Divergent trajectories of flows on homogeneous spaces and Diophantine approximation

By *S. G. Dani**) at Bombay

Let G be a connected Lie group and Γ be a lattice in G; that is, Γ is a discrete subgroup of G such that G/Γ admits a finite G-invariant measure. Let $\{g_t\}_{t \in \mathbb{R}}$ be a one-parameter subgroup of G. The action of $\{g_t\}$ on G/Γ (on the left) induces a flow on G/Γ. The ergodic theory of these flows is extensively studied and, at least from a certain point of view, satisfactorily understood (cf. [6] and its references). Thus, for instance, it is possible to determine, in terms of the position of $\{g_t\}$ in G relative to Γ, whether the flow admits dense trajectories $\{g_t \times \Gamma \mid t \geqq 0\}$, where $x \in G$, and whether a generic trajectory (either with respect to the measure or topologically) is dense in G/Γ. In general, however, there exist exceptional trajectories which are not dense, but to describe their set is a very difficult task; for an arbitrary one-parameter subgroup this is known only when G is a nilpotent Lie group (cf. [18] for that case and [16] for results on horocycle flows).

In this paper we assume G/Γ to be non-compact and investigate a special class of such exceptional trajectories: 'divergent' trajectories. A trajectory is said to be divergent if eventually it leaves every compact subset of G/Γ (cf. § 1 for precise definition). In §§ 2 and 3 we also get some results on bounded trajectories of certain flows.

It was proved by G. A. Margulis in [21] that if $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$ and $\{g_t\}$ is a one-parameter subgroup consisting of unipotent elements then there are no divergent trajectories (cf. [11] and [14] for stronger results). A similar situation can be seen to hold if all the eigenvalues of g_t, $t \in \mathbb{R}$ are of absolute value 1 (cf. Proposition 2. 6). However if g_1 (or any g_t, $t \neq 0$) has some eigenvalue λ with $|\lambda| \neq 1$ then there exist at least certain 'obvious' divergent trajectories. For instance, if $G = SL(2, \mathbb{R})$, $\Gamma = SL(2, \mathbb{Z})$ and $g_t = \operatorname{diag}(e^{-t}, e^t)$, then the trajectory starting from any point of $P\Gamma/\Gamma$, where P is the subgroup consisting of all upper triangular matrices in G, is divergent for simple geometric reasons. We call these degenerate divergent trajectories (cf. § 2 for details). In § 2 we also consider the one-parameter subgroups of G of the form

$$\operatorname{diag}(e^{-t}, \ldots, e^{-t}, \ e^{\lambda t}, \ldots, e^{\lambda t}),$$

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(with G and Γ as above) and relate the divergence of their trajectories to a question involving Diophantine approximation for certain systems of linear forms; specifically, if $g=\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$, where L is a $(n-p)\times p$ matrix (and I stands for identity matrices of appropriate sizes) then for the one-parameter subgroup as above, with $1\leqq p\leqq n-1$ and $\lambda=\frac{(n-p)}{p}$, the trajectory of $g\Gamma$ is divergent if and only if the system of $(n-p)$ linear forms in p variables is singular. A classical result in number theory then implies that while for $n=2$ all the divergent trajectories are degenerate, for $n\geqq 3$ there always exist non-degenerate trajectories, at least for $p=n-1$ (cf. Theorem 2. 7 and the subsequent remark). For these one-parameter subgroups we also find that boundedness of the trajectory of $g\Gamma$ as above is equivalent to the system of linear forms corresponding to L being badly approximable in the sense of [26]. By a result of W. Schmidt this implies that the set of points on bounded trajectories has full Hausdorff dimension (equal to that of the manifold) (cf. Corollary 2. 21).

If $\xi=(\xi_1,\ldots,\xi_p)$, where $1\leqq p\leqq n-1$ and $\xi_1,\ldots,\xi_p\in\mathbb{R}^n$ is an irrational p-frame (that is, $\xi_1,\ldots,\xi_p$ are linearly independent and the subspace spanned by them does not contain any non-zero rational vector) then there exists a sequence $\{\gamma_i\}$ in $SL(n,\mathbb{Z})$ such that $\gamma_i\xi\to 0$ (cf. [15] for stronger and general versions of this). In § 3 we relate divergence/boundedness of trajectories on $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ of the flows as above to the speed of the convergence $\gamma_i\xi\to 0$ in terms of the sizes of γ_i (cf. Theorems 3. 4 and 3. 5). This in particular reproves a part of a recent result of S. Raghavan in that direction.

One of the author's motivations in investigating divergence of trajectories is its application to orbits of horospherical subgroups (cf. § 1 for definition). Recently, D. S. Ornstein and M. Ratner obtained a simpler proof in the particular case of $SL(2,\mathbb{R})$ of the present author's classification (cf. [12]) of invariant measures of maximal horospherical flows. (The author is thankful to M. Ratner for communicating the proof.) The idea of the proof can be employed to prove the following: Let G be a Lie group, Γ a lattice, $\{g_t\}$ a one-parameter subgroup of G consisting of semisimple elements and let U be the horospherical subgroup corresponding to g_1 (or any g_t, $t>0$). Suppose that the U-action on G/Γ is ergodic. Then for $x\in G$, $Ux\Gamma/\Gamma$ is dense in G/Γ whenever $\{g_t x\Gamma \mid t\geqq 0\}$ is not a divergent trajectory (cf. Theorem 1. 6). It may be mentioned that for a certain class of U as above it can be proved by a similar method that any ergodic U-invariant measure on G/Γ other than the G-invariant measure is supported on the set of points whose trajectories under $\{g_t\}$ are divergent.

Now let G be a connected linear semisimple Lie group and let Γ be an irreducible lattice in G. The study of divergence of trajectories in the general case can be reduced to this case (cf. § 4). In § 5 we develop a natural extension of the notion of degeneracy of a divergent trajectory for flows induced by one-parameter subgroups of G on G/Γ. In § 6 we prove that if the "rank" of Γ is 1 then all the divergent trajectories are degenerate. This enables us to apply Theorem 1. 6 to determine for certain horospherical subgroups precisely which orbits are dense in G/Γ (cf. Corollary 6. 3).

As suggested by the rank 1 case it would be very nice if all divergent trajectories were degenerate! However, in the concluding section we show that at least if Γ is of maximal rank (equal to the $\mathbb{R}$-rank of G) and the rank exceeds 1 then for any one-parameter subgroup $\{g_t\}$ which admits any divergent trajectories at all, admits non-degenerate divergent trajectories (cf. Theorem 7. 3).

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§. 1. Divergent trajectories and horospherical orbits

Let G be a connected Lie group and Γ be a lattice in G; that is, Γ is a discrete subgroup such that G/Γ admits a finite G-invariant measure. A lattice Γ is said to be uniform or non-uniform according to whether G/Γ is compact or non-compact, respectively.

Let $\{g_t\}_{t\in\mathbb{R}}$ be a one-parameter subgroup of G and consider its action on G/Γ on the left. By a *trajectory* of $\{g_t\}$ on G/Γ we mean a curve of the form $\{g_t g\Gamma \mid t \geqq 0\}$, where $g \in G$. A trajectory $\{g_t g\Gamma \mid t \geqq 0\}$, $g \in G$ is said to be *divergent* if given any compact set C of G/Γ there exists $T \geqq 0$ such that for $t \geqq T$, $g_t g\Gamma \notin C$ or, equivalently, if for any sequence $\{t_i\}$ in $\mathbb{R}^+$ such that $t_i \to \infty$ the sequence $\{g_{t_i} g\Gamma\}$ has no limit point in G/Γ; we often write this as $g_t g\Gamma \to \infty$. We note that for a divergent trajectory to exist, G/Γ has to be non-compact.

1. 1. Proposition. *Let $h \in G$ be such that $\{g_t h g_{-t} \mid t \geqq 0\}$ is bounded (relatively compact in G). Then $\{g_t g\Gamma \mid t \geqq 0\}$, where $g \in G$, is a divergent trajectory if and only if $\{g_t h g\Gamma \mid t \geqq 0\}$ is a divergent trajectory.*

Proof. This is immediate from the equality $g_t h g\Gamma = (g_t h g_{-t})(g_t g\Gamma)$ for all $t \geqq 0$.

Thus, in particular, the set of points of G/Γ on divergent trajectories of $\{g_t\}$ is invariant under the centralizer of g_1 (the set of elements which commute with g_1). It is also invariant under the "horospherical subgroup" corresponding to g_1 (or, equivalently any g_t, $t > 0$).

1. 2. Definition. Let $g \in G$. Then the subgroup

$$U = \{h \in G \mid g^j h g^{-j} \to e \quad \text{as} \quad j \to \infty\}$$

where e is the identity, is called the *horospherical subgroup* corresponding to g.

The horospherical subgroup as above is also sometimes called the contracting horospherical subgroup; the one corresponding to g^{-1} is called the expanding horospherical subgroup. Any horospherical subgroup is a connected Lie subgroup of G (cf. [9] § 1, for instance). The group of 2×2 upper triangular unipotent matrices, viz. $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| \, t \in \mathbb{R}\right\}$, which defines the classical horocycle flow, is the horospherical subgroup in $SL(2, \mathbb{R})$ corresponding to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for $0 < \lambda < 1$.

1. 3. Remark. If G is a semisimple Lie group then a subgroup is horospherical (corresponding to some element) if and only if it is the unipotent radical of a parabolic subgroup of G. We also note that for such a G, and $\{g_t\}$ consisting of semisimple elements the subgroup $Q=\{h\in G\,|\,\{g_t h g_{-t}\,|\,t\geqq 0\}$ is bounded$\}$ is a parabolic subgroup and the horospherical subgroup corresponding to g_1 (or any g_t, $t>0$) is the unipotent radical of Q. It may also be recalled here that a parabolic subgroup coincides with the normalizer of its unipotent radical. Thus in this case Proposition 1. 1 can be stated as follows: the set of points of G/Γ on divergent trajectories of $\{g_t\}$ is invariant under the action of the normalizer of the horospherical subgroup corresponding to g_1.

1. 4. Definition. Let G be a connected Lie group and Γ be a lattice in G. The *commensurator* $C(\Gamma)$ of Γ is defined by

$$C(\Gamma)=\{\theta\in G \mid \theta\Gamma\theta^{-1}\cap\Gamma \text{ is of finite index in both } \Gamma \text{ and } \theta\Gamma\theta^{-1}\}.$$

It is easy to see that $C(\Gamma)$ is a subgroup of G.

1. 5. Proposition. *Let G be a connected Lie group and Γ be a lattice in G. Let $\{g_t\}$ be a one-parameter subgroup of G. Let $\theta\in C(\Gamma)$. Then for $g\in G$, $\{g_t g\Gamma\,|\,t\geqq 0\}$ is a divergent trajectory if and only if $\{g_t g\theta\Gamma\,|\,t\geqq 0\}$ is a divergent trajectory.*

Proof. We first observe that if Γ' is any subgroup of finite index in Γ then for any $g\in G$, $\{g_t g\Gamma\,|\,t\geqq 0\}$ is a divergent trajectory in G/Γ if and only if $\{g_t g\Gamma'\,|\,t\geqq 0\}$ is a divergent trajectory in G/Γ': this is because the canonical quotient map $\eta: G/\Gamma'\to G/\Gamma$ is a continuous surjective map such that for any $y\in G/\Gamma$. $\eta^{-1}(y)$ consists exactly of γ points, where γ is the index of Γ' in Γ.

Now let $\theta\in C(\Gamma)$. Then $\Gamma'=\Gamma\cap\theta\Gamma\theta^{-1}$ is of finite index in both Γ and $\theta\Gamma\theta^{-1}$ and hence by the above observation $\{g_t g\Gamma\,|\,t\geqq 0\}$ is a divergent trajectory in G/Γ if and only if $\{g_t g\theta\Gamma\theta^{-1}\,|\,t\geqq 0\}$ is a divergent trajectory in $G/\theta\Gamma\theta^{-1}$. But observe that the map $\Psi: G/\Gamma\to G/\theta\Gamma\theta^{-1}$, defined by $\Psi(x\Gamma)=x\Gamma\theta^{-1}=x\theta^{-1}(\theta\Gamma\theta^{-1})$, is a homeomorphism. Hence the above assertion implies that $\{g_t g\Gamma\,|\,t\geqq 0\}$ is a divergent trajectory if and only if the trajectory $\{g_t g\theta\Gamma\,|\,t\geqq 0\}$ is divergent.

As noted earlier, one of our motivations for studying divergence or otherwise of trajectories is its application to the study of orbits of horospherical flows (actions of horospherical subgroups) on G/Γ, and, dually, Γ-orbits under various linear actions of G.

