Entropy, leafwise measures and invariance

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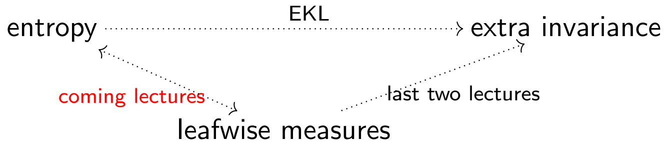
Properties of Leafwise measures

We have seen that leafwise measures μ_x^T describe properties of the measure μ along the direction of T-leaves. For instance:

- μ_x^T is trivial (i.e. $\mu_x^T \propto \delta_e$) a.e. if and only if there exists a measurable set $B \subset X$ with $\mu(X \setminus B) = 0$ such that $x, tx \in B$ for some $t \in T$ implies $t = e$ (i.e. B is a global cross-section).
- μ_x^T is infinite a.e. if and only if μ is T-recurrent.
- μ_x^T is the left Haar measure on T a.e. if and only if μ is T-invariant.

Recap:



Setting

Throughout, we let

- $G \subset \mathrm{SL}_n(\mathbb{R})$ be a closed connected real linear group.
- $\Gamma \subset G$ a discrete subgroup of G.
- $X := G/\Gamma$ with Riemannian metric induced from a right-invariant metric on G.
- there is a G-left action on X by left multiplication $x \mapsto gx$.
- μ be a (Borel probability) measure on X.

Sometimes we specialize to the following special case:

- Γ is a **lattice** if there is a G-invariant probability measure m_X (the **Haar measure**) on X.

Horospherical subgroup

Fix $a \in G$. Define the **stable horospherical subgroup for** a by

$$G^- := \{g \in G \mid a^n g a^{-n} \to e \quad \text{as} \quad n \to \infty\}.$$

Similarly define the **unstable horospherical subgroup** G^+ **for** a**.**
Note that for $x, gx \in X$ for some $g \in G^-$,

$$d(a^n x, a^n g x) = d(a^n x, (a^n g a^{-n}) a^n x) \leq d(e, a^n g a^{-n}) \to 0.$$

Thus we refer to $G^- x$ as the **stable manifold through** x.
Example: For $G := \mathrm{SL}_3(\mathbb{R}), \Gamma := \mathrm{SL}_3(\mathbb{Z})$ and $a := \mathrm{diag}(e^t, e^s, e^r) \in G$ with $t > s > r$,

$$G^- = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}, \qquad G^+ = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}.$$

Entropy contribution: Theorem 7.6

Theorem (Clay notes Thm. 7.6)

Fix $a \in G$. Let μ be an a-invariant probability measure on G/Γ. Let U be a closed subgroup of G^- normalized by a. Then

1. *The* **entropy contribution** *of U at x*

$$D_\mu(a, U)(x) := \lim_{n\to\infty} \frac{1}{n} \log \mu_x^U(a^{-n} B_1^U a^n)$$

exists for μ-a.e. x and defines an a-invariant function on X.

2. *For μ-a.e. x we have $D_\mu(a, U)(x) \leq h_{\mu_x^{\mathcal{E}}}(a)$, with equality if $U = G^-$. Here $\mathcal{E}$ denotes the σ-algebra of a-invariant sets.*
3. *For μ-a.e. x we have $D_\mu(a, U)(x) = 0$ iff μ_x^U is finite iff μ_x^U is trivial.*

Remark: we are NOT assuming: (i) a is diagonalizable, (ii) μ is a-ergodic (not even A-ergodic, which is what EKL assumes), (iii) Γ is a lattice in G.

Characterization of recurrence using entropy

From (b), we know that $D_\mu(a, G^-)(x) \leq h_{\mu_x^{\mathcal{E}}}(a)$. From (c), we know that $D_\mu(a, G^-)(x) = 0$ iff μ_x^U is finite. Therefore we have the corollary

Corollary

The measure μ is G^--recurrent iff $h_{\mu_x^{\mathcal{E}}}(a) > 0$ a.e.
Assuming μ is in addition a-ergodic, then μ is G^--recurrent iff $h_\mu(a) > 0$.

Example

Let $G := \mathrm{SL}_3(\mathbb{R}), \Gamma := \mathrm{SL}_3(\mathbb{Z})$ and $a := \mathrm{diag}(e^t, e^s, e^r) \in G$ with $t > s > r$. Let

$$U_1 := \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \subset \quad U_2 := \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{bmatrix} \quad \subset \quad U_3 := G^- = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$$

Let $\mu := m_X$ be the Haar measure on X. Note that

$$a^{-1} \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} a = \begin{bmatrix} 1 & 0 & 0 \\ xe^{t-s} & 1 & 0 \\ ye^{t-r} & ze^{s-r} & 1 \end{bmatrix}.$$

Since $\mu = m_X$ is invariant under G^-, $\mu_x^{U_i}$ is Haar on U_i a.e. for $i = 1, 2, 3$, thus we can calculate

$$D_{m_X}(a, U_1)(x) := \lim_{n \to \infty} \frac{1}{n} \log \mu_x^U(a^{-n} B_1^U a^n) = t - s,$$

Similarly $D_{m_X}(a, U_2)(x) = (t - s) + (t - r), \quad D_{m_X}(a, G^-)(x) = 2(t - r).$

Entropy contribution for Haar measure m_X

In general, it can be computed that if $\mu = m_X$ is Haar, then

$$D_{m_X}(a, G^-) = -\log|\det \mathrm{Ad}_a|_{\mathfrak{g}^-}|,$$

where $\mathrm{Ad}_a : \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation defined by $v \mapsto ava^{-1}$.

