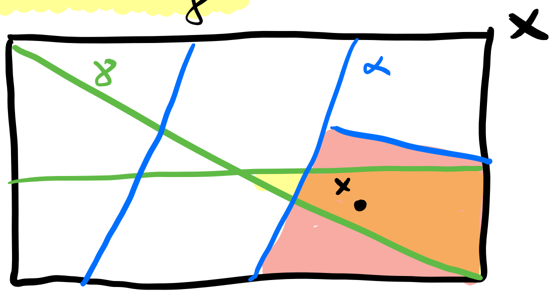
MORE (CONDITIONAL) ENTROPY. 10.12.2020

$(X, \mathcal{B}, \mu)$ Borel prob. space. ($(X,\mathcal{B})$ is std. Borel space)

Previously: (Triftach)



γ : countable m'ble partition.
α

$\gamma \vee \alpha = \{ C \cap A \mid A \in \alpha, C \in \gamma \}$.
↑ common ref.

$$I_\mu(\gamma \mid \alpha)(x) = -\log \frac{\mu \, [x]_{\gamma \vee \alpha}}{\mu [x]_\alpha} = -\log \mu|_{[x]_\alpha}([x]_\gamma)$$

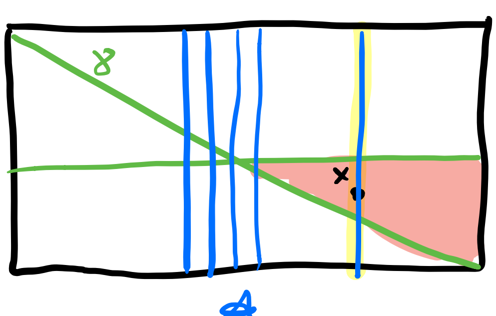
$[x]_{\gamma \vee \alpha} = [x]_\gamma \cap [x]_\alpha$

normalized restr.

$$H_\mu(\gamma \mid \alpha) = \int_X I_\mu(\gamma \mid \alpha) \, d\mu$$

Today: More general "partitions".

possibly uncountable, with lots of null sets.



countably generated σ-algebra.

$$I_\mu(\gamma \mid \mathcal{A})(x) = -\log \text{"}\mu|_{[x]_\mathcal{A}}\text{"}([x]_\gamma).$$

we need an adequate replacement of this in general.

! If $\mathcal{A}$ is a countably gen. σ-alg., then

$$[x]_{\mathcal{A}} := \bigcap_{x \in A \in \mathcal{A}} A = \bigcap_{x \in A \in \underbrace{\mathcal{B}'}} A \in \mathcal{A}.$$

(Exercise)

$\bar{c}$ algebra gen. by $\{A_1, A_2, \dots\}$ gen.

Atoms give measurable partition of X.

§1. Conditional measures:

Our main technical tool (from functional analysis) is conditional expectation.

Theorem: $(X, \mathcal{B}, \mu)$ prob. space
$\mathcal{A} \subset \mathcal{B}$ sub-σ-algebra

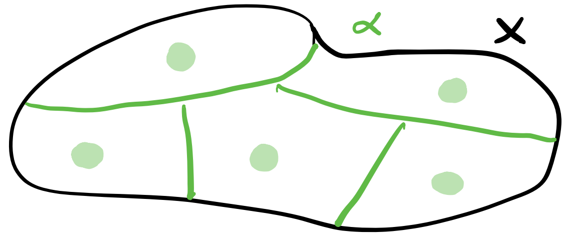
For every $f \in L^1(X, \mathcal{B}, \mu)$, there is a unique fct. $E(f \mid \mathcal{A}) \in L^1(X, \mathcal{A}, \mu)$ s.t.

$$\int_A \underbrace{E(f \mid \mathcal{A})}_{\text{conditional expectation}} \, d\mu = \int_A f \, d\mu. \qquad \forall A \in \mathcal{A}.$$

Moreover, $\mathbb{E}(-, \mathcal{A}) : L^1(X, \mathcal{B}, \mu) \to L^1(X, \mathcal{A}, \mu)$.
is a linear, positive op. of norm one.