1. 6. Theorem. *Let G be a connected Lie group and Γ be a lattice in G. Let $\{g_t\}$ be a one-parameter subgroup of G such that* $\operatorname{Ad} g_t$, $t\in\mathbb{R}$ *are diagonalisable over $\mathbb{C}$. Let U be the horospherical subgroup corresponding to g_1 (or any g_t, $t>0$). Suppose that the U-action on G/Γ is ergodic (with respect to the G-invariant probability measure on G/Γ). Let $g\in G$ be such that $\{g_t g\Gamma\,|\,t\geqq 0\}$ is not a divergent trajectory. Then $Ug\Gamma/\Gamma$, the U-orbit of $g\Gamma$, is dense in G/Γ.*

As stated in the introduction our proof of this is motivated by certain ideas of Ornstein and Ratner for horocycle flows on $SL(2,\mathbb{R})/\Gamma$ by which they reprove, in the particular case, the present author's classification theorem for invariant measures. One may expect the ideas to yield a similar classification for arbitrary horospherical flows (cf. [10] and [12] for such results). However, there seem to be several difficulties in doing that in general. Firstly, the proof seems to involve the fact that the action of $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, where $0<\lambda<1$ contracts the orbits of $\left\{\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix}\right\}$ "uniformly"; thus it is not clear whether it would work for the action of a horospherical subgroup corresponding to an element g in a semisimple group G (even $SL(n, \mathbb{R})$, $n \geqq 3$) such that $\operatorname{Ad} g$ has more than one eigenvalue with distinct absolute values each less than 1. Secondly, and perhaps more importantly, unlike in the case of $SL(2, \mathbb{R})$ when G is a semisimple Lie group of $\mathbb{R}$-rank $\geqq 2$ and Γ is a lattice of maximal rank (e.g. $G = SL(n, \mathbb{R})$, $n \geqq 3$ and $\Gamma = SL(n, \mathbb{Z})$) the set of points on divergent trajectories is rather complicated: there are "non-degenerate divergent trajectories" (cf. Theorem 7. 3) which are outside all the "geometrically nice" subsets which account for the invariant measures of at least the maximal horospherical flows (cf. [10] and [12]).

However, the idea does yield a proof of Theorem 1. 6 as above. We include the details in the Appendix. In the cases when we have a good description of the divergent trajectories the theorem can be used to study the orbits of horospherical flows (cf. Corollaries 2. 18 and 6. 3 for some applications). It may be worthwhile to note here that similar results were obtained by a different method in [15], § 4; in the situation studied there those results are stronger than can be obtained by the present method.

§ 2. Trajectories in lattice spaces and number theory

In this and the next section we discuss divergence and boundedness of trajectories of certain flows and lattice spaces in relation to certain notions in number theory.

(2. 1) Let $\mathscr{L}_n$ be the space of lattices in $\mathbb{R}^n$, where $n \geqq 2$, of determinant 1 (that is, the Lebesgue measure of a fundamental domain is 1). We denote by Λ_0 the lattice $\mathbb{Z}^n$. The space $\mathscr{L}_n$ can be identified with the homogeneous space $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ via the correspondence $g\,SL(n, \mathbb{Z}) \leftrightarrow g(\Lambda_0)$ for all $g \in SL(n, \mathbb{R})$, which can be easily checked to be a well-defined bijection. We note also that the correspondence is a $SL(n, \mathbb{R})$-equivariant homeomorphism when $\mathscr{L}_n$ is equipped with the usual topology (cf. [25], Chapter 1) and $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ is equipped with the quotient space topology. The following is a consequence of the well-known Mahler criterion (cf. [25], Corollary 10. 9).

2. 2. Proposition. *Let $\{g_i\}$ be a sequence in $SL(n, \mathbb{R})$. Then the sequence $\{g_i SL(n, \mathbb{Z})\}$ diverges (that is, has no limit point in $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$) if and only if for each neighbourhood Ω of 0 in $\mathbb{R}^n$ there exists i_0 such that for $i \geqq i_0$ the lattice $g_i(\Lambda_0)$ contains a non-zero element belonging to Ω.*

(2. 3) It is well-known that $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$, or equivalently $\mathscr{L}_n$ admits a finite $SL(n, \mathbb{R})$-invariant measure (cf. [25], Chapter 10). Let $\{g_t\}$ be a one-parameter subgroup of $SL(n, \mathbb{R})$. Then the action of $\{g_t\}$ on $\mathscr{L}_n$ is ergodic if and only if $\{g_t\}$ is not contained in any compact subgroup of $SL(n, \mathbb{R})$; further the action is mixing whenever it is ergodic. These assertions follow from a theorem of C. C. Moore (cf. [23], Theorem 1) and the fact that $SL(n, \mathbb{R})$ is a simple Lie group with finite center. The results in particular imply the following.

2. 4. Proposition. *Let $\{g_t\}$ be a one-parameter subgroup of $SL(n, \mathbb{R})$ which is not contained in any compact subgroup. Let $\{t_i\}$ be a sequence in $\mathbb{R}$ such that $t_i \to \infty$. Then for almost all $\Lambda \in \mathscr{L}_n$ (with respect to the $SL(n, \mathbb{R})$-invariant probability measure), $\{g_{t_i}(\Lambda)\}$ is dense in $\mathscr{L}_n$.*

We remark here that this proposition in particular implies the main theorem of W. Schmidt in [27], when we choose $g_t = \operatorname{diag}(e^{-t}, e^{-t}, \ldots, e^{-t}, e^{(n-1)t})$ ($\operatorname{diag}(\ldots)$ denotes the diagonal matrix with the parenthetical entries along the diagonal). We also remark that in fact for any sequence $\{g_i\}$ in $SL(n, \mathbb{R})$ which has no limit point in $SL(n, \mathbb{R})$, $\{g_i(\Lambda)\}$ is dense in $\mathscr{L}_n$ for almost all Λ. This can be deduced from Theorem 5. 2 in [30] or the (stronger) results in [19].

The significance of the proposition for the subject at hand is the following.

2. 5. Corollary. *Let $\{g_t\}$ be a one-parameter subgroup as in Proposition* 2. 4. *Then the set of $\Lambda \in \mathscr{L}_n$ such that the trajectory $\{g_t(\Lambda) \mid t \geqq 0\}$ is either divergent or bounded is of zero measure.*

While in general the description of the sets of bounded or divergent trajectories is a difficult task, thanks to a theorem of G. A. Margulis we have the following simple criterion for the latter set to be non-empty.

2. 6. Proposition. *Let $\{g_t\}$ be a one-parameter subgroup of $SL(n, \mathbb{R})$. Then there exists $\Lambda \in \mathscr{L}_n$ such that $\{g_t(\Lambda) \mid t \geqq 0\}$ is a divergent trajectory if and only if g_1 (or any g_t, $t \neq 0$) has an eigenvalue λ (possible complex) such that $|\lambda| \neq 1$.*

Proof. If $\{g_t\}$ is a unipotent one-parameter subgroup (that is, if all eigenvalues of g_t, $t \in \mathbb{R}$ are 1) then a theorem of G. A. Margulis (cf. [21] and also [11] and [14] for stronger results in a different direction) asserts that the trajectory $\{g_t(\Lambda) \mid t \geqq 0\}$ is never divergent. Now let $\{g_t\}$ be such that all eigenvalues of g_t, $t \in \mathbb{R}$ are of absolute value 1. By Jordan decomposition g_t, $t \in \mathbb{R}$ can be expressed as $g_t = c_t u_t$ where $\{u_t\}$ is a one-parameter subgroup consisting of unipotent elements, $\{c_t\}$ is a one-parameter subgroup contained in a compact subgroup and c_t, $t \in \mathbb{R}$ and u_t, $t \in \mathbb{R}$ commute with each other. The above special case together with boundedness of $\{c_t\}$ now implies that $\{g_t(\Lambda) \mid t \geqq 0\}$ is never divergent.

Conversely, suppose that $\{g_t\}$ is a one-parameter subgroup such that g_1 (or any g_t, $t \neq 0$) has an eigenvalue λ such that $|\lambda| \neq 1$; since $g_1 \in SL(n, \mathbb{R})$ we may choose λ so that $|\lambda| < 1$. Put

$$W(\{g_t\}) = \{v \in \mathbb{R}^n \mid g_t(v) \to 0 \text{ as } t \to \infty\}. \tag{2. 7}$$

It is a positive dimensional subspace of $\mathbb{R}^n$. Let $\Lambda \in \mathscr{L}_n$ be such that $\Lambda \cap W(\{g_t\}) \neq (0)$, say it contains $v \neq 0$. Then $g_t(v) \in g_t(\Lambda)$ and $g_t(v) \to 0$ as $t \to \infty$. Hence by Proposition 2. 2, $\{g_t(\Lambda) \mid t \geqq 0\}$ is a divergent trajectory, which proves the proposition.

The proof of the converse part above shows that when $W(\{g_t\})$, as defined by (2. 7) is non-zero there are certain too obvious divergent trajectories; viz. those of $\Lambda \in \mathscr{L}_n$ such that $\Lambda \cap W(\{g_t\}) \neq (0)$. A moment's reflection also suggests the following divergent trajectories to be almost as obvious:

Let $\mathbb{R}^n$ be equipped with the usual inner product. Also let any non-zero subspace V of $\mathbb{R}^n$ be equipped with the Lebesgue measure l_V such that a (any) parallelopiped $\{\sum \xi_i v_i \mid 0 \leqq \xi_i \leqq 1\}$ where $\{v_i\}$ is an orthonormal basis of V has measure 1. If Σ is a lattice in V let $d(\Sigma)=l_V(F)$, where F is a (any) fundamental domain for Σ in V. Now let $\{g_t\}$ be a one-parameter subgroup of $SL(n,\mathbb{R})$ and let $\Lambda \in \mathscr{L}_n$ be such that for some non-zero subgroup Σ of Λ, $d(g_t(\Sigma)) \to 0$ as $t \to \infty$. Then, by Minkowski's theorem (cf. [8], for instance), given any neighbourhood Ω of 0 there exists t_0 such that for $t \geqq t_0$, $g_t(\Sigma) \cap \Omega \neq (0)$. Hence by Proposition 2. 2, $\{g_t(\Lambda) \mid t \geqq 0\}$ is a divergent trajectory. Observe that when Σ is the cyclic subgroup generated by an element of $W(\{g_t\})$ this coincides with the examples as before. However, it is not difficult to construct, for various one-parameter subgroups $\{g_t\}$, examples of lattices Λ which contain subgroups Σ of rank $\geqq 2$ such that $d(g_t(\Sigma)) \to 0$, but $\Lambda \cap W(\{g_t\}) = (0)$; e.g. if $g_t = \operatorname{diag}(e^{\alpha_1 t}, e^{\alpha_2 t}, e^{\alpha_3 t}) \in SL(3,\mathbb{R})$, where $\alpha_1 < 0 < \alpha_2 < \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_3 = 0$, then this happens for any lattice Λ in $\mathbb{R}^3$ which intersects the plane of e_1 and e_2 in a lattice but does not contain any non-zero multiple of e_1, $\{e_1, e_2, e_3\}$ being the standard basis of $\mathbb{R}^3$.

2. 8. Definition. Let $\{g_t\}$ be a one-parameter subgroup of $SL(n,\mathbb{R})$ and let $\Lambda \in \mathscr{L}_n$ be such that $\{g_t(\Lambda) \mid t \geqq 0\}$ is a divergent trajectory. If there exists a non-zero subgroup Σ of Λ such that $d(g_t(\Sigma)) \to 0$ as $t \to \infty$ we say that $\{g_t(\Lambda) \mid t \geqq 0\}$ is a *degenerate* divergent trajectory; otherwise it is said to be a *non-degenerate* divergent trajectory.

2. 9. Proposition. *Let $\{g_t\}$ be a one-parameter subgroup in $SL(n,\mathbb{R})$. Let $\Lambda \in \mathscr{L}_n$. Then $\{g_t(\Lambda) \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if there exist $g \in SL(n\,\mathbb{R})$ and $1 \leqq p \leqq n-1$ such that $g\Lambda_0 = \Lambda$ and $\bigwedge^p(g_t g)(e_1 \wedge e_2 \wedge \cdots \wedge e_p) \to 0$ as $t \to \infty$, where $e_1, e_2, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. ($\bigwedge$ stands for exterior products.)*

Proof. The "if" part follows from the fact that if we put $\Sigma = g(\Sigma_0)$, where Σ_0 is the subgroup generated by $e_1, e_2, \ldots, e_p$, then the condition as in Definition 2. 8 is satisfied; indeed $d(g_t(\Sigma)) = \|\bigwedge^p(g_t g)(e_1 \wedge \cdots \wedge e_p)\|$ for a suitable norm on the space of exteriors (cf. [11], Lemma 1. 4). Conversely, suppose that there exists $\Sigma \subset \Lambda$ such that $d(g_t(\Sigma)) \to 0$. By replacing it by the largest subgroup of $\bigwedge$ generating the same subspace, we can assume that there exists a basis $\{v_1, \ldots, v_n\}$ of Λ such that $\{v_1, v_2, \ldots, v_p\}$ is a basis of Σ. Adjusting v_n suitably (up to sign) we may assume that there exists $g \in SL(n,\mathbb{R})$ such that $g(e_j) = v_j$ for all $j = 1, 2, \ldots, n$. Then evidently g has the required properties.