Clearly if μ_x^U is finite, then $D_\mu(a, U)(x) = 0$.

The calculation for general μ, however, is less clear.

Entropy contribution and Theorem 7.9

Fix $a \in G$. Define the **entropy contribution** of an a-normalized closed subgroup $U \subset G^-$.

$$h_\mu(a, U) := \int D_\mu(a, U)\, d\mu.$$

(Theorem 7.6(2) implies that $h_\mu(a, G^-) = h_\mu(a)$)

The next theorem in particular shows that the Haar measure m_X has maximal entropy contribution of G^- among a-invariant measures μ.

Theorem (Clay notes Thm. 7.9)

Let $U \subset G^-$ be an a-normalized closed subgroup of G^-, and $\mathfrak{u} := \mathrm{Lie}(U)$. Let μ be an a-invariant probability measure on $X = G/\Gamma$. Then

$$h_\mu(a, U) \leq -\log |\det \mathrm{Ad}_a|_{\mathfrak{u}}|,$$

and equality holds iff μ is U-invariant.

Corollary of Theorem 7.9: m_X is the unique MME

As a corollary of Theorem 7.9, one can show that in many cases, the Haar measure is the unique measure of maximal entropy.

Corollary (Clay notes Corollary 7.10)

Suppose Γ is a lattice in G, and let $X = G/\Gamma$. Suppose $a \in G$ is such that G is generated by G^+ and G^-.
Then m_X is the **unique** *measure of maximal entropy for the a-action on X, i.e. if μ is an a-invariant probability measure on X with $h_\mu(a) = h_{m_X}(a)$, then $\mu = m_X$.*

Note that the assumption on a is satisfied quite generally - for instance if G is a simple real Lie group and $\mathfrak{g}^-$ is nontrivial.

Entropy assumption $\implies$ extra invariance

To summarize, consider the case of $X = \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ and $a = \mathrm{diag}(t, s, r) \in \mathrm{SL}_3(\mathbb{R})$ with $t > s > r$.
Let A be the full diagonal subgroup of $\mathrm{SL}_3(\mathbb{R})$. Let m_X be Haar on X.

- Corollary 7.10 implies:
  if μ is a-invariant and $h_\mu(a) = h_{m_X}(a)$, then $\mu = m_X$.
- High entropy method of Einsiedler-Katok (Thm. 9.5) implies:
  if μ is A-ergodic and $h_\mu(a) > \frac{1}{2} h_{m_X}(a)$, then $\mu = m_X$.
- High entropy method can also show (Problem 9.13) that:
  if μ is A-ergodic and all $a \neq e \in A$ satisfies $h_\mu(a) > 0$, then $\mu = m_X$.
- High+low entropy by Einsiedler-Katok-Lindenstrauss (Thm. 11.5):
  if μ is A-ergodic and for some $\{a_t\} \subset A$, $h_\mu(a_1) > 0$, then $\mu = m_X$.

We write "A-ergodic" for A-invariant and A-ergodic.

Proofs

Starting the proofs: simplifying assumptions

Throughout, assume that

- $a \in G$,
- μ is an a-invariant probability measure on G/Γ,
- $U \subset G^- := \{g \in G \mid a^n g a^{-n} \to e \text{ as } n \to \infty\}$ is normalized by a.

For the proofs presented here, we further assume that

- μ is a-ergodic (the precise reduction is in 7.19-7.24 of the Clay notes)
  - This implies $\mu = \mu_x^{\mathcal{E}}$ for μ-a.e. x.
- a is of "class A", i.e.
  1. the eigenvalues of a as an element of $\mathrm{SL}_n(\mathbb{R})$ are all in $\mathbb{R}$.
  2. 1 is the only eigenvalue of Ad_a with absolute value 1.
  3. No two eigenvalues have the same absolute value.

  For instance $a = \mathrm{diag}(e^t, e^s, e^r) \in \mathrm{SL}_3(\mathbb{R})$ with $t > s > r$.