Pf:
$$L^2(X, \mathcal{A}, \mu) \underset{\text{closed}}{\subset} L^2(X, \mathcal{B}, \mu).$$

Ex:



$\mathcal{A} = \sigma(\alpha)$

$\mathcal{A}$ - measurable = const. on each of these parts.

$$\mathbb{E}(f \mid \alpha)(x) = \sum_{A \in \alpha} c_A(f) \cdot \mathbb{1}_A$$

$$c_A(f) = \fint_A f \, d\mu.$$

Theorem (conditional measures)

$(X, \mathcal{B}, \mu)$ Borel prob. space

$\mathcal{A} \subset \mathcal{B}$ sub-σ-algebra

Then $\exists \; X' \subset X$ $\mathcal{A}$-m'ble co-null

$$\mu^{\mathcal{A}} : X' \longrightarrow \mathrm{Prob}(X, \mathcal{B})$$
$$x \longmapsto \mu_x^{\mathcal{A}}$$

$\boxed{\mu^{\mathcal{A}} \text{ is } \mathcal{A}\text{-m'ble } \& \quad \mu = \int \mu_x^{\mathcal{A}} \, d\mu(x)}$

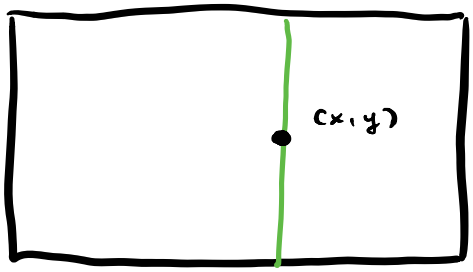
s.t. $\displaystyle\int_X f \, d\mu_x^{\mathcal{A}} = \mathbb{E}(f \mid \mathcal{A})(x)$ a.e. x

for all $f \in \mathcal{L}^1(X, \mathcal{B}, \mu)$.

- $\mu^{\mathcal{A}}$ is uniquely defined up to X' ($\mathcal{A}$-conull set)
- $\mathcal{A} \overset{\mu}{=} \mathcal{C} \implies \mu^{\mathcal{A}} \overset{\mu}{=} \mu^{\mathcal{C}}$.
- If $\mathcal{A}$ is countably generated $\mu_x^{\mathcal{A}}([x]_{\mathcal{A}}) = 1$.

Pf Dense subset of $C(\overline{X})$ $\square$

Ex.



$I = [0,1]$

$\mathcal{B}$ = Borel

λ = Lebesgue.

$(I^2, \mathcal{B}^2, \mu = \lambda^{\otimes 2})$.

$\mathcal{A} = \mathcal{B} \times \{\emptyset, I\}$. $\quad [(x,y)]_{\mathcal{A}} = \{x\} \times I$

$$\mathbb{E}(f \mid \mathcal{A})(x,y) = \int_{I^2} f(s,t)\, d\underbrace{(\delta_x \otimes \lambda)}_{\mu^{\mathcal{A}}_{(x,y)}}(s,t)$$

$$= \underline{\int_I f(x,t)\, d\lambda(t)}$$

Important properties

- If $N \in \mathcal{B}$ null, then $\mu^{\mathcal{A}}_x(N) = 0$ a.e.
- If $\mathcal{A}' \subset \underline{\mathcal{A}} \subset \mathcal{B}$ are c.g.:

  $[x]_{\mathcal{A}} \subset [x]_{\mathcal{A}'}$ for all x

  $(\mu^{\mathcal{A}}_x)^{\mathcal{A}'}_z = \mu^{\mathcal{A}'}_z$ for a.e $z \in [x]_{\mathcal{A}'}$.
- If $\phi : (X, \mathcal{B}_X, \mu) \xrightarrow{\text{measure preserving}} (Y, \underline{\mathcal{B}_Y}, \nu)$, $\quad \mathcal{B}_Y \supset \mathcal{A}$

  $\phi^{-1}\mathcal{A} \subset \mathcal{B}_X$

  $\phi_* \mu_x^{\phi^{-1}\mathcal{A}} = \nu^{\mathcal{A}}_{\phi(x)}$ for a.e x.