In § 5 we extend the notion of degenerate divergent trajectories to homogeneous spaces of arbitrary Lie groups and show that if the homogeneous space is of "rank 1" all divergent trajectories are degenerate, while in the general case there exist non-degenerate divergent trajectories. In the rest of the present section we restrict to a special class of one-parameter subgroups of $SL(n,\mathbb{R})$ and relate their orbit behaviour to certain number theoretic notions.

(2. 10). For concreteness we view $\mathbb{R}^n$ as the space of n-rowed column vectors with real entries. Let e_i, $i = 1, 2, \ldots, n$ be the column vector with 1 in the ith row and 0 elsewhere. Let ρ be an index between 1 and $n-1$ and let $D_p(t)$ be the diagonal matrix such that $D_p(t)\, e_i = e^{-t} e_i$ if $1 \leqq i \leqq p$ and $D_p(t)\, e_i = e^{\lambda t} e_i$ if $p+1 \leqq i \leqq n$, where $\lambda = \dfrac{p}{(n-p)}$. Then $\{D_p(t)\}$ is a one-parameter subgroup of $SL(n,\mathbb{R})$.

(2. 11) Any matrix $g \in SL(n, \mathbb{R})$ can be expressed in the form

$$g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \sigma$$

where A, B, D and L are matrices of sizes $p \times p$, $p \times (n-p)$, $(n-p) \times (n-p)$ and $(n-p) \times p$ respectively, I and 0 stand for identity and zero matrices of appropriate sizes and $\sigma \in SL(n, \mathbb{R})$ is such that $\sigma(e_i) \in \{\pm e_j \mid 1 \leqq j \leqq n\}$; that is, σ is a "permutation matrix" except for signs. Note that for a matrix h of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, $\{D_p(t)\, h D_p(-t) \mid t \geqq 0\}$ is bounded. Therefore in view of Proposition 1. 1 and the fact that $\sigma \in SL(n, \mathbb{Z})$, we get the following.

2. 12. Proposition. *Let* $g \in SL(n, \mathbb{R})$ *be expressed as in* (2. 11). *Then*

$$\{D_p(t)\, g(\Lambda_0) \mid t \geqq 0\}$$

is a divergent (*or respectively bounded*) *trajectory if and only if* $\left\{D_p(t) \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} (\Lambda_0) \mid t \geqq 0\right\}$ *is a divergent* (*resp. bounded*) *trajectory.*

In view of the proposition in studying divergence or boundedness of a trajectory of $\{D_p(t)\}$ we may restrict to orbits of Λ of the form $\begin{pmatrix} I & 0 \\ L & I \end{pmatrix} (\Lambda_0)$. We relate this to certain number theoretic properties of L, or more precisely of the set (or system) of linear forms corresponding to L. Here and in the sequel by the set (or system) of linear forms corresponding to L we mean the $n-p$ linear forms $\sum l_{ij} x_j$, $i = 1, 2, \ldots, n-p$, where $L = (l_{ij})$.

In the rest of § 2 we use the following notation: For any $t \in \mathbb{R}$ let

$$\rho(t) = \min\{|t-n| \mid n \text{ integer}\} \quad \text{and} \quad \nu(t) = \min\{n \mid |t-n| = \rho(t)\}.$$

Observe that for any t, $t - \nu(t) = \pm \rho(t)$. For any $x = {}^t(x_1, \ldots, x_n) \in \mathbb{R}^n$ (^{t} stands for "transpose of") we denote $\max\{|x_i| \mid 1 \leqq i \leqq n\}$ by $|x|$. A vector $x = {}^t(x_1, \ldots, x_n)$ is said to be integral if $x_1, x_2, \ldots, x_n$ are integers.

2. 13. Definition. Let $L_1, L_2, \ldots, L_k$ be a system of k linear forms in l variables. It is said to be *singular* if for every $\varepsilon > 0$ there exists N_0 such that for all $N \geqq N_0$ the set of inequalities

$$\rho(L_i(x)) < \varepsilon N^{-\frac{l}{k}}, \quad i = 1, 2, \ldots, k$$

and

$$|x| < N$$

have a (common) non-zero integral solution x. The system is said to be *regular* if it is not singular. The system is said to be *badly approximable* if there exists $c > 0$ such that

$$\max_{1 \leqq i \leqq k} (\rho(L_i(x)) > c\,|x|^{-\frac{l}{k}}$$

for all non-zero integral vectors $x \in \mathbb{R}^l$.

The reader may refer to [7] and [26] for discussions pertaining to these notions.

2. 14. Theorem. *Let L be a $(n-p)\times p$ matrix, where $1\leqq p\leqq n-1$, and let $L_1, L_2,\dots, L_{(n-p)}$ be the corresponding system of linear forms. Let $\Lambda=\begin{pmatrix} I & 0\\ L & I\end{pmatrix}(\Lambda_0)$. Then $\{D_p(t)(\Lambda)\mid t\geqq 0\}$ is a divergent trajectory if and only if $\{L_1, L_2,\dots, L_{(n-p)}\}$ is singular.*

Proof. Suppose that $\{L_1, L_2,\dots, L_{(n-p)}\}$ is singular. For $i=1, 2,\dots, n$ let

$$f_i=\begin{pmatrix} I & 0\\ L & I\end{pmatrix}e_i.$$

For any $\delta>0$ let

$$B_\delta=\{\sum \xi_i e_i \mid |\xi_i|<\delta \quad\text{for all}\quad i=1, 2,\dots, n\}.$$

In view of Proposition 2. 2 it is enough to show that for any $\delta>0$ there exists T such that for all $t\geqq T$, $D_p(t)(\Lambda)\cap B_\delta\neq(0)$. Let $1>\delta>0$ be given and choose ε such that $0<\varepsilon<\delta^n$. Since $\{L_1,\dots, L_{(n-p)}\}$ is singular there exists N_0 such that for $N\geqq N_0$ the inequalities

$$\rho\big(L_i(x)\big)<\varepsilon N^{-\frac{p}{(n-p)}},\quad i=1, 2,\dots, (n-p)$$

and

$$|x|<N$$

have a common non-zero integral solution. Fix $N\geqq N_0$ and let $x={}^t(x_1, x_2,\dots, x_p)$ be such a solution. For $j=1, 2,\dots, (n-p)$ put $x_{p+j}=\nu\big(-L_j(x)\big)$ so that

$$L_j(x)+x_{p+j}=\pm\rho\big(L_j(x)\big).$$

Then

$$y=\sum_{i=1}^{n} x_i f_i=\sum_{i=1}^{p} x_i e_i+\sum_{j=1}^{n-p}\big(L_j(x)+x_{p+j}\big)e_{p+j}$$

is an element of Λ and $D_p(t)(y)\in B_\delta$ whenever

$$e^{-t}N<\delta \quad\text{and}\quad \varepsilon e^{\lambda t}N^{-\lambda}<\delta$$

where as before $\lambda=\dfrac{p}{(n-p)}$. Let I_N be the open interval $\left(\log\dfrac{N}{\delta},\ \lambda^{-1}\log\dfrac{\delta N^\lambda}{\varepsilon}\right)$, which is non-empty since $\varepsilon<\delta^n$. Then we have $D_p(t)(\Lambda)\cap B_\delta\neq(0)$ for all $t\in I_N$, which is true for all $N\geqq N_0$. Again since $\varepsilon<\delta^n$, for all sufficiently large N, I_N and I_{N+1} overlap and consequently $\bigcup I_N$ contains an interval of the form $[T,\infty)$. Hence $D_p(t)(\Lambda)\cap B_\delta\neq(0)$ for all $t\geqq T$ as required.

Conversely, suppose that $\{D_p(t)(\Lambda)\mid t\geqq 0\}$ is not divergent. Then there exists $\delta>0$ and a sequence $\{t_k\}$ such that $t_k\to\infty$ and $D_p(t_k)(\Lambda)\cap B_\delta=(0)$ for all $k=1, 2,\dots$. Let $N_k=\nu(\delta e^{t_k})=\delta e^{t_k}\pm\rho(\delta e^{t_k})$. Let $x={}^t(x_1, x_2,\dots, x_p)$ be an integral vector such that

(2. 15) $|x|<N_k$ and $\rho\big(L_i(x)\big)<2^{-\lambda}\delta N_k^{-\lambda}$ for all $i=1, 2,\dots, (n-p)$

and put $x_{p+j}=\nu\big(-L_j(x)\big)$ $\big(=-L_j(x)\pm\rho\big(L_j(x)\big)\big)$ for all $j=1,\dots, (n-p)$. Let

$$y=\sum_{i=1}^{n} x_i f_i=\sum_{i=1}^{p} x_i e_i+\sum_{j=1}^{n-p}\rho\big(L_j(x)\big)e_{p+j}\in\Lambda.$$

Then by (2. 15) we have $D_p(t_k)(y) \in B_\delta$. Since $D_p(t_k)(\Lambda) \cap B_\delta = (0)$ we conclude that $y=0$ or equivalently $x=0$. Thus the inequalities (2. 15) have no non-zero integral solution for any N_k. Since $N_k \to \infty$ this means that $\{L_1, L_2, \ldots, L_{(n-p)}\}$ is not singular.

2. 16. Remark. It is straightforward to verify that for the one-parameter subgroup $D_{n-1}(t)$ a divergent trajectory $\{D_{n-1}(t)(\Lambda) \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if $\Lambda \cap W(\{D_{n-1}(t)\}) \neq (0)$ (notation as in (2. 7)); (if every non-zero element of Λ is of the form $\sum_{i=1}^{n} \xi_i e_i$ with $\xi_n \neq 0$, then for any $k \leqq n-1$ and $x^1, x^2, \ldots, x^k \in \Lambda$, $x^1 \wedge x^2 \wedge \cdots \wedge x^k$ is not contained in $W(\bigwedge^k D_{n-1}(t)\})$, unless it is zero). For Λ as in the theorem (with $p=n-1$) this happens if and only if the subgroup generated by the coefficients of the corresponding (single) linear form L_1 does not contain any rational element; in other words, if and only if $l_1, l_2, \ldots, l_{n-1}$ and 1 are linearly independent over $\mathbb{Q}$, $l_1, l_2, \ldots, l_{n-1}$ being the coefficients of L_1 (or entries of L). Thus in this case Theorem 14 of [7] implies the following.

2. 17. Theorem. *For $n=2$, the degenerate divergent trajectories are the only divergent trajectories for the flow on $\mathscr{L}_2$ induced by $\{D_1(t)\}$. However, for $n \geqq 3$ there always exists $\Lambda \in \mathscr{L}_n$ such that $\{D_{n-1}(t)(\Lambda) \mid t \geqq 0\}$ is a non-degenerate divergent trajectory.*

In $SL(2, \mathbb{R})$, up to conjugacy and scaling (linear change in t) $\{D_1(t)\}$ is the only one-parameter subgroup for which not all eigenvalues under the adjoint action are of absolute value 1. Thus Proposition 2. 6 together with the first part of Theorem 2. 17 constitute a complete description of divergent trajectories of actions of one-parameter subgroups of $SL(2, \mathbb{R})$ on $\mathscr{L}_2$. We shall achieve a similar description for all flows on homogeneous spaces of "rank 1" (cf. Corollary 6. 2).

The discussion on p. 94 of [7] does not seem to imply the second part of Theorem 2. 17 for $\{D_p(t)\}$, $p<n-1$ (in the place of $n-1$) since in that case analogue of Remark 2. 16 is not valid. However, interpreting the ideas in the proof of Theorem 14 in [7] geometrically, we shall uphold existence of non-degenerate divergent trajectories on all homogeneous spaces of rank $\geqq 2$ (cf. Theorem 7. 3).

Combining Theorem 1. 6 and Proposition 2. 12 we get the following.

2. 18. Corollary. *Let $G=SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$ and*

$$U = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \middle| X \text{ a } p \times (n-p) \text{ matrix} \right\}$$

and

$$g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \sigma,$$

the latter being a decomposition as in (2. 11). *Suppose that the system $\{L_1, \ldots, L_{(n-p)}\}$ of linear forms corresponding to L is regular. Then $Ug\Gamma/\Gamma$ is dense in G/Γ.*

Proof. U as above is indeed the horospherical subgroup corresponding to $D_p(t)$ for any $t>0$. Hence the result follows from Theorem 1. 6 and Proposition 2. 12.

2. 19. Remark. Let the notation be as in Corollary 2. 18. It is easy to see that if $h=\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$ is such that $\{D_p(t)\,h\Gamma \mid t\geqq 0\}$ is a degenerate divergent trajectory then $Uh\Gamma/\Gamma$ is not dense in G/Γ. Since by Theorem 2. 17 for $n=2$ all divergent trajectories of $\{D_1(t)\}$ are degenerate, Theorems 1. 6 and 2. 17 constitute a necessary and sufficient condition for density of the U-orbit. (See Corollary 6. 3 for a more general result.)