  This implies that $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$, where

$$\mathfrak{g}_0 := \mathrm{Lie}(C_G(a)), \qquad \mathfrak{g}_- := \mathrm{Lie}(G^-), \qquad \mathfrak{g}_+ := \mathrm{Lie}(G^+).$$

Proof of Theorem 7.6(i)

We first prove

Theorem (Clay notes Thm. 7.6(i))

1. *The* **entropy contribution** *of U at x*

$$D_\mu(a, U)(x) := \lim_{n\to\infty} \frac{1}{n} \log \mu_x^U(a^{-n} B_1^U a^n)$$

exists for μ-a.e. x and defines an a-invariant function on X.

Outline:

1. Assume normalization $\mu_x^U(B_1^U) = 1$ for all x (wherever μ_x^U is well-defined). Then

$$\mu_x^U(a^{-n} B_1^U a^n) = \prod_{i=0}^{n-1} \mu_{a^i x}^U(a^{-1} B_1^U a).$$

2. Take log and apply the pointwise ergodic theorem (shows both existence and a-invariance).

Proof of Theorem 7.6(i)

Thus it suffices to show that given normalization $\mu_x^U(B_1^U) = 1$, we have

$$\prod_{i=0}^{n-1} \mu_{a^i x}^U(a^{-1} B_1^U a) = \mu_x^U(a^{-n} B_1^U a^n). \tag{1}$$

We need the following fact from Weikun's lecture:

Lemma

$$\mu_{ax}^U \propto (\mathrm{Conj}_a)_* \mu_x^U,$$

where $\mathrm{Conj}_a : G \to G$ *is defined by conjugation by* a: $g \mapsto aga^{-1}$.

Proof of (1): By Lemma,

$$\frac{\mu_{a^i x}^U(a^{-1} B_1^U a)}{\mu_{a^i x}^U(B_1^U)} = \frac{(\mathrm{Conj}_{a^i})_* \mu_x^U(a^{-1} B_1^U a)}{(\mathrm{Conj}_{a^i})_* \mu_x^U(B_1^U)} = \frac{\mu_x^U(a^{-i-1} B_1^U a^{i+1})}{\mu_x^U(a^{-i} B_1^U a^i)}.$$

Now use $\mu_{a^i x}^U(B_1^U) = 1$ and take product $0 \le i \le n-1$ yields (1). 7.6(i) ✓

Interlude

To show the other statements, we need the construction of a suitable σ-algebra $\mathcal{A}$. We say that

- $\mathcal{A}$ is **subordinate to** U if for μ-a.e. x, there exists $\delta = \delta(x) > 0$ such that
$$B_\delta^U(x) \subset [x]_\mathcal{A} \subset B_{\delta^{-1}}^U(x).$$
- $\mathcal{A}$ is a-**descending** if $a^{-1}\mathcal{A} \subset \mathcal{A}$.

Lemma ("Good" σ-algebra)

There exists a countably generated σ-algebra $\mathcal{A}$ on G/Γ that is subordinate to U and a-descending.

For now on, assume the lemma and fix one such "good" σ-algebra $\mathcal{A}$. Also, by a-ergodicity, $D_\mu(a, U)(x)$ is constant μ-a.e. - let it be $h_\mu(a, U)$.

Proposition

Suppose $\mathcal{A}$ is a countably generated σ-algebra subordinate to U and a-descending. Then $h_\mu(a, U) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})$, or equivalently,

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_x^U(a^{-n} B_1^U a^n) = \int -\log \mu_x^{a^{-1}\mathcal{A}}([x]_\mathcal{A})\, d\mu(x).$$

Outline:

1. Show that
$$\lim_{n\to\infty} -\frac{1}{n} \log \mu_x^{a^{-n}\mathcal{A}}([x]_\mathcal{A}) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}).$$
2. Use the fact that $\mathcal{A}$ is subordinate to U to get some set $Y \subset G/\Gamma$ with $\mu(Y) > 0$ and some $\delta > 0$ such that for all $x \in Y$, we have
$$\mu_x^U(a^{-n} B_\delta^U a^n) \le c(x)\mu_x^{a^{-n}\mathcal{A}}([x]_\mathcal{A})^{-1} \le \mu_x^U(a^{-n} B_{\delta^{-1}}^U a^n).$$
3. Take log and let $n \to \infty$, noting that in the definition of $D_\mu(a, U)(x)$ we can replace the radius 1 by any $r > 0$.

Step 1

We first show that for $I_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})(x) := -\log \mu_x^{a^{-1}\mathcal{A}}([x]_\mathcal{A})$,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu_x^{a^{-n}\mathcal{A}}([x]_\mathcal{A}) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}) := \int I_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})\, d\mu(x).$$

Proof: Recall by the compatibility property of conditional measures and $a^{-1}\mathcal{A} \subset \mathcal{A}$, we have

$$\frac{\mu_x^{a^{-i}\mathcal{A}}([x]_\mathcal{A})}{\mu_x^{a^{-i+1}\mathcal{A}}([x]_\mathcal{A})} = \mu_x^{a^{-i}\mathcal{A}}([x]_{a^{-i+1}\mathcal{A}}).$$

