}^{\infty} \tilde{S}^{-i}\{f^{-1}(U), X \backslash f^{-1}(U)\}$$

is simply a finite partition, which implies $h_{\tilde{\mu}}(\tilde{S}, \{f^{-1}(U), X \backslash f^{-1}(U)\}) = 0$.

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Now we note that the characteristic functions $\chi_{\{x \in \mathbb{Z}_2 \mid |x-k|_2 \leq 2^{-l}\}} : \mathbb{Z}_2 \to \mathbb{R}$ are continuous and span dense set in $C(\mathbb{Z}_2)$, hence $\chi_{B_l(k)}$ span a dense set in $C(G)$.

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Thus

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Hence by Kolmogorov-sinai theorem for sequences we obtain

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-End of second talk-

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We saw that $[x]_{\mathcal{A}} = x + G$ and $[x]_{\xi_X \vee \mathcal{A}} = x + T_2 G$, where $G \leq X$ is the closed subgroup $G \overset{\text{def}}{=} [0]_{\mathcal{A}} = \left\{x \in X \mid x_{m,n} = 0, \ \forall\, (m,n) \in \mathbb{N} \times \mathbb{Z}\right\}$.

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We consider the probability measures $\nu_x = -x + \mu_x^{\mathcal{A}}$ (namely, $\nu_x(B) \overset{\text{def}}{=} \mu_x^{\mathcal{A}}(x+B),\ B \in \mathcal{B}_G)$ which are supported on G.

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So we have

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We will now show that the pinsker σ-algebras of T_2 and T_3 are the same.

Let $\mathcal{S}$ be a σ-algebra invariant under T_2 and T_3. In the following we will use that $h_{\mu_X}(T_i, \xi_X \mid \mathcal{S}) = h_{\mu_X}(T_i \mid \mathcal{S})$, for $i \in \{1, 2\}$ (which we proved last time-see correction before the proof).

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Proposition (entropy formula-key to pinsker meausrability). Let $\mathcal{S}$ be a σ-algebra invariant under T_2 and T_3, then

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In fact,

$$H_\mu(\xi_1 \mid \xi_2) = \sum_{P \in \xi_1} \mu(P) H_{\mu|_P}(\xi_2) \leq \sum_{P \in \xi_1} \mu(P) \log(N) = \log(N)$$

If ξ_1 and ξ_2 partitions of $\mathbb{T}$ comprised of intervals of length l_1 and l_2 correspondingly such that

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Now observe that $\bigvee_{i=0}^{n} S_2^{-i}\{\left[0, \frac{1}{6}\right), \left[\frac{1}{6}, \frac{2}{6}\right), ..., \left[\frac{5}{6}, 1\right)\}$ is composed of intervals of length $l_2(n) = \frac{1}{3 \cdot 2^n}$ and $\bigvee_{i=0}^{m} S_3^{-i}\{\left[0, \frac{1}{6}\right), \left[\frac{1}{6}, \frac{2}{6}\right), ..., \left[\frac{5}{6}, 1\right)\}$ is composed of intervals of length $l_3(m) = \frac{1}{2 \cdot 3^m}$.

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Assume that $m = \left\lfloor \frac{\log 3}{\log 2} n \right\rfloor$, then $2^m \leq 3^n \leq 2^{m+1}$, which yields $\frac{1}{2} l_2(m) \leq l_3(n) \leq \frac{3}{2} l_2(n)$. Hence

$$H_{\mu_X}(\bigvee_{i=0}^{n-1} T_3^{-i}\xi_X \mid \bigvee_{i=0}^{m-1} T_2^{-i}\xi_X),\ H_{\mu_X}(\bigvee_{i=0}^{m-1} T_2^{-i}\xi_X \mid \bigvee_{i=0}^{n-1} T_3^{-i}\xi_X) \leq \log(3)$$

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Hence

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Reversing the role of T_3 and T_2 we obtain the opposite inequality and conclude the result.

Entropy formula+Abramov-Rokhlin formula imply Pinsker algebras are the same

Corollary. The maps T_2 and T_3 of $(X, \mathcal{B}_X, \mu_X)$ have the same Pinsker algebra.

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In fact, since T_2 and T_3 are commuting and T_3 preserves μ_X we have

$$h_{\mu_X}(T_2, \{B, X\backslash B\}) = h_{\mu_X}(T_2, T_3^{-1}\{B, X\backslash B\})$$

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Since T_3 is invertible, we get $T_3^{-1}\mathcal{P}(T_2) = \mathcal{P}(T_2)$.

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Namely, here we deduce that there exists a space $(Y, \mathcal{B}_Y, \nu, S)$ and a factor map $\phi : X \to Y$ (measurable preserving map such that $\phi \circ T_3 = S \circ \phi$) such that $\phi^{-1}(\mathcal{B}_Y) = \mathcal{P}(T_2)$ modulo μ_X.

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So if $A \in \mathcal{P}(T_2)$, then $A = \phi^{-1}(\tilde{A})$ and we have

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which proves $A \in \mathcal{P}(T_3)$.

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The other inclusion follows by the same arguments after we exchange 3 with 2.

Last arguments - proof that $\nu_x = m_G$ for a.e. x

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Namely $T_2(G) \subsetneqq \text{Support}(\nu_x)$.

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Now if $\tilde{x} \in \text{Support}(\nu_x) \backslash T_2(G)$ then the image of $\tilde{x}$ in $\mathbb{Z}_2$ is of the form $1 + \sum_{i=1}^{\infty} a_i 2^i$.

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Such elements generate a dense (cyclic) subgroup of $\mathbb{Z}_2$.

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Now if $\tilde{x} \in \text{Support}(\nu_x) \setminus T_2(G)$ then the image of $\tilde{x}$ in $\mathbb{Z}_2$ is of the form $1 + \sum_{i=1}^{\infty} a_i 2^i$.
Such elements generate a dense (cyclic) subgroup of $\mathbb{Z}_2$.
Hence we obtain that a dense subset of G stabilizes ν_x, which implies that $\nu_x = m_G$.

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The end :)