2. 20. Theorem. *Let L be a $(n-p)\times p$ matrix, where $1\leqq p\leqq n-1$, and let $L_1, L_2,\dots, L_{(n-p)}$ be the corresponding system of linear forms. Let $\Lambda=\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}(\Lambda)$. Then the trajectory $\{D_p(t)(\Lambda) \mid t\geqq 0\}$ is bounded if and only if $\{L_1, L_2,\dots, L_{(n-p)}\}$ is badly approximable.*

Proof. Suppose that the system $\{L_1,\dots, L_{(n-p)}\}$ is badly approximable. Let $0<c<\frac{1}{2}$ be such that

$$\max_{1\leqq i\leqq n-p}\left(\rho\left(L_i(x)\right)\right)>c\,|x|^{-\lambda}$$

for all non-zero integral x, where as before $\lambda=\dfrac{p}{(n-p)}$. Choose $\delta>0$ so that $\delta^{(1+\lambda)}<c$. Let $y\in\Lambda$ and suppose that $D_p(t')(y)\in B_\delta$ for some $t'\geqq 0$. Then y can be expressed as $y=y_1+y_2$ where $y_1=\sum_{i=1}^{p} x_i e_i$ and $y_2=\sum_{j=1}^{n-p}\left(L_j(x)+x_j\right)e_{p+j}$, with $x_1, x_2,\dots, x_n$ integers. Observe that $|D_p(t)(y_1)|^\lambda\,|D_p(t)(y_2)|=|y_1|^\lambda\,|y_2|$ for all $t\geqq 0$. As $D_p(t')(y)\in B_\delta$ we see that $D_p(t')(y_1)$ and $D_p(t')(y_2)$ belong to B_δ and consequently

$$|y_1|^\lambda\,|y_2|=|D_p(t')(y_1)|^\lambda\,|D_p(t')(y_2)|<\delta^{(1+\lambda)}.$$

Also since $D_p(t')(y_2)\in B_\delta$, $|L_j(x)+x_j|<e^{-\lambda t'}\delta\leqq\delta$ and consequently $|L_j(x)+x_j|=\rho\left(L_j(x)\right)$. Hence $|y_1|^\lambda\,|y_2|=|x|^\lambda\max_{1\leqq j\leqq n-p}\left(\rho\left(L_j(x)\right)\right)<\delta^{(1+\lambda)}<c$ which contradicts the choice of c unless $x=0$; in the latter case $y=0$. Hence $D_p(t)(\Lambda)\cap B_\delta=(0)$ for all $t\geqq 0$. By Proposition 2. 2 this implies that $\{D_p(t)(\Lambda) \mid t\geqq 0\}$ is bounded.

Conversely, suppose that the trajectory $\{D_p(t)(\Lambda) \mid t\geqq 0\}$ is bounded. By Proposition 2. 2 there exists $\delta>0$ such that $D_p(t)(\Lambda)\cap B_\delta=(0)$ for all $t\geqq 0$. Now let $x={}^t(x_1,\dots, x_p)$ be an integral p-tuple, $x_{p+j}=\nu\left(-L_j(x)\right)=-L_j(x)\pm\rho\left(L_j(x)\right)$ for all $j=1, 2,\dots, (n-p)$ and $y=\sum x_i f_i\in\Lambda$ where $f_i=\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}e_i$. Then $D_p(t)(y)\notin B_\delta$ for any $t\geqq 0$. Choose t to be such that $e^{-t}|x|=\frac{\delta}{2}$. Since $D_p(t)(y)\notin B_\delta$ we must have $e^{\lambda t}\rho\left(L_i(x)\right)\geqq\delta$ for some $i=1, 2,\dots, (n-p)$. In other words,

$$\max\rho\left(L_i(x)\right)>2\left(\frac{\delta}{2}\right)^{1+\lambda}|x|^{-\lambda}>c\,|x|^{-\lambda}$$

for a suitable $c>0$. Thus $\{L_1,\dots, L_{(n-p)}\}$ is badly approximable.

In view of Corollary 2. 5 and Proposition 2. 12, the theorem implies the well-known fact that the set of badly approximable systems of $(n-p)$ linear forms in p variables is of zero measure in the $p(n-p)$ dimensional vector space of all systems of $(n-p)$ linear forms in p variables. More interestingly, arguing the other way and using a theorem of W. Schmidt (cf. [26]) together with Proposition 2. 12 we conclude the following.

2. 21. Corollary. *The Hausdorff dimension of the set*

$$\{\Lambda \in \mathscr{L}_n \mid \{D_p(t)(\Lambda) \mid t \geqq 0\} \text{ is a bounded trajectory}\}$$

is n^2-1 (*viz. the dimension of* $\mathscr{L}_n$).

2. 22. Remark. For $n=2$ the result involved in the above corollary is classical (cf. [26] for details). Also, in that case the flow is the geodésic flow associated to the surface $K \backslash SL(2, \mathbb{R})/SL(2, \mathbb{Z})$ with respect to the Poincaré metric, where K is the subgroup of $SL(2, \mathbb{R})$ consisting of all rotations. Thus the set of points whose (forward) trajectories under the geodesic flow (as above) is of Hausdorff dimension 3. One may ask whether a similar assertion holds for the geodesic flow associated to any surface of constant negative curvature and finite area. We note however that for the horocycle flows associated to these surfaces, by a theorem of Hedlund (cf. [16] for reference and a stronger result), the set of points on bounded trajectories is a two dimensional submanifold.

§ 3. Orbits of Euclidean frames

A p-tuple $(\xi_1, \xi_2, \dots, \xi_p)$, where $1 \leqq p \leqq n-1$, and $\xi_1, \xi_2, \dots, \xi_p \in \mathbb{R}^n$ is called a *p-frame* if $\xi_1, \xi_2, \dots, \xi_p$ are linearly independent. A p-frame $(\xi_1, \xi_2, \dots, \xi_p)$ is said to be *irrational* if the subspace spanned by $\{\xi_1, \xi_2, \dots, \xi_p\}$ does not contain any integral vector. We shall view a p-frame $\xi = (\xi_1, \xi_2, \dots, \xi_p)$ also as an $n \times p$ matrix in the obvious way (with ξ_i as the ith column). For any matrix θ we denote by $\|\theta\|$ the maximum of the absolute values of its entries.

The classical Kronecker theorem implies that if ξ is an irrational p-frame and $\varepsilon > 0$ there exists an integral row matrix $x = (x_1, \dots, x_n)$ such that $\|x\xi\| < \varepsilon$. In [15] we proved a matrix analogue of the theorem: for ξ as above and $\varepsilon > 0$ there exists $\gamma \in SL(n, \mathbb{Z})$ such that $\|\gamma\xi\| < \varepsilon$. In fact it is shown that there exist sequences $\{\gamma_i\}$ in $SL(n, \mathbb{Z})$ and $\lambda_i \in \mathbb{R}^+$ such that $\gamma_i \xi \to 0$ and $\lambda_i \gamma_i \xi \to \eta$ where η is a p-frame (the latter condition means that the sequence $\{\gamma_i\}$ does not totally suppress the "shape of ξ"). This was applied to deduce the inhomogeneous form of the above assertion; viz. if ξ is a p-frame then the $SL(n, \mathbb{Z})$-orbit of ξ (componentwise action) is dense in the space of p-frames, viewed as a subspace of $\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ (p copies). Thus any p-frame can be approximated by p-frames of the form $\gamma\xi$ with $\gamma \in SL(n, \mathbb{Z})$. (A similar result is also proved for $Sp(2n, \mathbb{Z})$.)

Motivated by the recent quantitative versions of the usual Kronecker theorem (cf. [29]) one may ask how fast the sequences $\{\gamma_i\}$ in the above discussions have to grow (in terms of the approximation achieved). We shall not directly concern ourselves with this question; but rather, relate the rates of growth of optimally chosen sequences to the behaviour of certain trajectories of $\{D_p(t)\}$ as defined in the last section: The question itself has been studied in a recent preprint [24] by S. Raghavan. Our results overlap with those of [24] for a class of p-frames (specifically, when the index as defined in [24] is zero, which incidentally is the case when the results are optimal).

In the sequel we consider p fixed. We denote by ξ_0 the frame $(e_1, e_2, \ldots, e_p)$, where as before $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Observe that any p-frame ξ can be expressed as $g\xi_0$. Also, as before, through the rest of this section we put

$$\lambda = \frac{p}{(n-p)}.$$

3. 1. Proposition. *Let $\xi = g\xi_0$ be a p-frame, where $g \in SL(n, \mathbb{R})$. Let K be a compact subset of $\mathscr{L}_n$. Then there exists a constant $C > 0$ such that the following holds: if $\{t_i\}$ is a sequence in $\mathbb{R}^+$ such that $t_i \to \infty$ and $D_p(t_i)\, g^{-1}(\Lambda_0) \in K$ for all i, then there exists a sequence $\{\gamma_i\}$ in $SL(n, \mathbb{Z})$ such that for all large i, $\|\gamma_i\| \leqq Ce^{\lambda t_i}$ and $\|\gamma_i \xi\| < Ce^{-t_i}$ and further, there exists a subsequence of $\{e^{t_i}\gamma_i \xi\}$ which converges to a p-frame η.*

Proof. Let $g^{-1} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \sigma$ be the decomposition as in § 2. 11 for g^{-1} in the place of g. Then for $t > 0$, $D_p(t)\, g^{-1}(\Lambda_0) \in K$ only if $D_p(t) \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} (\Lambda_0) \in K'$ where $K' = \left\{ D_p(t) \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} D_p(-t) \mid t \geqq 0 \right\}^{-1} \cdot K$ is a compact set. On the other hand if for some $\gamma \in SL(n, \mathbb{Z})$ and $t > 0$ we have $\left\| \gamma \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} \xi_0 \right\| < Ce^{-t}$ then we have

$$\begin{aligned} \|\gamma\sigma\xi\| &= \left\| \gamma\sigma \cdot \sigma^{-1} \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} \xi_0 \right\| \\ &= \left\| \gamma \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} \xi_0 A^{-1} \right\| \leqq n \|A^{-1}\| \left\| \gamma \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} \xi_0 \right\| \\ &< n \|A^{-1}\|\, Ce^{-t}. \end{aligned}$$

In view of these observations it is evident that it is enough to prove the Proposition only for g of the form $\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$ where L is an $(n-p) \times p$ matrix; hence we shall suppose

$$g = \begin{pmatrix} I & 0 \\ L & I \end{pmatrix}.$$

Recall that $\mathscr{L}_n$ can be identified with $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$. Thus there exists a compact subset K_1 of $SL(n, \mathbb{R})$ such that $K = K_1 SL(n, \mathbb{Z})/SL(n, \mathbb{Z})$. Put

$$C = 3n \|L\| \sup \{\|h\| \mid h \in K_1^{-1}\}$$

which is finite in view of compactness of K_1. Now let $t_i \to \infty$ be a sequence such that $D_p(t_i)\, g^{-1}(\Lambda_0) \in K$ for all i. In view of the above, this means that there exists a sequence $\{\gamma_i\}$ in $SL(n, \mathbb{Z})$ such that $D_p(t_i)\, g^{-1} \gamma_i^{-1} \in K_1$ for all i. In view of the compactness of K_1 it is enough to show that the conclusion of the Proposition holds for the subsequences of $\{\gamma_i\}$ for which $D_p(t_i)\, g^{-1}\gamma_i^{-1}$ converges. In other words, we may assume $D_p(t_i)\, g^{-1}\gamma_i^{-1}$ to be convergent; say $D_p(t_i)\, g^{-1}\gamma_i^{-1} \to h \in K_1$. Then $\gamma_i g D_p(-t_i) \to h^{-1} \in K_1^{-1}$. Comparing the first p columns of this convergence (or, equivalently, operating the sequence and the limit on the frame ξ_0) we conclude that $e^{t_i}\gamma_i \xi \to \eta$ where η is the p-frame formed by the first p columns of h^{-1}. Since $\|\eta\| \leqq \|h^{-1}\| < C$ the convergence also implies that for all large i, $\|\gamma_i \xi\| < Ce^{-t_i}$. Comparing the last $(n-p)$ columns we conclude that for all large i, $\|\gamma_i \xi^0\| < 2\|h^{-1}\|\, e^{\lambda t_i}$, where ξ^0 is the $(n-p)$-frame $(e_{p+1}, e_{p+2}, \ldots, e_n)$.