Also $\mu_{ax}^{\mathcal{A}} = a_* \mu_x^{a^{-1}\mathcal{A}}$. Take log on both side and sum over $1 \le i \le n$, we get by pointwise ergodic theorem that

$$\begin{aligned} -\frac{1}{n} \log \mu_x^{a^{-n}\mathcal{A}}([x]_\mathcal{A}) &= -\frac{1}{n} \sum_{i=1}^{n} \log \mu_x^{a^{-i}\mathcal{A}}([x]_{a^{-i+1}\mathcal{A}}) \\ &= \frac{1}{n} \sum_{i=1}^{n} I_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})(a^{i-1}x) \to H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}). \end{aligned}$$

Step 2: Squeeze argument

Since $\mathcal{A}$ is subordinate to U, there exists $\delta > 0$ such that the set

$$Y := \{x \mid B_\delta^U(x) \subset [x]_\mathcal{A} \subset B_{\delta^{-1}}^U(x)\}$$

has positive μ-measure. By pointwise ergodic theorem (and a-ergodicity of μ), there is a sequence $n_j \to \infty$ such that $a^{n_j}x \in Y$. For these $n = n_j$, we have

$$a^{-n}B_\delta^U a^n x \subset [x]_{a^{-n}\mathcal{A}} = a^{-n}[a^n x]_\mathcal{A} \subset a^{-n}B_{\delta^{-1}}^U a^n x.$$

Let $V_x \subset U$ so that $[x]_\mathcal{A} = V_x x$ and $a^{-n}[a^n x]_\mathcal{A} = a^{-n}V_{a^n x}a^n x$. Then

$$\mu_x^U(a^{-n}B_\delta^U a^n) \le \mu_x^U(a^{-n}V_{a^n x}a^n) = \mu_x^U(V_x)\mu_x^{a^{-n}\mathcal{A}}([x]_\mathcal{A})^{-1} \le \mu_x^U(a^{-n}B_{\delta^{-1}}^U a^n).$$

Now proceed to **Step 3**: take log average and limit $n_j \to \infty$, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_x^U(a^{-n}B_1^U a^n) = \lim_{n\to\infty} -\frac{1}{n} \log \mu_x^{a^{-n}\mathcal{A}}([x]_\mathcal{A}) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}).$$

Thus $D_\mu(a, U)(x) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})$.

Proof of Theorem 7.6(iii)

Now we are ready to prove Theorem 7.6(iii).

Theorem (Clay notes Thm. 7.6(iii))

3. *For μ-a.e. x we have $h_\mu(a, U) = 0$ iff μ_x^U is finite iff μ_x^U is trivial.*

Proof: Clearly μ_x^U is trivial $\Rightarrow$ μ_x^U is finite $\Rightarrow$ $h_\mu(a, U) = 0$.
It remains to show that $h_\mu(a, U) = 0$ implies $\mu_x^U \propto \delta_e$ a.e.
By Proposition,

$$\int -\log \mu_x^{a^{-1}\mathcal{A}}([x]_{\mathcal{A}})\, d\mu(x) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}) = h_\mu(a, U) = 0.$$

Thus $\mu_x^{a^{-1}\mathcal{A}}([x]_{\mathcal{A}}) = 1$ a.e. $\quad\Rightarrow\quad \mu_x^{a^{-m}\mathcal{A}}([x]_{a^m\mathcal{A}}) = 1$ a.e.
This implies $\mu_x^U(V_{-m,x} \setminus V_{m,x}) = 0$ a.e. where $[x]_{a^m\mathcal{A}} =: V_{m,x}x$.
Yet $V_{-m,x} \nearrow U$ and $V_{m,x} \searrow \{e\}$ as $m \to \infty$, therefore $\mu_x^U \propto \delta_e$. 7.6(iii) ✓

Left to show

Theorem

7.6(ii): *We have* $h_\mu(a, U) \le h_\mu(a)$*, with equality if* $U = G^-$*.*
7.9: $h_\mu(a, U) \le -\log|\det \mathrm{Ad}_a|_{\mathfrak{u}}|$*, and equality holds iff* μ *is* U*-invariant.*

We used the following "good" σ-algebra to prove Theorem 7.6 (iii).

Lemma ("Good" σ-algebra)

There exists a countably generated σ*-algebra* $\mathcal{A}$ *on* G/Γ *that is a-descending and subordinate to* U*. Then* $h_\mu(a, U) = H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A})$*.*

To show Theorem 7.6 (ii), we need the construction of a "good" partition.

Proposition ("Good" partition (Clay notes Prop. 7.43))

There exists a countable partition $\mathcal{P}$ *of* G/Γ *with finite entropy such that*

- $\mathcal{P}$ *is a generator of a mod* μ*,*
- σ*-algebra* $\mathcal{A} := \bigvee_{n \ge 0} a^{-n}\mathcal{P}$ *is a-descending and subordinate to* G^-*.*

Proof of Theorem 7.6(ii)

Assume there exists a countable partition $\mathcal{P}$ of G/Γ with finite entropy such that (a): $\mathcal{P}$ is a generator of a mod μ, and (b): σ-algebra $\mathcal{A} := \bigvee_{n\geq 0} a^{-n}\mathcal{P}$ is a-descending and subordinate to G^-.