§2. Conditional entropy

Def: $(X, \mathcal{B}, \mu)$ Borel prob.
$\mathcal{A}, \mathcal{C} \subset \mathcal{B}$ countably gen.

$$I_\mu(\mathcal{C} \mid \mathcal{A})(x) = -\log \mu_x^{\mathcal{A}}([x]_{\mathcal{C}}).$$

(*) $x \mapsto I_\mu(\mathcal{C} \mid \mathcal{A})(x)$ is Borel.

$$H_\mu(\mathcal{C} \mid \mathcal{A}) = \int_X I_\mu(\mathcal{C} \mid \mathcal{A})\, d\mu .$$

Exercise:
- $I_\mu(\mathcal{C} \mid \mathcal{A}) = I_\mu(\mathcal{A} \vee \mathcal{C} \mid \mathcal{A})$
- $\mathcal{C} \overset{\mu}{=} \mathcal{C}' \implies I_\mu(\mathcal{C} \mid \mathcal{A}) = I_\mu(\mathcal{C}' \mid \mathcal{A})$.
- $\mathcal{A} \overset{\mu}{=} \mathcal{A}' \implies I_\mu(\mathcal{C} \mid \mathcal{A}) = I_\mu(\mathcal{C} \mid \mathcal{A}')$
- This def. extends the one from Tiftach's lecture.
- If $\mathcal{N} = \{\emptyset, X\}$:

  $I_\mu(\mathcal{C} \mid \mathcal{N})(x) =: I_\mu(\mathcal{C})(x) = -\log \mu [x]_{\mathcal{C}}$.

  $H_\mu(\mathcal{C} \mid \mathcal{N}) =: H_\mu(\mathcal{C}) = \int_X I_\mu(\mathcal{C})\, d\mu$

  $H_\mu(\mathcal{C}) < \infty \iff \mathcal{C} = \sigma(\xi)$

  ξ is m'ble partition w/ finite entropy.

Proposition $[\, I_\mu(-|\mathcal{A}) \,,\ H_\mu(-|\mathcal{A}) \,]$

$\mathcal{A}, \underbrace{\mathcal{C}_n, \mathcal{C}}_{\text{countably gen}} \subset \mathcal{B}$ sub-σ-alg. $(n \in \mathbb{N})$

Then

(1) $x \longmapsto I_\mu(\mathcal{C}|\mathcal{A})(x)$ is $\sigma(\mathcal{A} \cup \mathcal{C})$-measurable.

(2) $\mathcal{C}_1 \subset \mathcal{C}_2 \implies I_\mu(\mathcal{C}_1|\mathcal{A}) \leq I_\mu(\mathcal{C}_2|\mathcal{A})$.

(3) If $\mathcal{C}_n \nearrow \mathcal{C}$ then

(i) $I_\mu(\mathcal{C}_n|\mathcal{A}) \nearrow I_\mu(\mathcal{C}|\mathcal{A})$.

(ii) $\underline{H_\mu(\mathcal{C}_n|\mathcal{A}) \nearrow H_\mu(\mathcal{C}|\mathcal{A})}$.

$$\cdots \subset \mathcal{C}_n \subset \mathcal{C}_{n+1} \subset \cdots,$$
$$\mathcal{C} = \sigma\Big(\bigcup_n \mathcal{C}_n\Big).$$

Pf: (2) $[x]_{\mathcal{C}_2} \subset [x]_{\mathcal{C}_1}$

(3) (i) $[x]_{\mathcal{C}} = \bigcap_n [x]_{\mathcal{C}_n}$

$$\implies \mu_x^{\mathcal{A}}[x]_{\mathcal{C}_n} \searrow \mu_x^{\mathcal{A}}[x]_{\mathcal{C}}.$$

---

(1) $\mathcal{C} = \sigma(\{C_1, C_2, \ldots\})$

$$\boxed{\begin{array}{l} \mathcal{C}_n := \sigma(\{C_1, \ldots, C_n\}) \\ x \longmapsto I_\mu(\mathcal{C}_n|\mathcal{A})(x) \quad \text{is mble.} \end{array}}$$

$\square$

$[I_\mu(\xi \mid -), \quad H_\mu(\xi \mid -)].$

Proposition:

ξ countable m'ble partition of X.

$\mathcal{A}_n \nearrow \mathcal{A}_\infty$ sub-σ-alg. of $\mathcal{B}$

or $\mathcal{A}_n \searrow \mathcal{A}_\infty$.

Then (i) $I_\mu(\xi \mid \mathcal{A}_n) \longrightarrow I_\mu(\xi \mid \mathcal{A}_\infty)$.