Since $g=\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$ and consequently $\xi=\begin{pmatrix} I \\ L \end{pmatrix}$ we see that $\gamma_i\xi=\gamma_i\xi_0+\gamma_i\xi^0 L$ for all i. Hence

$$\|\gamma_i\xi_0\| \leqq \|\gamma_i\xi^0 L\| + \|\gamma_i\xi\| \leqq (n-p)\,\|L\|\,\|\gamma_i\xi^0\| + \|\gamma_i\xi\| \leqq 2n\|L\|\,\|h^{-1}\|\,e^{\lambda t_i} + \|\gamma_i\xi\| \leqq Ce^{\lambda t_i}$$

for all large i, since $\gamma_i\xi \to 0$. Hence $\|\gamma_i\| = \max\{\|\gamma_i\xi_0\|, \|\gamma_i\xi^0\|\} \leqq Ce^{\lambda t_i}$.

A proof of the second part is evidently contained in the above.

Before considering applications of the above to special trajectories we shall prove a (stronger) converse. As before, for any $\delta>0$ let $B_\delta=\{\sum \alpha_i e_i \mid |\alpha_i|<\delta \mid \text{for all } i=1,2,\dots,n\}$. Recall also that $\Lambda_0=\mathbb{Z}^n$.

3. 2. Lemma. *There exists a constant β depending only on n such that the following holds: if $h\in SL(n,\mathbb{R})$ and $\delta>0$ are such that $hB_\delta \cap \Lambda_0 \neq (0)$ then ${}^t h^{-1} B_{\delta^{-1}\beta}$ does not contain n linearly independent elements of Λ_0.*

Proof. The set hB_δ is a parallelopiped and ${}^t h^{-1} B_{\delta^{-1}}$ is its reciprocal (polar) body. Hence the result follows from Theorem 4A of [28].

3. 3. Proposition. *Let $\xi=g\xi_0$ be a p-frame, where $g\in SL(n,\mathbb{R})$. Let $\{t_i\}$ be a sequence in $\mathbb{R}^+$, such that $t_i\to\infty$. Suppose that there exists a constant $C\geqq 1$ and a sequence $\{\gamma_i\}$ in $SL(n,\mathbb{Z})$ such that $\|\gamma_i\xi\|\leqq e^{-t_i}$ and $\|\gamma_i\|\leqq Ce^{\lambda t_i}$. Then $\{D_p(t_i)\,g^{-1}(\Lambda_0)\}$ is contained in a compact subset of $\mathscr{L}_n$.*

Proof. As in the proof of Proposition 3. 1, without loss of generality we may assume $g=\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}$, where L is a $(n-p)\times p$ matrix. Suppose that the conclusion in the proposition is not true. By passing to a subsequence we may assume that $\{D_p(t_i)\,g^{-1}(\Lambda_0)\}$ has no limit point. Put $\delta=\dfrac{\beta}{2C}$ where β is as in Lemma 3. 2. In view of Proposition 2. 2 our assumption implies that for all large i, say $i\geqq i_0$, $D_p(t_i)\,g^{-1}(\Lambda_0)\cap B_\delta \neq (0)$, hence $gD_p(-t_i)\,B_\delta\cap\Lambda_0\neq(0)$. Using Lemma 3. 2 and substituting for δ we conclude that for $i\geqq i_0$, ${}^t(gD_p(-t_i))^{-1}\,B_{2C}$ does not contain n linearly independent elements of Λ_0. But observe that

$$ {}^t(gD_p(-t_i))\,{}^t\gamma_i = {}^t\left\{\gamma_i\begin{pmatrix} I & 0 \\ L & I \end{pmatrix} D_p(-t_i)\right\}$$
$$= {}^t(e^{t_i}\gamma_i\xi,\ e^{-\lambda t_i}\gamma_i\xi^0)$$

so that $\|{}^t(gD_p(-t_i)\,{}^t\gamma_i\|\leqq C$. Consequently ${}^t(gD_p(-t_i)\,{}^t\gamma_i(e_k)\in B_{2C}$ for all $k=1,2,\dots,n$. Hence ${}^t\gamma_i(e_k)\in {}^t(gD_p(-t_i))^{-1}\,B_{2C}$ for all $k=1,\dots,n$. But since for any i, ${}^t\gamma_i(e_k)$, $k=1,\dots,n$ are n linearly independent elements of Λ_0, this contradicts our earlier observation, for all $i\geqq i_0$. Hence $D_p(t_i)\,g^{-1}(\Lambda_0)$ must have a limit point in $\mathscr{L}_n$.

We now apply the propositions to divergent and bounded trajectories.

3. 4. Theorem. *Let $\xi=g\xi_0$ be a p-frame. Then the following conditions are equivalent:*

i) *$\{D_p(t)\,g^{-1}(\Lambda_0)\mid t\geqq 0\}$ is not a divergent trajectory.*

ii) *There exists a constant $C'>0$ and a sequence $\{\gamma_i\}$ in $SL(n,\mathbb{Z})$ such that $\|\gamma_i\|\leqq C'\|\gamma_i\xi\|^{-\lambda}$ and $\gamma_i\to 0$.*

(iii) *There exist a constant* $C' > 0$ *and sequences* $\{\gamma_i\}$ *in* $SL(n, \mathbb{Z})$ *and* $\{\lambda_i\}$ *in* $\mathbb{R}^+$ *such that* $\|\gamma_i\| \leqq C' \|\gamma_i \xi\|^{-\lambda}$ *for all* i, $\lambda_i \to \infty$ *and* $\lambda_i \gamma_i \xi \to \eta$, *where* η *is a p-frame.*

Proof. i) $\Rightarrow$ iii) Since the trajectory is not divergent there exists a compact subset K of $\mathscr{L}_n$ and a sequence $\{t_i\}$ such that $D_p(t_i)\, g^{-1}(\Lambda_0) \in K$ for all i. Let C be the constant and $\{\gamma_i\}$ be the sequence in $SL(n, \mathbb{Z})$ given by Proposition 3. 1 corresponding to the above data. Then for all large i we have $\|\gamma_i\| \leqq C e^{\lambda t_i} < C^2 \|\gamma_i \xi\|^{-\lambda}$. This, together with the last part of the conclusion of Proposition 3. 1, implies that iii) holds when we replace $\{\gamma_i\}$ as above by a suitable subsequence.

iii) $\Rightarrow$ ii) is trivial.

ii) $\Rightarrow$ i) Let $t_i = -\log \|\gamma_i \xi\|$ then $t_i \to \infty$ and we have $\|\gamma_i \xi\| \leqq e^{-t_i}$ and $\|\gamma_i\| \leqq C' e^{\lambda t_i}$. Hence by Proposition 3. 3, $\{D_p(t_i)\, g^{-1}(\Lambda_0)\}$ is contained in a compact subset; in other words $\{D_p(t)\, g^{-1}(\Lambda_0) \mid t \geqq 0\}$ is not a divergent trajectory.

3. 5. Theorem. *Let* $\xi = g\xi_0$ *be a p-frame where* $g \in SL(n, \mathbb{R})$. *Then the following conditions are equivalent*:

i) $\{D_p(t)\, g^{-1}(\Lambda_0) \mid t \geqq 0\}$ *is a bounded trajectory in* $\mathscr{L}_n$.

ii) *There exists a constant* C *such that for any* $\varepsilon > 0$ *there exists* $\gamma \in SL(n, \mathbb{Z})$ *such that* $\|\gamma\| < C\varepsilon^{-\lambda}$ *and* $\|\gamma \xi\| < \varepsilon$.

iii) *There exists a constant* C' *such that given a sequence* $\{\varepsilon_i\}$, *where* $\varepsilon_i \searrow 0$, *there exists a sequence* $\{\gamma_i\}$ *in* $SL(n, \mathbb{Z})$ *such that* $\|\gamma_i\| \leqq C' \varepsilon_i^{-\lambda}$, $\|\gamma_i \xi\| < \varepsilon_i$ *and for a suitable sequence* $\{\lambda_i\}$ *of positive real numbers* $\lambda_i \gamma_i \xi \to \eta$, *a p-frame.*

Proof. i) $\Rightarrow$ iii) Let K be the closure of the bounded trajectory $\{D_p(t)\, g^{-1}(\Lambda_0) \mid t \geqq 0\}$ and let $C > 0$ be the constant given by Proposition 3. 1 for the compact set K. Choose $t_i = \log\left(\dfrac{C}{\varepsilon_i}\right)$. Then the condition of Proposition 3. 1 holds. Let $\{\gamma_i\}$ be a sequence in $SL(n, \mathbb{Z})$ for which the conclusion of that proposition holds. Then $\|\gamma_i\| \leqq C e^{\lambda t_i} = C^{1+\lambda} \varepsilon_i^{-\lambda}$ and $\|\gamma_i \xi\| < C e^{-t_i} = \varepsilon_i$. The last part follows from the corresponding assertion in Proposition 3. 1.

iii) $\Rightarrow$ ii) is trivial.

ii) $\Rightarrow$ i) If the trajectory is not bounded there exists a sequence t_i such that $t_i \to \infty$ and the sequence $\{D_p(t_i)\, g^{-1}(\Lambda_0)\}$ has no limit point in $\mathscr{L}_n$. But since by ii) there exists $\gamma_i \in SL(n, \mathbb{Z})$ such that $\|\gamma_i \xi\| \leqq e^{-t_i}$ and $\|\gamma_i\| \leqq C e^{\lambda t_i}$, Proposition 3. 3 implies that the sequence must have a limit point. Hence i) must hold.

Combining Theorems 3. 4 and 3. 5 with Theorems 2. 14 and 2. 20 of the last section, in particular, we get the following.

3. 6. Theorem. *Let* L *be an* $(n-p) \times p$ *matrix and let* $\{L_1, L_2, \ldots, L_{(n-p)}\}$ *be the corresponding system of linear forms. Let* $\xi = \begin{pmatrix} I \\ L \end{pmatrix}$ *where* I *is the* $p \times p$ *identity matrix. Recall that* $\lambda = \dfrac{p}{(n-p)}$. *Then we have the following.*

i) *There exist a constant* $C>0$ *and a sequence* $\{\gamma_i\}$ *in* $SL(n,\mathbb{Z})$ *such that* $\|\gamma_i\xi\|\to 0$ *and* $\|\gamma_i\|\leqq C\|\gamma_i\xi\|^{-\lambda}$ *for all i if and only if* $\{L_1, L_2,\ldots, L_{(n-p)}\}$ *is regular.*

ii) *There exists a constant* $C>0$ *such that for any* $\varepsilon>0$ *the inequalities* $\|\gamma\xi\|<\varepsilon$ *and* $\|\gamma\|\leqq C\varepsilon^{-\lambda}$ *have a solution* $\gamma\in SL(n,\mathbb{Z})$ *if and only if* $\{L_1, L_2,\ldots, L_{(n-p)}\}$ *is badly approximable.*

One of the implications in ii), viz. badly approximable $\Rightarrow$ existence of the solution γ for the inequalities, overlaps with a recent result of S. Raghavan (cf. [24], Theorem 4). There the author allows a weaker condition than badly approximable and concludes the existence of a constant C such that for all $\varepsilon>0$ there exists $\gamma\in SL(n,\mathbb{Z})$ for which $\|\gamma\xi\|<\varepsilon$ and $\|\gamma\|<C\varepsilon^{-\lambda-\delta}$, where $\delta\geqq 0$ is the "proper index" of the system. We note however that like badly approximable systems the systems satisfying the weaker condition involved in the above also form a set of measure zero in the $p(n-p)$-dimensional vector space of all systems of all $(n-p)$ linear forms in p variables. It would be interesting to know whether the weaker condition or solvability of the above (weaker) system of inequalities signifies something about the trajectory $\{D_p(t)\,g^{-1}(\Lambda_0)\mid t\geqq 0\}$, where $g\in SL(n,\mathbb{R})$ is such that $\xi=g\xi_0$.

Since, for $n\geqq 3$ there exist singular systems of linear forms (cf. Theorem 2.17), there exist p-frames ξ for which there exists no constant C such that $\gamma_i\xi\to 0$ and $\|\gamma_i\|<C\|\gamma_i\xi\|^{-\lambda}$ for a suitable sequence $\{\gamma_i\}$ in $SL(n,\mathbb{Z})$. It was shown to the author by E. Bombieri that, in fact, given any monotonically increasing function $\omega(t)$ on $\mathbb{R}^+$ there exists an $(n-1)$-frame ξ such that there is no sequence $\{\gamma_i\}$ in $SL(n,\mathbb{Z})$ satisfying $\gamma_i\xi\to 0$ and $\|\gamma_i\|<\omega(\|\gamma_i\xi\|^{-1})$ for all i; the proof is based on Theorem 14 from [7] and Theorem 4A from [28]. As in the previous paragraph it would be interesting to know if this phenomenon signifies something about the dynamics of the trajectories involved.

§ 4. Divergent trajectories on homogeneous spaces

Let G be a connected Lie group and Γ be a lattice in G. Let $\{g_t\}$ be a (continuous) one-parameter subgroup of G. In the remaining sections we shall discuss the divergent trajectories of the flow induced by $\{g_t\}$ on G/Γ.