Theorem (Clay notes Thm. 7.6(ii))

2. *We have $h_\mu(a, U) \leq h_\mu(a)$, with equality if $U = G^-$.*

Outline:

1. $h_\mu(a, G^-) \overset{(b)}{=} H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}) = h_\mu(a, \mathcal{P}) \overset{(a)}{=} h_\mu(a)$.
2. Construct $\mathcal{P}_U$ such that (c): $\mathcal{A}_U := \bigvee_{n\geq 0} a^{-n}\mathcal{P}_U$ is subordinate to U,

   (d): $\mathcal{A} \subset \mathcal{A}_U$, and (e): $\mathcal{A} \vee a^{-1}\mathcal{A}_U = \mathcal{A}_U$ mod μ.
3. Use the "good" σ-algebra $\mathcal{A}_U$ to show that $h_\mu(a, U) \leq h_\mu(a, G^-)$:

$$h_\mu(a, U) \overset{(c)}{=} H_\mu(\mathcal{A}_U \mid a^{-1}\mathcal{A}_U) \overset{(e)}{=} H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}_U)$$
$$\overset{(d)}{\leq} H_\mu(\mathcal{A} \mid a^{-1}\mathcal{A}) = h_\mu(a, G^-).$$

Step 2 in the proof of Theorem 7.6(ii)

Claim

One can construct $\mathcal{P}_U$ from $\mathcal{P}$ such that:

1. $\mathcal{A}_U := \bigvee_{n\geq 0} a^{-n}\mathcal{P}_U$ *is subordinate to* U,
2. $\mathcal{A} \subset \mathcal{A}_U$, *and* $\mathcal{A} \vee a^{-1}\mathcal{A}_U = \mathcal{A}_U$ *mod* μ.

Proof sketch (detailed in Clay notes 7.38):

1. Let $P \in \mathcal{P}$ small enough. Lift $P \subset G/\Gamma$ to G and cut P into pieces of U-orbits and push back down to G/Γ.
2. Let $\mathcal{P}_U$ be the σ-algebra whose elements are unions of such pieces of U-orbits. Then $\mathcal{P} \subset \mathcal{P}_U \Rightarrow \mathcal{A} \subset \mathcal{A}_U$.
3. $\mathcal{A}$ is subordinate to $G^- \Rightarrow \mathcal{A}_U$ is subordinate to U.
4. Show $[x]_{\mathcal{A}} \cap [x]_{a^{-1}\mathcal{A}_U} \overset{\mu}{=} [x]_{\mathcal{A}_U}$ for μ-a.e. x by construction. 7.6(ii)✓

Remark: (1): This also shows the existence of a "good" σ-algebra $\mathcal{A}_U$.
(2): With more care one can show for a-normalized subgroups $U \subset V \subset G^-$ (see Clay notes 7.40), $h_\mu(a, U) \leq h_\mu(a, V)$.

Proof of Theorem 7.9

Theorem (Clay notes Thm. 7.9)

$h_\mu(a, U) \leq -\log|\det \mathrm{Ad}_a|_{\mathfrak{u}}| =: J$, *and equality holds iff* μ *is* U*-invariant.*

The main idea is to compare leafwise measures w.r.t μ with those w.r.t. Haar, and then use strict convexity of log. Let m be Haar on G/Γ, and m_U be Haar on U. Let $\mathcal{A}$ be a σ-algebra subordinate to U and a-descending. Outline:

1. Note that: $m_U(a^{-1}Ba) = e^J m_U(B)$ for any measurable $B \subset U$.
2. Write $[x]_\mathcal{A} =: V_x x$ and $[x]_{a^{-1}\mathcal{A}} = a^{-1}[ax]_\mathcal{A} = a^{-1}V_{ax}ax$. Then

$$m_x^{a^{-1}\mathcal{A}}([x]_\mathcal{A}) = \frac{m_U(V_x)}{m_U(a^{-1}V_{ax}a)} = \frac{m_U(V_x)}{m_U(V_{ax})}e^{-J}.$$

3. Take log and apply the pointwise ergodic theorem

$$\int \log m_x^{a^{-1}\mathcal{A}}([x]_\mathcal{A})\, d\mu(x) = -J \left(\text{cf. } \int -\log \mu_x^{a^{-1}\mathcal{A}}([x]_\mathcal{A})\, d\mu = h_\mu(a, U)\right.$$

Outline continued

Outline (cont'd):

4. Since $\mathcal{A}$ refines $a^{-1}\mathcal{A}$, both subordinate to U, for μ-a.e. x,

$$[x]_{a^{-1}\mathcal{A}} = N_x \cup \bigcup_{i=0}^{\infty} [x_i]_{\mathcal{A}}$$

where N_x is $\mu_x^{a^{-1}\mathcal{A}}$-null but not necessarily $m_x^{a^{-1}\mathcal{A}}$-null.