(ii) $H_\mu(\xi \mid \mathcal{A}_n) \longrightarrow H_\mu(\xi \mid \mathcal{A}_\infty)$, if ξ finite.

Pf: (i) Martingale convergence

$\mathcal{A}_n \nearrow \mathcal{A}_\infty$, $\searrow$: $\mathbb{E}(f \mid \mathcal{A}_n) \to \mathbb{E}(f \mid \mathcal{A}_\infty)$.
$f \in L^1(X, \mathcal{B}, \mu)$

$$I_\mu(\xi \mid \mathcal{A}_n)(x) = \sum_{P \in \xi} -\mathbb{1}_P(x) \log \mu_x^{\mathcal{A}_n}(P) = \sum_{P \in \xi} -\mathbb{1}_P(x) \log \underline{\mathbb{E}(\mathbb{1}_P \mid \mathcal{A}_n)}(x)$$

(ii) <u>Lemma</u>: $H_\mu(\xi \mid \mathcal{A}) = \int_X \underbrace{H_{\mu_x^{\mathcal{A}}}(\xi)}_{\leq \log |\xi|.} \, d\mu(x)$

$\square$

Proposition: $\mathcal{A}, \mathcal{C}_1, \mathcal{C}_2$ countably generated.

- $\begin{cases} I_\mu(\mathcal{C}_1 \vee \mathcal{C}_2 \mid \mathcal{A}) = I_\mu(\mathcal{C}_1 \mid \mathcal{A}) + I_\mu(\mathcal{C}_2 \mid \mathcal{C}_1 \vee \mathcal{A}) \\ H_\mu(\mathcal{C}_1 \vee \mathcal{C}_2 \mid \mathcal{A}) = H_\mu(\mathcal{C}_1 \mid \mathcal{A}) + H_\mu(\mathcal{C}_2 \mid \mathcal{C}_1 \vee \mathcal{A}) \end{cases}$
- $H_\mu(\mathcal{C}_2 \mid \mathcal{C}_1 \vee \mathcal{A}) \leq H_\mu(\mathcal{C}_2 \mid \mathcal{A})$
- $H_\mu(\mathcal{C}_1 \vee \mathcal{C}_2 \mid \mathcal{A}) \leq H_\mu(\mathcal{C}_1 \mid \mathcal{A}) + H_\mu(\mathcal{C}_2 \mid \mathcal{A})$

MORE & MORE ENTROPY. 17.12.2020

Last time: $(X, \mathcal{B}, \mu)$ Borel prob.
$\mathcal{C}, \mathcal{A} \subset \mathcal{B}$ sub σ alg., $\mathcal{C}$ c.g.

$$I_\mu(\mathcal{C}|\mathcal{A})(x) := -\log \mu_x^{\mathcal{A}}\, [x]_{\mathcal{C}} .$$

$$H_\mu(\mathcal{C}|\mathcal{A}) := \int_X I_\mu(\mathcal{C}|\mathcal{A})\, d\mu .$$

Properties:

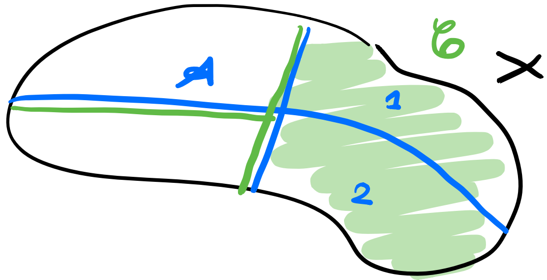
- If $\mathcal{C}_n \nearrow \mathcal{C}$ c.g., then $H_\mu(\mathcal{C}_n|\mathcal{A}) \nearrow H_\mu(\mathcal{C}|\mathcal{A})$.
- If $\mathcal{A}_n \nearrow \mathcal{A}$ & ξ is a finite partition, or $\mathcal{A}_n \searrow \mathcal{A}$ then $H_\mu(\xi|\mathcal{A}_n) \longrightarrow H_\mu(\xi|\mathcal{A})$.
- If $\mathcal{A}_n \nearrow \mathcal{A}$ & ξ a countable partition, also get $H_\mu(\xi|\mathcal{A}_n) \longrightarrow H_\mu(\xi|\mathcal{A})$.
- $\mathcal{C}_1, \mathcal{C}_2, \mathcal{A}$ are c.g.
  $H_\mu(\mathcal{C}_1 \vee \mathcal{C}_2|\mathcal{A}) = H_\mu(\mathcal{C}_1|\mathcal{A}) + H_\mu(\mathcal{C}_2|\mathcal{C}_1 \vee \mathcal{A})$.
  $H_\mu(\mathcal{C}_2|\mathcal{C}_1 \vee \mathcal{A}) \le H_\mu(\mathcal{C}_2|\mathcal{A})$.