As G is a connected Lie group, the class of closed (not necessarily connected) normal subgroups of G which do not contain any non-compact semisimple subgroup has a unique maximal element M. G/M is a semisimple Lie group with trivial center and no non-trivial compact factors; (in fact, G/M is the maximal among such quotients). Further, for any lattice Γ, $M\Gamma$ is a closed subgroup of G and the natural quotient map $\eta: G/\Gamma\to G/M\Gamma$ is proper (cf. [25]). We conclude from this the following.

4.1. Proposition. *For* $g\in G$ *the trajectory* $\{g_t g\Gamma\mid t\geqq 0\}$ *in* G/Γ *is divergent if and only if the trajectory* $\{g_t g M\Gamma\mid t\geqq 0\}$ *in* $G/M\Gamma$ *is divergent.*

Since $G/M\Gamma$ is the quotient of G/M by the lattice $M\Gamma/M$, the proposition signifies that in studying divergence of trajectories of flows there is no loss of generality if we restrict to (connected) semisimple Lie groups G with trivial center and non-trivial compact factors. Such a group is linear (that is, a subgroup of $GL(n,\mathbb{R})$ for some n). In the following sections we shall, in general, only assume G to be a connected linear semisimple Lie group. We next motivate further restrictions on Γ.

A lattice Γ in a semisimple Lie group G is said to be *irreducible* if for any normal subgroup F of positive dimension, $F\Gamma$ is dense in G. We recall the following result on "decomposition" of lattices into irreducible components.

4. 2. Proposition. *If G is a semisimple Lie group with trivial center and no non-trivial compact factors and Γ is a lattice in G then there exist normal (closed) subgroups $G_1, G_2, \ldots, G_k$ of G such that*

i) $G = G_1 \cdot G_2 \cdots G_k$ *(direct product)*;

ii) *for each $i = 1, 2, \ldots, k$, $L \cap G_i = \Gamma_i$, say, is an irreducible lattice in G_i*;

iii) *$\Gamma_1 \cdot \Gamma_2 \cdots \Gamma_k$ is of finite index in Γ.*

(4. 3) Thus the homogeneous space $\prod G_i/\Gamma_i$ is canonically isomorphic to $G/\Gamma_1\Gamma_2 \cdots \Gamma_k$. It is evident that a trajectory $\{g_t g\Gamma \mid t \geqq 0\}$ is divergent if and only if $\{g_t g \Gamma_1 \Gamma_2 \cdots \Gamma_k \mid t \geqq 0\}$ is divergent and that holds if and only if the projection on any one of G_i/Γ_i, $1 \leqq i \leqq k$ is divergent. We thus find that in studying divergence as above we may (further) restrict to irreducible lattices.

(4. 4) Now let G be a semisimple Lie group with trivial center and no non-trivial compact factors and let Γ be an irreducible lattice in G. Then by Margulis' arithmeticity theorem (cf. [22]) there are only the following three (not mutually exclusive) possibilities: i) G/Γ is compact; ii) Γ is an arithmetic lattice in the sense that G is the connected component of the identity in the group of $\mathbb{R}$-elements of an algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and Γ is commensurable with $\mathbf{G}_{\mathbb{Z}}$ (that is, $\Gamma \cap \mathbf{G}_{\mathbb{Z}}$ has finite index in Γ and $\mathbf{G}_{\mathbb{Z}}$); or iii) G is of $\mathbb{R}$-rank 1.

The *rank* of an irreducible lattice Γ (or equivalently of the homogeneous space G/Γ) as above is defined to be 0 if i) holds, 1 if iii) holds and i) does not and to be the $\mathbb{Q}$-rank of the algebraic group $\mathbf{G}$ if ii) holds. It is well-known that this is consistent; (it follows for instance from Theorem 13. 1 and Corollary 15. 3 of [2]). If G has non-trivial center C (but other conditions are as before) then the rank of an irreducible lattice Γ in G is defined to be the rank of the (irreducible) lattice $C\Gamma/C$ in G/C.

In analogy to Propositions 2. 4 and 2. 6 in general we have the following.

4. 5. Proposition. *Let G be a connected semisimple Lie group with finite center. Let Γ be an irreducible lattice in G. Let $\{g_t\}$ be a one-parameter subgroup of G which is not contained in any compact subgroup of G. If $\{t_i\}$ is a sequence in $\mathbb{R}$ such that $t_i \to \infty$ then for almost all $g\Gamma \in G/\Gamma$, $\{g_{t_i} g\Gamma\}$ is dense in G/Γ. If all eigenvalues of $\operatorname{Ad} g_1$ are of absolute value 1 then no trajectory of $\{g_t\}$ on G/Γ is divergent. If Γ is non-uniform and $\operatorname{Ad} g_1$ is diagonalisable over $\mathbb{C}$ and has an eigenvalue λ with $|\lambda| \neq 1$ then $\{g_t\}$ has a divergent trajectory on G/Γ.*

Proof. As before the first part follows from the results of C. C. Moore (cf. [23], Theorem 1). For an arithmetic lattice Γ and a one-parameter subgroup $\{g_t\}$ consisting of unipotent elements (that is, $\operatorname{Ad} g_t$ is unipotent) the non-existence of divergent trajectories can be conclude using Margulis' result as involved in Proposition 2. 6, using Proposition 9. 3 of [25] (cf. [11] for an idea of the proof). For a non-arithmetic lattice this (and even a stronger assertion) is proved in [14]. (Actually, a simpler proof is also possible if we only need non-divergence.) As in the proof of Proposition 2. 6, non-existence of divergent trajectories when all eigenvalues of $\operatorname{Ad} g_1$ are of unit absolute value can be concluded from the above special case, using the Jordan decomposition.

For a large class of homogeneous spaces we shall actually find divergent trajectories for $\{g_t\}$ when $\operatorname{Ad} g_1$ has an eigenvalue λ with $|\lambda| \neq 1$ (cf. Theorem 7. 3 and Proposition 5. 6). However, to prove the converse assertion sought after, we follow an indirect argument. If $\operatorname{Ad} g_1$ is diagonalisable over $\mathbb{C}$ and has an eigenvalue λ with $|\lambda| \neq 1$, then the corresponding horospherical subgroup U is non-trivial and consequently not contained in any compact subgroup of G. Since Γ is an irreducible lattice, this implies that the action of U on G/Γ is ergodic. Hence if $\{g_t\}$ has no divergent trajectories, then by Theorem 1. 6 this implies that all U-orbits on G/Γ are dense. But when G/Γ is non-compact, this is false even for a maximal horospherical subgroup (cf. [12], Corollary 2. 4 and Remark 2. 5 in the arithmetic case and condition d) of Theorem 5. 2 of the present paper for the case of lattices in simple Lie groups of $\mathbb{R}$-rank 1).

It seems to the author that the converse assertion would be true without the assumption that $\operatorname{Ad} g_1$ is diagonalisable over $\mathbb{C}$.

As suggested by Theorem 2. 17 it will turn out that while for lattices of rank 1 all divergent trajectories are "degenerate", when the rank is $\geqq 2$ there always exist "non-degenerate" divergent trajectories. This will be taken up in §§ 6 and 7 respectively. We conclude this section by proving an abstract characterization of degenerate divergent trajectories in $SL(n, \mathbb{R})/SL(n, \mathbb{Z})$.

4. 6. Definition. Let G be a connected semisimple Lie group and let Γ be a lattice in G. A parabolic subgroup Q of G is said to be *Γ-rational* if the unipotent radical N of Q intersects Γ in a lattice; that is, $N \cap \Gamma$ is a lattice in N.

Let Q_i, where $1 \leqq i \leqq n-1$, be the subgroup of $SL(n, \mathbb{R})$ consisting of all the elements which under the ith exterior action (natural if $i=1$) leaves invariant the subspace spanned by $e_1 \wedge e_2 \wedge \cdots \wedge e_i$, where as before $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. It is well-known that any maximal parabolic subgroup Q of $SL(n, \mathbb{R})$ is conjugate to (a unique) Q_i, $i \leqq n-1$; Q is Γ-rational if it is conjugate to Q_i by an element of $SL(n, \mathbb{Q})$; that is, $Q = qQ_iq^{-1}$ for some $q \in SL(n, \mathbb{Q})$ and $1 \leqq i \leqq n-1$.

4. 7. Theorem. *Let $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$. Let $\{g_t\}$ be a one-parameter subgroup and let $g \in G$. The trajectory $\{g_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if the following holds: there exist a representation $\rho: G \longrightarrow GL(V)$ over a finite dimensional vector space and a $v \in V-(0)$ such that the following conditions are satisfied:*

i) *The subgroup $Q = \{x \in G \mid \rho(x)\, v = \chi(x)\, v$ for some $\chi(x) \in \mathbb{R}^*\}$ is a maximal Γ-rational parabolic subgroup of G, and*

ii) *there exists $\theta \in C(\Gamma) = SL(n, \mathbb{Q})$ such that $\rho(g_t g\theta) \longrightarrow 0$ as $t \longrightarrow \infty$ (note that θ is understood to be independent of t).*

Proof. Suppose that $\{g_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory (in the sense of § 1). Then by Proposition 2.9 there exists i, $1 \leqq i \leqq n-1$ such that if $\rho = \bigwedge^i$ the ith exterior representation (natural, if $i=1$) then $\rho(g_t g\gamma)(e_1 \wedge e_2 \wedge \cdots \wedge e_i) \to 0$ for a suitable $\gamma \in \Gamma$. This together with the remark preceding the statement of the theorem implies that conditions i) and ii) are satisfied for $\rho = \bigwedge^i$, $v = e_1 \wedge e_2 \wedge \cdots \wedge e_i$ and $\theta = \gamma$.

Now let ρ be a representation of G over a finite dimensional vector space V and let $v \in V$ be such that conditions i) and ii) are satisfied. By the remark preceding the theorem, the subgroup Q as in condition i) is of the form $qQ_i q^{-1}$ for some $q \in SL(n, \mathbb{Q})$ and $1 \leqq i \leqq n-1$ (with Q as defined there). Then $\rho(Q_i)$ leaves invariant the subspace spanned by $w = \rho(q^{-1})(v)$. Also $\rho(g_t g\theta q)(w) = \rho(g_t g\theta)(v) \to 0$ as $t \to \infty$. Since

$$\theta q \in SL(n, \mathbb{Q}) = C(\Gamma),$$

this means that by replacing v by w in the hypothesis we may assume Q as in condition i) to be Q_i for some i, $1 \leqq i \leqq n-1$.

Since any representation of $SL(n, \mathbb{R})$ is completely reducible, by passing to one of the components we may assume ρ to be the irreducible one; note that since Q_i is a maximal parabolic subgroup the subspace spanned by at least one of the components of v is invariant only under the action of elements of Q_i.

Recall that an irreducible representation is determined completely by its heighest weight. Also since Q_i is a maximal parabolic subgroup, all the possible highest weights of irreducible representations for which there is a Q_i-invariant 1-dimensional subspace in the representation space are multiples of each other (note that one-zero vectors in the Q_i-invariant one-dimensional subspace have to be highest weight vectors). Using the fact that the highest weight has to be "integral" (cf. [20]) and that the highest weight corresponding to the ith exterior representation is a fundamental weight we conclude that the highest weight of ρ as above must be a positive integral multiple, say k times, of the highest weight corresponding to the ith exterior representation.

Consider the kth symmetric power ρ_i^k of the ith exterior representation. Let $w = (e_1 \wedge e_2 \wedge \cdots \wedge e_i)^k$, the kth symmetric power, and let W be the smallest $\rho_i^k(SL(n, \mathbb{R}))$-invariant subspace containing w. In view of the choice of k and the uniqueness of an irreducible representation with a given highest weight, it follows that W can be expressed as a direct sum of ρ_i^k-invariant subspaces $W_1, W_2, \ldots, W_l$ such that the restriction of ρ_i^k to each of the subspaces is isomorphic to the representation ρ. Further if $w_1, w_2, \ldots, w_l$ be the components of w in $W_1, W_2, \ldots, W_l$ respectively they are highest weight vectors and hence under this isomorphism they correspond to a scalar multiple of $v \in V$. Hence by our hypothesis $\rho_i^k(g_t g\theta)(w_j) \to 0$ as $t \to \infty$ for all $j = 1, 2, \ldots, l$. Hence $\rho_i^k(g_t g\theta)(w) \to 0$ as $t \to \infty$; that is, the kth symmetric power $(\bigwedge^i(g_t g\theta)(e_1 \wedge \cdots \wedge e_i))^k \to 0$ as $t \to \infty$. It is straightforward to verify by looking at the components that this implies that $\bigwedge^i(g_t g)(e_1 \wedge \cdots \wedge e_i) \to 0$. It is well-known, and easy to see, that $SL(n, Q) \subset SL(n, \mathbb{Z}) \cdot P$, where P is the subgroup consisting of all upper triangular matrices. In particular, θ can be expressed as $\theta = \gamma \cdot p$, where $\gamma \in SL(n, \mathbb{Z})$ and $p \in P \subset Q_i$. Then in view of the above, evidently $\bigwedge^i(g_t g\gamma)(e_1 \wedge \cdots \wedge e_i) \to 0$. By Proposition 2.9 this implies that $\{g_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory.