5. By convexity of log,

$$\begin{aligned}
&\int \log m_x^{a^{-1}\mathcal{A}}([x]_{\mathcal{A}}) - \log \mu_x^{a^{-1}\mathcal{A}}([x]_{\mathcal{A}})\, d\mu_x^{a^{-1}\mathcal{A}} \\
&= \sum_{i=0}^{\infty} \log \left(\frac{m_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})}{\mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})} \right) \mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}}) \\
&\leq \log \sum_{i=0}^{\infty} \left(\frac{m_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})}{\mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})} \right) \mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}}) \\
&\leq 0.
\end{aligned}$$

Outline (cont'd):

5.
$$
\begin{aligned}
&\int \log m_x^{a^{-1}\mathcal{A}}([x]_{\mathcal{A}}) - \log \mu_x^{a^{-1}\mathcal{A}}([x]_{\mathcal{A}}) \; d\mu_x^{a^{-1}\mathcal{A}} \\
&= \sum_{i=0}^{\infty} \log \left(\frac{m_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})}{\mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})} \right) \mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}}) \\
&\leq \log \sum_{i=0}^{\infty} \left(\frac{m_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})}{\mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})} \right) \mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}}) \\
&\leq 0.
\end{aligned}
$$

6. Thus by integrating over all G/Γ,
$$
-J + h_\mu(a, U) = \int \log m_x^{a^{-1}\mathcal{A}}([x]_{\mathcal{A}}) - \log \mu_x^{a^{-1}\mathcal{A}}([x]_{\mathcal{A}}) \; d\mu(x) \leq 0.
$$

7. Equality iff N_x are $m_x^{a^{-1}\mathcal{A}}$-null and $m_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}}) = \mu_x^{a^{-1}\mathcal{A}}([x_i]_{\mathcal{A}})$ μ-a.s.

Outline for equality case

Outline for equality implies U-invariance:

7. Equality iff N_x are $m_x^{a^{-1}\mathcal{A}}$-null and $m_x^{a^{-1}\mathcal{A}}([x_i]_\mathcal{A}) = \mu_x^{a^{-1}\mathcal{A}}([x_i]_\mathcal{A})$ μ-a.s.
8. Similarly, we have $m_x^{a^{-\ell}\mathcal{A}}([x_i]_{a^k\mathcal{A}}) = \mu_x^{a^{-\ell}\mathcal{A}}([x_i]_{a^k\mathcal{A}})$ μ-a.s.
9. For fixed ℓ, the atoms of $a^k\mathcal{A}$ for all $k \geq 0$ generate the Borel σ-algebra of each $a^{-\ell}\mathcal{A}$-atom.
10. Thus $\mu_x^{a^{-\ell}\mathcal{A}} = m_x^{a^{-\ell}\mathcal{A}}$ for μ-a.e. x.
11. Use this for all ℓ, we see that $\mu_x^U = m_x^U = m_U$ for μ-a.e. x.
12. This implies μ is U-invariant. 7.9 ✓

It remains to show the existence of a "good" partition $\mathcal{P}$.

"Good" partition

It remains to show

Proposition (Clay notes Prop. 7.43)

There exists a countable partition $\mathcal{P}$ of G/Γ with finite entropy such that

- *$\mathcal{P}$ is a generator of a mod μ,*
- *σ-algebra $\mathcal{A} := \bigvee_{n\geq 0} a^{-n}\mathcal{P}$ is a-descending and subordinate to G^-.*

We will first construct such a partition $\mathcal{P}$ (in fact a finite partition) in the case when

- G/Γ is compact, and
- a is expansive (i.e. there exists $\delta > 0$ such that for all $x \neq y \in G/\Gamma$, there exists $n \in \mathbb{Z}$ such that $d(a^n x, a^n y) > \delta$.)

For instance a hyperbolic toral automorphism a on $\mathbb{T}^n$ (i.e. $a \in \mathrm{SL}_n(\mathbb{Z})$ such that all eigenvalues have absolute value $\neq 1$).

G/Γ compact and a is expansive

In the case when G/Γ is compact and a is expansive, the following are clear: For any partition $\mathcal{P}$ whose atoms have small enough diameter,

1. (**Generating**) $\mathcal{P}$ is generator for a,
2. (a-**descending**) $\mathcal{A} = \bigvee_{n=0}^{\infty} a^{-n}\mathcal{P}$ is a-descending.
3. (**Upper bound**) For μ-a.e. x, there exists $\delta = \delta(x) > 0$ such that

$$[x]_{\mathcal{A}} \subset B_{\delta^{-1}}^{G^-}(x).$$

It remains to show the **lower bound**: for μ-a.e. x, for some $\delta = \delta(x) > 0$,

$$B_{\delta}^{G^-}(x) \subset [x]_{\mathcal{A}}.$$

This is not true for every partition $\mathcal{P}$ as the boundaries of the atoms may have nontrivial μ-mass.