- Extremal values of entropy

Prop: $\mathcal{C}, \mathcal{A}$ c.g.

$$H_\mu(\mathcal{C}|\mathcal{A}) = 0 \iff \mathcal{C} \underset{\mu}{\subset} \mathcal{A} .$$

$\forall\, C \in \mathcal{C} : \exists\, A \in \mathcal{A}$
w/ $\mu(A \,\Delta\, C) = 0$



$\subset 1 \cup 2.$

If $\mathcal{C}$ is c.g.
$\mathcal{C} \subseteq_{\mu} \mathcal{A}$ if
$\exists$ N null set in $\mathcal{B}$:
for every $C \in \mathcal{C}$
$\exists\, A \in \mathcal{A}$ s.t.
$A \setminus N = C \setminus N$.

Pf: $$H_\mu(\mathcal{C} \mid \mathcal{A}) = \int I_\mu(\mathcal{C} \mid \mathcal{A})\, d\mu = 0$$

$$\iff \mu_x^{\mathcal{A}}[x]_{\mathcal{C}} = 1 \quad \text{a.e. } x \in X$$

$(\Leftarrow)$ $$[x]_{\mathcal{A}} \setminus N \subset [x]_{\mathcal{C}} \quad \forall\, x.$$

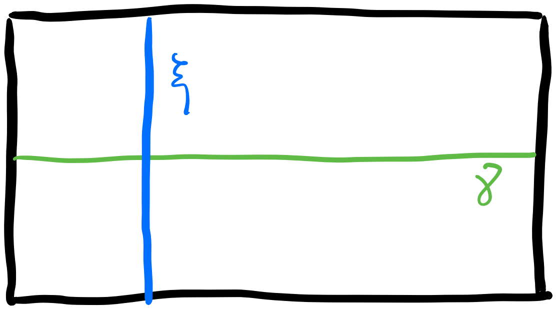
$$\Rightarrow \underset{\substack{\| \\ 1}}{\mu_x^{\mathcal{A}}[x]_{\mathcal{A}}} \leq \mu_x^{\mathcal{A}}[x]_{\mathcal{C}} \quad \text{a.e.}$$

$(\Rightarrow)$ $$\underset{\substack{\| \\ \mathbb{E}(\mathbb{1}_C \mid \mathcal{A})(x)}}{\mu_x^{\mathcal{A}}(C)} = \mathbb{1}_C(x) \quad \text{for every } C \in \mathcal{C} \text{ \& almost every } x \in C.$$

$$\Rightarrow C \in_{\mu} \mathcal{A}. \qquad \square$$

As with partitions, we say $\mathcal{A}, \mathcal{C} \subset \mathcal{B}$ are independent $(\mathcal{A} \perp \mathcal{C})$ if

$$(*) \quad \mu(A \cap C) = \mu(A) \cdot \mu(C) \quad \forall\, A \in \mathcal{A},\ C \in \mathcal{A}.$$



Proposition: $\mathcal{C}$ countably gen.

ξ countable m'ble partition w/ $< \infty$ entropy.

$P \in \mathcal{B}$.

(i) $P \perp \mathcal{C} \iff \mu_x^{\mathcal{A}}(P) = \mu(P)$ a.e.

(ii) $\xi \perp \mathcal{C} \iff H_\mu(\xi \mid \mathcal{C}) = H_\mu(\xi)$.