§ 5. Fundamental domains and degeneracy

In this section we recall the results on fundamental domains of lattices (to be used in the sequel following the same notation) and motivate the notion of degeneracy in the general context.

(5. 1) Let G be a connected linear semisimple Lie group and Γ be an irreducible non-uniform lattice in G; in particular this means that G can have no compact factors. We shall assume that there exists, and consider fixed, an algebraic group $\mathbf{G}$ defined over $\mathbf{k}$, where $\mathbf{k}=\mathbb{Q}$ or $\mathbb{R}$, such that $G=\mathbf{G}_{\mathbb{R}}^{\circ}$ (notation as in [3]) and at least one of the following conditions is satisfied:

a) $\mathbf{k}=\mathbb{Q}$ and Γ is commensurable with $\mathbf{G}_{\mathbb{Z}}$ or

b) $\mathbf{k}=\mathbb{R}$ and $\mathbb{R}$-rank of G is 1.

Recall that in view of the reductions in § 4, the above assumptions do not involve any loss of generality in the study of divergent trajectories on homogeneous spaces of Lie groups by lattices.

Let $\mathbf{S}$ be a maximal $\mathbf{k}$-split torus in $\mathbf{G}$ and let $S=\mathbf{S}_{\mathbb{R}}^{\circ}$. Let $\mathfrak{G}$ be the (algebraic or equivalently, complex) Lie algebra of $\mathbf{G}$. Relative to the adjoint action of $\mathbf{S}$ on $\mathfrak{G}$ we have the decomposition

$$\mathfrak{G}=\mathfrak{Z}(\mathbf{S})+\sum_{\lambda\in\Phi}\mathfrak{G}^{\lambda}$$

where $\mathfrak{Z}(\mathbf{S})$ is the Lie algebra corresponding to the centralizer $\mathbf{Z}(\mathbf{S})$ of $\mathbf{S}$ in $\mathbf{G}$, Φ is the system of $\mathbf{k}$-roots relative to $\mathbf{S}$ and for each $\lambda\in\Phi$

$$\mathfrak{G}^{\lambda}=\{\xi\in\mathfrak{G}\mid(\operatorname{Ad}s)\,\xi=\lambda(s)\,\xi\quad\text{for all}\quad s\in\mathbf{S}\}.$$

Let $\mathbf{P}$ be a minimal $\mathbf{k}$-parabolic subgroup of $\mathbf{G}$ and let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}$. There exists a unique order on Φ such that $\mathbf{U}$ is the subgroup generated by $\{\exp\mathfrak{G}^{\lambda}\mid\lambda\in\Phi^{+}\}$, Φ^{+} being the set of positive roots relative to the order. We denote by Δ the set of simple roots with respect to the order. For any $\tau>0$ put

$$S_{\tau}=\{s\in S\mid\alpha(s)<\tau\quad\text{for all}\quad\alpha\in\Delta\}.$$

We note also that $\mathbf{P}$ can be expressed as $\mathbf{P}=\mathbf{S}\cdot\mathbf{M}\cdot\mathbf{U}$, where $\mathbf{M}$ is a reductive algebraic subgroup of $\mathbf{Z}(\mathbf{S})$ which is defined and anisotropic over $\mathbf{k}$. Put $P=\mathbf{P}_{\mathbb{R}}\cap G$, $M=\mathbf{M}_{\mathbb{R}}^{\circ}$ and $U=\mathbf{U}_{\mathbb{R}}$. Then we get the decomposition $P^{\circ}=SMU$. Finally, let r denote the rank of Γ (cf. § 4. 4).

5. 2. Theorem. *Let the notations be as above. Further let K be a maximal compact subgroup of G. Then there exists a compact subset C of MU, a finite subset J of $\mathbf{G}_{\mathbf{k}}\cap G$ and $\tau>0$ such that the following conditions are satisfied:*

a) $G=KS_{\tau}CJ\Gamma$.

b) *Let $J_1=J$ if $\mathbf{k}=\mathbb{R}$ and any finite subset of $\mathbf{G}_{\mathbb{R}}$ if $\mathbf{k}=\mathbb{Q}$; then for any compact subset D of MU and $\sigma>0$ the set $\{\gamma\in\Gamma\mid(KS_{\sigma}DJ_1)\,\gamma\cap(KS_{\sigma}DJ_1)\neq\emptyset\}$ is finite.*

c) *If $r=1$ then for any compact subset D of MU there exists $\sigma>0$ such that the following holds: if $j,j_1\in J$ and $\gamma\in\Gamma$ are such that $(KS_{\sigma}Dj\gamma)\cap(KS_{\sigma}Dj_1)$ is non-empty then $j_1=j$ and $j\gamma j^{-1}\in P$.*

d) *For all $j\in J$, $j^{-1}Uj\cap\Gamma$ is a lattice in $j^{-1}Uj$.*

e) *If Q is a parabolic subgroup such that the unipotent radical N of Q intersects Γ in a lattice (that is, $N \cap \Gamma$ is a lattice in N) then there exists $j \in J$ and $\gamma \in \Gamma$ such that Q contains $\gamma^{-1} j^{-1} P j \gamma$.*

For arithmetic lattices conditions a), b) and c) as above follow from Theorem 13.1, Corollary 15.3 and Proposition 17.9 respectively of [2]. Condition d) follows from the fact that $J \subset \mathbf{G}_{\mathbb{R}}$ and $\mathbf{U}$ is a unipotent algebraic group defined over $\mathbb{Q}$. Condition e) follows from conjugacy of minimal $\mathbb{Q}$-parabolic subgroups and the fact that if the unipotent radical of a parabolic subgroup Q of G intersects Γ in a lattice then $Q = \mathbf{Q}_{\mathbb{R}} \cap G$ for an appropriate $\mathbb{Q}$-parabolic subgroup of $\mathbf{G}$. (The unipotent radical must be defined over $\mathbb{Q}$ and hence so is $\mathbf{Q}$, being its normalizer.)

For lattices in simple Lie groups of $\mathbb{R}$-rank 1 conditions a), b), c) and d) follow from Theorem 0.6 of [17]. To uphold condition e) we proceed as follows. Observe that in view of condition d) as in the theorem, for any $x \in PJ\Gamma$, the orbit of $Ux\Gamma/\Gamma$ under the U-action is compact. It turns out that $PJ\Gamma/\Gamma$ is precisely the set of all the points with this property; this may be deduced from Theorem 3.4 in [12]. A simpler proof can also be obtained using Theorem 1.6 and the "only if part" of Theorem 6.1 of the present paper (cf. Corollary 6.3). We note that the latter does not involve condition e) (or any of its consequences). Now let Q be a proper parabolic subgroup. Since G is of $\mathbb{R}$-rank 1, Q must be conjugate to P, say $Q = gPg^{-1}$ for some $g \in G$. Then gUg^{-1} is the unipotent radical of Q. If, as in the condition, $gUg^{-1} \cap \Gamma$ is a (necessarily uniform) lattice in gUg^{-1}, then $U \cap g^{-1}\Gamma g$ is a uniform lattice in U and consequently $Ug^{-1}\Gamma/\Gamma$ is a compact U-orbit. Hence by the above remark, $g^{-1} \in PJ\Gamma$; let $g^{-1} = pj\gamma$, where $p \in P$, $j \in J$ and $\gamma \in \Gamma$. Then $Q = gPg^{-1} = \gamma^{-1} j^{-1} p^{-1} P p j \gamma = \gamma^{-1} j^{-1} P j \gamma$.

A set J for which the conditions of Theorem 5.2 are satisfied is called a *sufficient set of cusp elements.*

(5.3) Let the notation be as in (5.1). For any subset Σ of Δ let $\langle \Sigma \rangle$ be the set of all elements of Φ which are contained in the subgroup of the character group generated by Σ. For $\Sigma \subset \Delta$ let $\mathbf{P}_\Sigma$ be the subgroup generated by $\mathbf{Z}(\mathbf{S}) \cup \{\exp \mathfrak{G}^\lambda \mid \lambda \in \langle \Sigma \rangle \cup \Phi^+\}$, where $\mathbf{Z}(\mathbf{S})$ is the centralizer of $\mathbf{S}$. Note that $\mathbf{P}_\phi = \mathbf{P} \cdot \mathbf{P}_\Sigma$, $\Sigma \subset \Delta$ are all the $\mathbf{k}$-parabolic subgroups containing $\mathbf{P}$ (cf. [3]). They are called standard parabolic $\mathbf{k}$-subgroups with respect to the order. Any parabolic $\mathbf{k}$-subgroup is conjugate to a standard parabolic $\mathbf{k}$-subgroup, by an element of $\mathbf{G}_\mathbf{k}$. For each $\alpha \in \Delta$, $\mathbf{P}_{\Sigma - \{\alpha\}}$ is a maximal standard parabolic subgroup.

Let $\Sigma \subset \Delta$. The standard parabolic subgroup $\mathbf{P}_\Sigma$ can be (Levi) decomposed as $\mathbf{P}_\Sigma = \mathbf{Z}_\Sigma \cdot \mathbf{N}_\Sigma$ where $\mathbf{N}_\Sigma$ is the unipotent radical of $\mathbf{P}_\Sigma$ and $\mathbf{Z}_\Sigma$ is the reductive $\mathbb{Q}$-subgroup generated by $\mathbf{Z}(\mathbf{S})$ and $\{\exp \mathfrak{G}^\lambda \mid \lambda \in \langle \Sigma \rangle\}$. $\mathbf{Z}_\Sigma$ can be further decomposed as $\mathbf{Z}_\Sigma = \mathbf{T}_\Sigma \cdot \mathbf{H}_\Sigma$ where $\mathbf{T}_\Sigma$ is a $\mathbf{k}$-split torus in the center of $\mathbf{Z}_\Sigma$ and $\mathbf{H}_\Sigma$ is a reductive algebraic group defined over $\mathbf{k}$ on which there is no character defined over $\mathbf{k}$. This, in particular, implies that if $H = (\mathbf{H}_\Sigma)^\circ_{\mathbb{R}}$ then $H \cap \Gamma$ is a lattice in H (cf. [2], Corollary 13.2 for the arithmetic case; in the other case it is obvious since either H is compact or $H = G$). If $\Sigma = \phi$ then $H = M$. In this case $M \cap \Gamma$ is a uniform lattice in H. We note for later use that since $J \subset \mathbf{G}_\mathbb{Q}$ if $\mathbf{k} = \mathbb{Q}$ and M is compact if $\mathbf{k} = \mathbb{R}$, for all $j \in J$, $j^{-1} M j \cap \Gamma$ is a uniform lattice in $j^{-1} M j$.

(5.4) Let Q be a parabolic subgroup of G and let N be the unipotent radical of Q. Recall that Q is said to be Γ-rational if $N \cap \Gamma$ is a lattice in N. If Γ is an arithmetic lattice then a parabolic subgroup Q of G is Γ-rational if and only if $Q = \mathbf{Q}_{\mathbb{R}} \cap G$ for a suitable (unique) parabolic $\mathbb{Q}$-subgroup $\mathbf{Q}$ of $\mathbf{G}$.

Recall that in (5. 1) while choosing the order on Φ we started with a minimal parabolic $\mathbf{k}$-subgroup $\mathbf{P}$; that is, $\mathbf{P}$ is $\mathbb{Q}$-parabolic if Γ is arithmetic, but only $\mathbb{R}$-parabolic otherwise. We shall henceforth assume the choice to be such that in the latter case (also) P is Γ-rational. By condition e) this implies that there exists $\gamma \in \Gamma$ and $j_0 \in J$ (the corresponding set of cusp elements) such that $j_0 \gamma$ normalizes P. Since a parabolic subgroup is its own normalizer we get that $j_0 \gamma \in P$. P can be decomposed as $M'SU$ where S and U are as above and $M' = \{k \in K \mid ks = sk \text{ for all } s \in S\}$. Hence the last conclusion implies that the assertions as in the theorem remain valid, for a suitably modified C and τ, if j_0 is replaced by e, the identity. In other words we may assume without loss of generality, as we do in the sequel, that $e \in J$.

5. 5. Definition. Let $\rho: G \to GL(V)$ be a representation of G over a finite dimensional vector space V and let $v \in V-(0)$. The triple (ρ, V, v) is said to be an *admissible chart at ∞ for G/Γ* if the following conditions are satisfied:

i) The subgroup $Q = \{g \in G \mid \rho(g)(v) \in \mathbb{R}^* v\}$ is a Γ-rational parabolic subgroup of G.

ii) The subgroup $Q_0 = \{g \in G \mid \rho(g)(v) = v\}$ intersects Γ in a lattice; that is, $Q_0 \cap \Gamma$ is a lattice in Q_0.

If (ρ, V, v) is an admissible chart at ∞ then the subgroup Q as in condition i) is called the corresponding *projective isotropy subgroup*.