μ-thin boundary

Therefore we need a general assumption quantifying that having mass at boundary is the main obstruction.
Let X be a locally compact metric space, μ a Radon measure on X. For measurable $B \subset X$, let

$$\partial_\delta B := \{y \in X \mid \inf_{z \in B} d(y,z) + \inf_{z \notin B} d(y,z) < \delta\}.$$

Definition

We say that $B \subset X$ has **μ-thin boundary** if there exists $c > 0$ such that for all small $\delta > 0$, we have $\mu(\partial_\delta B) \leq c\delta$.

Proposition (Clay notes Lemma 7.31)

Suppose $\mathcal{P}$ is a finite partition of X such that each atom has μ-thin boundary. Then for μ-a.e. $x \in X$, there exists $\delta = \delta(x) > 0$ such that

$$B_\delta^{G^-}(x) \subset [x]_{\mathcal{A}}.$$

Proposition (Clay notes Lemma 7.31)

Suppose $\mathcal{P}$ is a finite partition of G/Γ with μ-thin boundary. Then for μ-a.e. $x \in G/\Gamma$, there exists $\delta = \delta(x) > 0$ such that $B_\delta^{G^-}(x) \subset [x]_\mathcal{A}$.

Outline:

1. **Lemma:** There exists $\alpha > 0$ and $C > 0$ such that for all $r \in (0,1)$ and $n \geq 1$, we have

$$a^n(B_r^{G^-})a^{-n} \subset B^G_{Ce^{-n\alpha}r}.$$

2. For $\delta > 0$ and $n \geq 1$, let $E_n(\delta) := a^{-n}\partial_{Ce^{-n\alpha}\delta}\mathcal{P}$.
3. By construction, $\mu(\bigcup_{n\geq 0} E_n(\delta)) \leq cC(\sum_{n\geq 0} e^{-n\alpha})\delta$.
4. Thus for μ-a.e. x, there exists $\delta(x) > 0$ such that $x \notin \bigcup_{n\geq 0} E_n(\delta)$.
5. Assume the contrary that $hx \in B_\delta^{G^-}(x)$ and $hx \notin [x]_\mathcal{A}$.
6. There exists $n \geq 0$ such that $a^n x$ and $a^n hx$ are in distinct atoms of $\mathcal{P}$.
7. By **Step 1**, $d(a^n x, a^n hx) \leq Ce^{-n\alpha}\delta$, thus $a^n x$ and $a^n hx$ are in $\partial_{Ce^{-n\alpha}\delta}\mathcal{P}$, i.e. $x \in E_n(\delta)$, a contradiction.

Existence of partition with μ-thin boundary

How do we know that G/Γ admits a finite partition $\mathcal{P}$ with μ-thin boundary? We have the following general lemma: Let X be a locally compact metric space, μ be a Radon measure on X.

Lemma (Clay notes Leamm 7.27)

For every $x \in X$, and Lebesgue-a.e. $r > 0$, there exists $c = c(x, r) > 0$ such that $\mu(\partial_\delta B_r(x)) \leq c\delta$.

Now cover the compact G/Γ by finitely many such balls with μ-thin boundary, and let $\mathcal{P}$ be the resulting partition.

For compact G/Γ and a expansive ✓

In the construction we can specify a universal upper bound $R > 0$ on the atoms (will be useful).

General case

In general, one needs to handle noncompact G/Γ (say $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$) and nonexpansive a (say $a = \mathrm{diag}(e^t, e^s, e^r)$).
We construct a countable partition $\mathcal{P}$ with finite entropy in 4 steps.

1. Fix an open subset $\Omega \subset G/\Gamma$ of compact closure, positive μ-measure and μ-thin boundary. (with also assume the diameter of Ω is at most $r/16$, where r is the injectivity radius of Ω.)
2. Let $\mathcal{Q} := \{\Omega, X \setminus \Omega\}$. Ω has μ-thin boundary, so we have the **lower bound**: for μ-a.e. x, there exists $\delta > 0$ with $B_\delta^{G^-}(x) \subset [x]_{\bigvee_{n\geq 0} a^{-n}\mathcal{Q}}$.
3. Define a partition $\widetilde{\mathcal{Q}} := \{Q_i \mid i = 0, 1, 2, \ldots\}$ as follows: $Q_0 := X \setminus \Omega$, and for $i \geq 1$,

$$Q_i := \{x \in \Omega \mid i \text{ is the first return time of } x \text{ to } \Omega\}.$$

Note that $\widetilde{\mathcal{Q}} \subset \bigvee_{n\geq 0} a^{-n}\mathcal{Q}$ (since $Q_i = (\Omega \setminus \cup_{k=1}^{i-1} a^{-k}\Omega) \cap a^{-i}\Omega$). Thus $\bigvee_{n\geq 0} a^{-n}\mathcal{Q} = \bigvee_{n\geq 0} a^{-n}\widetilde{\mathcal{Q}}$, which implies the **lower bound** for $\widetilde{\mathcal{Q}}$.