Pf: (i) $(\Longleftarrow)$ $C \in \mathcal{C}$.

$$\mu(P \cap C) = \int_C 1\!\!1_P \, d\mu = \int_C \underbrace{\mathbb{E}(1\!\!1_P \mid \mathcal{C})}_{\mu_x^{\mathcal{A}}(P) \,=\, \mu(P)}{}^{(*)} \, d\mu(x)$$

$$= \int_C \mu(P) \, d\mu$$

$$= \mu(P)\,\mu(C).$$

$(\Longrightarrow)$ For any $C \in \mathcal{C}$

$$\int_C \mu(P)\, d\mu = \mu(P \cap C) = \int_C \mathbb{1}_P \, d\mu .$$

$$= \int_C \underset{\underset{\mu_x^{\mathcal{A}}(P)}{\shortparallel}}{\mathbb{E}(\mathbb{1}_P \mid \mathcal{A})(x)} \, d\mu(x)$$

$$\Longrightarrow \mu_x^{\mathcal{A}}(P) = \mu(P) \quad \text{a.e.}$$

(ii) $$H_\mu(\xi \mid \mathcal{B}) = \int H_{\mu_x^{\mathcal{B}}}(\xi)\, d\mu(x)$$

$$= -\sum_{P \in \xi} \int \phi\left(\mu_x^{\mathcal{B}}(P)\right) d\mu(x)$$

$$\phi(x) = x \log x$$

$$\leq -\sum_{P \in \xi} \phi\Big(\underbrace{\int_X \mu_x^{\mathcal{B}}(P)\, d\mu(x)}_{= \mu(P)}\Big)$$

$$= H_\mu(\xi)$$

$=$ iff $\mu_x^{\mathcal{B}}(P) = \mu(P)$ a.e.

$$\Updownarrow$$

$$\mu(P \cap C) = \mu(P)\mu(C).$$

$\square$

§ 3. Conditional entropy of a transf.

$(X, \mathcal{B}_X, \mu, T)$ mps, $\mathcal{A} \subset \mathcal{B}_X$ sub-σ-alg.

Borel prob. sp.

Notation : ξ measurable partition of X

$$\xi_m^n := \bigvee_{k=m}^{n} T^{-k} \xi$$

Def : Assume that $T^{-1}\mathcal{A} = \mathcal{A}$ ($\mathcal{A}$ is T-inv).

We define :

- $h_\mu(T, \xi \mid \mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_0^{n-1} \mid \mathcal{A})$
$$= \inf_{n \geq 1} \frac{1}{n} H_\mu(\xi_0^{n-1} \mid \mathcal{A}).$$

- $h_\mu(T \mid \mathcal{A}) = \sup_{\xi : H_\mu(\xi) < \infty} h_\mu(T, \xi \mid \mathcal{A}).$

is OK : Fekete's lemma + T-invariance of H_μ

Lemma

$$I_\mu(T^{-1}\mathcal{C} \mid T^{-1}\mathcal{A}) = I_\mu(\mathcal{C} \mid \mathcal{A}) \circ T.$$
$$H_\mu(T^{-1}\mathcal{C} \mid T^{-1}\mathcal{A}) = H_\mu(\mathcal{C} \mid \mathcal{A}).$$

Pf : $I_\mu(\mathcal{C} \mid \mathcal{A})(Tx) = -\log \mu_{Tx}^{\mathcal{A}} [Tx]_{\mathcal{C}}$

$$= -\log T_* \mu_x^{T^{-1}\mathcal{A}} [Tx]_{\mathcal{C}}$$
$$= -\log \mu_x^{T^{-1}\mathcal{A}} [x]_{T^{-1}\mathcal{C}}.$$
$$= I_\mu(T^{-1}\mathcal{C} \mid T^{-1}\mathcal{A})(x) \quad \text{a.e.} \quad \square$$

Properties $\mathcal{A}$ T-inv ; ξ, η w/ $H_\mu < \infty$.

bounds

- $h_\mu(T, \xi \mid \mathcal{A}) \leq H_\mu(\xi \mid \mathcal{A})$. trivial
- $h_\mu(T, \xi \vee \eta \mid \mathcal{A}) \leq h_\mu(T, \xi \mid \mathcal{A}) + h_\mu(T, \eta \mid \mathcal{A})$. subadditivity
- $h_\mu(T, \eta \mid \mathcal{A}) \leq h_\mu(T, \xi \mid \mathcal{A}) + H_\mu(\eta \mid \xi \vee \mathcal{A})$. continuity

iterates

- $h_\mu(T, \xi \mid \mathcal{A}) = h_\mu(T, \xi_0^{k} \mid \mathcal{A}) \quad \forall k$.
  $= h_\mu(T^{-1}, \xi \mid \mathcal{A})$; $-k$ if T is invertible.
- $h_\mu(T^k \mid \mathcal{A}) = k\, h_\mu(T \mid \mathcal{A}) \quad \forall k$.
- $h_\mu(T \mid \mathcal{A}) = h_\mu(T^{-1} \mid \mathcal{A})$ if T is invertible.
- (**future formula**)
$$h_\mu(T, \xi \mid \mathcal{A}) = H_\mu(\xi \mid \xi_1^\infty \vee \mathcal{A}).$$
- (**additivity**) T invertible
$$h_\mu(T, \xi \vee \eta \mid \mathcal{A}) = h_\mu(T, \xi \mid \mathcal{A}) + h_\mu(T, \eta \mid \xi_{-\infty}^\infty \vee \mathcal{A})$$
$$= h_\mu(T, \xi \mid \mathcal{A}) + H_\mu(\eta \mid \eta_1^\infty \vee \xi_{-\infty}^\infty \vee \mathcal{A}).$$

Pf:
$$h_\mu(T, \xi \mid \mathcal{A}) = \lim_{n\to\infty} \underline{\frac{1}{n} H_\mu(\xi_0^{n-1} / \mathcal{A})}$$

$$\frac{1}{n} H_\mu(\underbrace{\xi_0^{n-1}}_{= \xi_1^{n-1} \vee \xi} \mid \mathcal{A}) = \frac{1}{n}\Big(\underbrace{H_\mu(\xi_1^{n-1} \mid \mathcal{A})}_{= H_\mu(T^{-1}\xi_0^{n-2} / T^{-1}\mathcal{A})} + H_\mu(\xi \mid \xi_1^{n-1} \vee \mathcal{A}) \Big)$$

$$= \frac{1}{n}\left(H_\mu(\xi_0^{n-2} \mid \mathcal{A}) + H_\mu(\xi \mid \xi_1^{n-1} \vee \mathcal{A}) \right)$$

$$= \cdots = \underline{\frac{1}{n} \sum_{i=0}^{n-1} \underline{H_\mu(\xi \mid \xi_1^i \vee \mathcal{A})}}$$

$$\xrightarrow{\text{cesaro}} H_\mu(\xi \mid \xi_1^\infty \vee \mathcal{A}).$$

$\square$

<u>Recall</u>: $(Y, \mathcal{B}_Y, \nu, S)$ is a <u>factor</u> of X if

$\phi : X \longrightarrow Y$ measure-pres. s.t. $\phi(Tx) = S\phi(x)$ for almost all $x \in X$.

In that case: $h_\nu(S) \leq h_\mu(T)$.

There is a correspondence between factors of X & invariant sub-σ-algebras of $\mathcal{B}_X$:

i) $\phi^{-1}\mathcal{B}_Y =: \mathcal{A} \subset \mathcal{B}_X$ is T-invariant.

ii) $\mathcal{A} \subset \mathcal{B}_X \longrightarrow \exists\ \phi : X \xrightarrow{\text{m.p.}} (Y, \mathcal{B}_Y, \nu, S)$ s.t. $\mathcal{A} \underset{\mu}{=} \phi^{-1}\mathcal{B}_Y$.

$$\left(\begin{array}{ll} Y = \mathrm{Prob}(\overline{X}), & \phi(x) = \mu_x^{\mathcal{A}}, \\ \nu = \phi_* \mu, & S\nu = T_*\nu. \end{array} \right).$$

$$h_\nu(S) = \sup_{\substack{\xi \subset \phi^{-1}\mathcal{B}_Y \\ H_\mu(\xi)<\infty}} h_\mu(T,\xi)$$

$$\leq \sup_{\substack{\xi, \\ H_\mu(\xi)<\infty}} h_\mu(T,\xi) = h_\mu(T).$$

<u>Theorem</u> [Abramov – Rokhlin formula].

$(X, \mathcal{B}_X, \mu, T)$ be an invertible mps.

$\phi : X \longrightarrow (Y, \mathcal{B}_Y, \nu, S)$ factor map.

Then

$$h_\mu(T) = h_\nu(S) + h_\mu(T \mid \mathcal{A})$$

where $\mathcal{A} = \phi^{-1}\mathcal{B}_Y \subset \mathcal{B}_X$.