3. We have the partition $\widetilde{\mathcal{Q}} := \{Q_i \mid i = 0, 1, 2, \dots\}$ where $Q_0 := X \setminus \Omega$, and for $i \geq 1$, $Q_i := \{x \in \Omega \mid i \text{ is the first return time of } x \text{ to } \Omega\}$. So far we know that $\widetilde{\mathcal{Q}}$ has the **lower bound**. Can also compute that $\widetilde{\mathcal{Q}}$ has **finite entropy**: $H_\mu(\widetilde{\mathcal{Q}}) < \infty$ (see Clay notes 7.50).

4. We construct a refinement $\mathcal{P}$ of $\widetilde{\mathcal{Q}}$ as follows:
   - **Lemma:** There exists $\alpha = \alpha(G, a) > 0$ such that for all $r > 0$ and $n \in \mathbb{Z}$, we have $a^n(B^G_{e^{-|n|\alpha} r})a^{-n} \subset B^G_r$ ($\alpha \approx$ top exponent of Ad_a).

   For $i \geq 1$, define a partition of Q_i by $\mathcal{P}_i := \{P_{ij} \mid 1 \leq j \leq N(i)\}$ such that P_{ij} has diameter $\leq e^{-\alpha i} r/8$. Here $r =$ injectivity radius of Ω.

   Let countable partition $\mathcal{P} := \{X \setminus \Omega\} \cup \{P_{ij} \mid i \geq 1, 1 \leq j \leq N(i)\}$.

   Can show **finite entropy**: $H_\mu(\mathcal{P}) = H_\mu(\widetilde{\mathcal{Q}}) + H_\mu(\mathcal{P} \mid \widetilde{\mathcal{Q}}) < \infty$
   (by showing that one can take $N(i) \leq O(e^{\kappa i})$, see Clay notes 7.46 and 7.52).

Upper bound on $\mathcal{P}$

It remains to show that $\mathcal{P}$ has the **upper bound**: $[x]_{\bigvee_{n\geq 0} a^{-n}\mathcal{P}} \subset B^{G^-}_{\delta^{-1}}(x)$.

The argument in the compact G/Γ and expansive a case only gives for μ-a.e. $x \in \Omega$,

$$[x]_{\bigvee_{n\geq 0} a^{-n}\mathcal{P}} \subset B^{G^-}_{r/2} B^{G^0}_{r/2}(x),$$

(recall $G^0 := C_G(a)$ is the centralizer of a in G.)

1. **Eliminate G^0:** let $x \in \Omega$, and $g_- g_0 x \in [x]_{\bigvee_{n\geq 0} a^{-n}\mathcal{P}}$ with $g_- \in B^{G^-}_{r/2}$ and $g_0 \in B^{G^0}_{r/2}$. Suppose $\mu(Q_{n_0}) > 0$ for infinitely many $n_0 \geq 1$.
   see Clay notes 7.53 if this fails. Then $a^n x \in Q_{n_0}$ for infinitely many $n \geq 0$.
   Now $d(a^n g_- g_0 x, a^n x) \leq e^{-\alpha n_0 r/8}$ as they are in the same $\mathcal{P}$-atom.
   The displacement is $a^n g_- g_0 a^{-n} = (a^n g_- a^{-n}) g_0 \in B^{G^-}_{Ce^{-\alpha n_0} r} g_0$.
   Thus $g_0 \in B^{G}_{2Ce^{-\alpha n_0} r}$. Since this holds for infinitely many n_0, $g_0 = e$.
2. **Extend to μ-a.e. $x \in G/\Gamma$:**
   for μ-a.e. $x \in G/\Gamma$, $\exists n \geq 0$ with $a^n x \in \Omega$ and $[a^n x]_{\mathcal{A}} \subset B^{G^-}_r a^n x$.
   Then $[x]_{\mathcal{A}} = a^{-n}[a^n x]_{\mathcal{A}} \subset B^{G^-}_s(x)$ for some $s = s(n) > 0$. good $\mathcal{P}$ ✓

Lower bound on $\mathcal{P}$

... not quite done yet. By refining $\widetilde{\mathcal{Q}}$ we have destroyed the **lower bound**!

The remedy is to "thicken" each atom P_{ij} in Q_i in the G^--direction.

More precisely, replace each P_{ij} by $B^{G^-}_{r/4} P_{ij}$, then intersects with Q_i and remove overlaps to get a partition P'_{ij} of each Q_i.

Each P'_{ij} still has diameter $\leq e^{-\alpha i} r/8$ in the $G^0 G^+$ direction, so the **upper bound** argument still works, and we have preserved the **lower bound**.
details: Clay notes 7.54. good $\mathcal{P}$ ✓✓