We need the following version of Kolmogorov-Sinai theorem :

<u>Theorem</u> : $(X, \mathcal{B}, \mu, T)$ mps.

$(\xi_k)_k$ generating sequence of partitions

i.e. $\mathcal{B} \underset{\mu}{=} \bigvee_{k=0}^{\infty} (\xi_k)_0^\infty$ & $(\xi_k)_0^\infty \subset (\xi_{k+1})_0^\infty$ for all k.

Then if $\mathcal{A} \subset \mathcal{B}$ is T-invariant

$$h_\mu(T \mid \mathcal{A}) = \lim_{k\to\infty} h_\mu(T, \xi_k \mid \mathcal{A})$$
$$= \sup_k h_\mu(T, \xi_k \mid \mathcal{A}).$$

Pf:

1) Use continuity estimate

For any part. ξ of finite entropy, for all k

$$h_\mu(T, \xi \mid \mathcal{A}) \le h_\mu(T, (\xi_k)_0^k \mid \mathcal{A}) = h_\mu(T, \xi_k \mid \mathcal{A})$$
$$+ H_\mu(\xi \mid (\xi_k)_0^k \vee \mathcal{A}).$$

2) $\xi = \xi_j \quad 1 \le j < k$.

$$h_\mu(T, \xi_j \mid \mathcal{A}) \le h_\mu(T, \xi_k \mid \mathcal{A})$$
$$\Rightarrow \underbrace{\lim_k h_\mu(T, \xi_k \mid \mathcal{A})}_{\text{exists}} = \sup_k h_\mu(T, \xi_k \mid \mathcal{A}).$$

3) for any ξ

$$h_\mu(T, \xi \mid \mathcal{A}) \le \lim_{k\to\infty} \Big(h_\mu(T, \xi_k \mid \mathcal{A}) + H_\mu(\xi \mid (\xi_k)_0^k \vee \mathcal{A}) \Big)$$
$$\le \lim_{k\to\infty} h_\mu(T, \xi_k \mid \mathcal{A}).$$

$\blacksquare$

Pf of AR formula: Pick sequences of countable partitions

$$\eta_m \nearrow \mathcal{B}_Y, \qquad \xi_n \nearrow \mathcal{B}_X .$$

$$h_\mu(T, \xi_n \vee \phi^{-1}\eta_m) = h_\mu(T, \phi^{-1}\eta_m) + H_\mu\left(\xi_n \mid (\xi_n)_1^\infty \vee (\phi^{-1}\eta_m)_{-\infty}^\infty\right)$$

Assume $h_\mu(T) < \infty$:

For $\varepsilon > 0$ & large enough n:

$$h_\mu(T) - \varepsilon \underset{\substack{\uparrow \\ \text{KS}}}{<} h_\mu(T, \xi_n) \le h_\mu(T, \xi_n \vee \phi^{-1}\eta_m)$$

$$\le h_\mu(T)$$

Hence if $m \to \infty$

$$h_\mu(T) - \varepsilon < h_\nu(S) + h_\mu(T, \xi_n \mid \overset{\mathcal{A}}{\overset{\|}{\phi^{-1}\mathcal{B}_Y}})$$

$$\le h_\mu(T)$$

$n \to \infty$

$$h_\mu(T) - \varepsilon < h_\nu(S) + h_\mu(T \mid \mathcal{A})$$

$$\le h_\mu(T).$$

∎

§4. Pinsker algebra

$(X, \mathcal{B}, \mu, T)$ invertible mps

$$\mathcal{P}(T) = \{ B \in \mathcal{B} \mid h_\mu(T, \{B, X \setminus B\}) = 0 \}$$
$$= \{ B \in \mathcal{B} \mid B \underset{\mu}{\in} \{B, X\setminus B\}_1^\infty \}.$$
$$\{B, X\setminus B\}_1^\infty = \bigvee_{k=1}^{\infty} T^{-k}\{B, X\setminus B\}.$$

is an invariant sub-σ-algebra of $\mathcal{B}$.

$\rightsquigarrow$

$(X_{\mathcal{P}}, \mathcal{B}_{\mathcal{P}}, \mu_{\mathcal{P}}, T_{\mathcal{P}})$ Pinsker factor.

$h_{\mu_{\mathcal{P}}}(T_{\mathcal{P}}) = 0$; is maximal factor with this property.