5. 6. Proposition. *Let $\{x_t\}_{t \geq 0}$ be a curve in G. Let (ρ, V, v) be an admissible chart at ∞ for G/Γ such that $\rho(x_t)(v) \to 0$ as $t \to \infty$. Then the following conditions hold:*

a) *There exists an admissible chart at ∞ for G/Γ, say (ρ', V', v') such that the corresponding projective isotropy subgroup is a maximal Γ-rational parabolic subgroup and $\rho'(x_t)(v') \to 0$ as $t \to \infty$.*

b) *$x_t \Gamma \to \infty$ as $t \to \infty$.*

c) *Let $\{g_t\}$ be a one-parameter subgroup of G and*

$$E = \{x \in G \mid \rho(g_t x)(v) \to 0 \quad \text{as} \quad t \to \infty\}.$$

Then given an $x \in E$ and a compact set F of G/Γ, there exists a neighbourhood Ω of x in E and a $T \geq 0$, such that $g_t g \Gamma \notin F$ whenever $t > T$ and $g \in \Omega$.

Proof. Let Q be the projective isotropy subgroup corresponding to (ρ, V, v). Since Q is Γ-rational, without loss of generality (by modifying the given representation suitably) we may assume that Q contains the subgroup P involved in Theorem 5. 2 and (5. 4). Let $\mathbf{Q}$ be the parabolic subgroup of $\mathbf{G}$ such that $Q = \mathbf{Q}_{\mathbb{R}} \cap G$. Then $\mathbf{Q}$ is a standard parabolic $\mathbf{k}$-subgroup ($\mathbf{k}$ as in (5. 1)); $\Sigma \subset \Delta$ be such that $\mathbf{Q} = \mathbf{P}_\Sigma$. Let $\mathbf{H}_\Sigma$ and $\mathbf{N}_\Sigma$ be as in § 5. 3. Let $H = (\mathbf{H}_\Sigma)_{\mathbb{R}}^\circ$, $T = (\mathbf{T}_\Sigma)_{\mathbb{R}}^\circ$ and $N = (\mathbf{N}_\Sigma)_{\mathbb{R}}$. Recall that $H \cap \Gamma$ is a lattice in H. Also by Γ-rationality of Q, $N \cap \Gamma$ is a lattice in N.

Let $Q_0 = \{q \in Q \mid \rho(q)(v) = v\}$. We claim that $HN \subset Q_0$. Recall that since (ρ, V, v) is an admissible chart $Q_0 \cap \Gamma$ is a lattice in Q_0. Since Q/Q_0 is one-dimensional and $Q \cap \Gamma$ cannot be a lattice in Q (as Q is not unimodular) it follows that $Q \cap \Gamma / Q_0 \cap \Gamma$ is finite. Hence there exist subgroups of finite index in $H \cap \Gamma$ and $N \cap \Gamma$ which are contained in Q_0. By Zariski-density of $N \cap \Gamma$ in N (cf. [13], § 4) this implies that $N \subset Q_0$. Also $H \cap \Gamma$ is Zariski-dense in a co-compact normal subgroup of H (cf. [13], § 4).

Hence $\rho(H)(v)$ is compact. But since the 1-dimensional subspace spanned by v is $\rho(H)$-invariant and H is connected, this is impossible unless $\rho(H)$ fixes v; that is, H is contained in Q_0.

Now let $\| \ \|$ be a $\rho(K)$-invariant norm on V such that $\|v\|=1$. We have $G=KQ^\circ=KTHN$ and if $x=ksy$, where $k\in K$, $s\in T$ and $y\in HN$ then

$$\|\rho(x)(v)\| = \|\rho(ksy)(v)\| = \|\rho(s)(v)\| = \chi(s)$$

where $\chi: S\to \mathbb{R}^+$ is the character defined by $\rho(s)(v)=\chi(s)v$. Let $\{x_t\}_{t\geq 0}$ be a curve in G as in the hypothesis. Writing $x_t=k_t s_t y_t$, where $k_t\in K$, $s_t\in T\subset S$ and $y_t\in HN$ for all t, we see that $\chi(s_t)=\|\rho(x_t)(v)\|\to 0$ as $t\to\infty$.

Observe that v is a highest weight vector (when we consider the complexified representation) with respect to any Borel subgroup contained in $\mathbf{Q}$. Since the corresponding weight has to be dominant integral it follows in particular that the character χ can be expressed as

$$\chi(s) = \prod_{\alpha\in\Delta} \alpha(s)^{m_\alpha}$$

where $m_\alpha\geqq 0$ for all $\alpha\in\Delta$. Since $H\subset Q_0$ we get that $m_\alpha=0$ for all $\alpha\in\Sigma$. On the other hand since $\chi(s)\to 0$ there exists $\beta\in\Delta$ such that $\beta(s_t)\to 0$. Evidently $\beta\in\Delta-\Sigma$ and $m_\beta>0$. Let $\mathbf{Q}'=\mathbf{P}_{\Delta-\{\beta\}}$ and $Q'=\mathbf{Q}'_{\mathbb{R}}\cap G$. Let N' be the unipotent radical of Q' and let l be the dimension of N'. Let $V'=\bigwedge^l\mathfrak{G}$, the lth exterior power of $\mathfrak{G}$ as a vector space. Let ρ' be the lth exterior power of the adjoint representation of G over $\mathfrak{G}$ and let v' be a non-zero element of the one-dimensional subspace of V' corresponding to the Lie subalgebra of N'. It is straightforward to verify that (ρ', V', v') is an admissible chart at ∞ for G/Γ. Now let $\| \ \|$ be a $\rho'(K)$-invariant norm on V' such that $\|v'\|=1$. Let $H'=(\mathbf{H}_{\Delta-\{\beta\}})^\circ_{\mathbb{R}}$ and $N'=(\mathbf{N}_{\Delta-\{\beta\}})_{\mathbb{R}}$. Then it is easy to see that $H\subset H'$ and $N\subset H'\cdot N'$ and that $\rho'(H'N')$ fixes v'. The argument applied to ρ above now shows that if $x=ksy$ where $k\in K$, $s\in S$ and $y\in H'N'$ then $\|\rho'(x)(v')\|=\beta(s)^m$ for some $m>0$. Let k_t, s_t and y_t where $t\geqq 0$ be as chosen above. Since $y_t\in HN\subset H'N'$ we get that $\|\rho'(x_t)(v')\|=\beta(s_t)^m\to 0$ as $t\to\infty$ since by choice of β, $\beta(s_t)\to 0$. This proves assertion a) of the Proposition.

To prove assertion b) we may revert to the initial notation and assume $\mathbf{Q}$ to be a maximal Γ-rational parabolic $\mathbf{k}$-subgroup. Then T is one-dimensional and the character χ has the form $\chi(s)=\beta(s)^m$ for some $\beta\in\Delta$ and $m>0$.

First consider the case of (non-uniform) lattices in simple Lie group of $\mathbb{R}$-rank 1. Then $HN=MU$ and $HN\cap\Gamma$ is a uniform lattice in HN. Hence there exists a compact subset D of MU such that $HN\subset D\Gamma$. Using the decomposition $x_t=k_t s_t y_t$ as before we see that $x_t\in Ks_tD\Gamma$, where $\beta(s_t)\to 0$ (β now being the unique element of Δ). Since $e\in J$ (cf. § 5. 4) by condition b) of Theorem 5. 2 this implies that $x_t\Gamma\to\infty$ as $t\to\infty$.

Next consider the case of arithmetic lattices; that is, $\mathbf{k}=\mathbb{Q}$. Put $\mathbf{S}'=\mathbf{S}\cap\mathbf{H}_{\Delta-\{\beta\}}$, $S'=(\mathbf{S}'_{\mathbb{R}})^\circ$ and $U'=U\cap H$. Let K' be a maximal compact subgroup of H and let K be a maximal compact subgroup of G containing K'. By Theorem 13. 1 of [2] (actually condition a) in Theorem 5. 2, except that H may have compact factors) there exist a compact subset D_1 of MU', a finite subset J_1 of $H\cap\mathbf{G}_{\mathbb{Q}}$ and $\sigma>0$ such that

$$H=K'(S'\cap S_\sigma)\,D_1J_1(H\cap\Gamma).$$

Since $N \cap \Gamma$ is a uniform lattice in N there exists a compact subset D_2 of N such that $N = D_2(N \cap \Gamma)$. Hence

$$HN = K'(S' \cap S_\sigma)\, D_1 J_1 (H \cap \Gamma) \cdot N = K'(S' \cap S_\sigma)\, D_1 J_1 N (H \cap \Gamma)$$
$$\subset K'(S' \cap S_\sigma)\, D_1 J_1 D_2 \Gamma \subset K'(S' \cap S_\sigma)\, D J_1 \Gamma,$$

where $D = D_1 \left(\bigcup_{j \in J_1} j D_2 j^{-1} \right)$. We note also that $\{s_t\} \subset T$ and hence centralizes K'. Thus in this case we have $x_t = k_t s_t y_t \in K s_t K'(S' \cap S_\sigma)\, D J_1 \Gamma = K s_t (S' \cap S_\sigma)\, D J_1 \Gamma$ and $\beta(s_t) \to 0$. Since J_1 is a finite subset of $\mathbf{G}_\mathbb{Q}$, by condition b) of Theorem 5.2 this implies that $x_t \Gamma \to \infty$.

Finally, to prove assertion c) we proceed as follows. Let the notation be as in the proof of a). By assertion b) given the compact subset F of G/Γ there exists $\varepsilon > 0$ such that for $x \in G$, the condition $\|\rho(x)(v)\| < \varepsilon$ implies that $x\Gamma \notin F$. We note also that since $\rho(g_t)$ is a one-parameter subgroup of matrices if $W = \{w \mid \rho(g_t)(w) \to 0 \text{ as } t \to \infty\}$ then $\rho(g_t)(w) \to 0$ uniformly on compact subsets of W. Now let $x \in E$. Then $\rho(x)(v) \in W$ and hence there exists $T \geqq 0$ such that $\|\rho(g_t)(w)\| < \varepsilon$ for all $t \geqq T$ and w in a neighbourhood, say Ω', of $\rho(x)(v)$. Now assertion c) evidently holds for $\Omega = \{g \in G \mid \rho(g)(v) \in \Omega'\}$.

Proposition 5.6 and Theorem 4.7 motivate the following definition.

5.7. Definition. Let G and Γ be as before. Let $\{g_t\}$ be a one-parameter subgroup of G and let $g \in G$. If there exists an admissible chart (ρ, V, v) at ∞ for G/Γ such that $\rho(g_t g)(v) \to 0$ as $t \to \infty$ then $\{g_t g\Gamma \mid t \geqq 0\}$ is said to be a *degenerate divergent trajectory* of $\{g_t\}$ on G/Γ.

Proposition 5.6 shows that such trajectories are indeed divergent. Further it also shows that for checking degeneracy we may restrict to those charts at ∞ for which the corresponding projective isotropy subgroup is a maximal Γ-rational parabolic subgroup. If Γ is a lattice such that the rank of Γ equals the $\mathbb{R}$-rank of G (that is, $\mathbb{Q}$-rank $=$ $\mathbb{R}$-rank if Γ is arithmetic) then condition i) as in Definition 5.5 automatically implies condition ii) (to see this follow the proof of $HN \cap \Gamma$ is a lattice in HN, as before). Since the condition on rank holds for $SL(n, \mathbb{Z})$ as a lattice in $SL(n, \mathbb{R})$, this explains why Theorem 4.7 does not involve condition ii) as in Definition 5.5. More generally this shows the following.

5.8. Proposition. *Let the notation be as before and suppose that the rank of $\Gamma =$ the $\mathbb{R}$-rank of G. Let $\{g_t\}$ be a one parameter subgroup of G and $g \in G$. Then $\{g_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if there exists a representation ρ of G on a finite dimensional vector space V and a vector $v \in V - (0)$ such that $\{q \in G \mid \rho(q)(v) \in \mathbb{R}^* v\}$ is a maximal Γ-rational parabolic subgroup and $\rho(g_t g)(v) \to 0$ as $t \to \infty$.*

However, in general (that is, if $\mathbb{Q}$-rank of $\mathbf{G} <$ $\mathbb{R}$-rank of $\mathbf{G}$), there exist maximal $\mathbb{Q}$-parabolic subgroups which are not maximal as $\mathbb{R}$-parabolic subgroups. In that case existence of ρ, V and v as in Proposition 5.8 may not imply divergence of $\{g_t g\Gamma \mid t \geqq 0\}$. We will not go into the proof of this; it suffices to note that at the other extreme, that is, $\mathbb{Q}$-rank $= 0$, there are uniform lattices for which no trajectory is divergent. The intermediary cases are a combination of the two extremes.

5. 9. Proposition. *Let G and Γ be as before. Let $\{g_t\}$ be a one-parameter subgroup of G. Let P be a Γ-rational parabolic subgroup of G. Let N be the horospherical subgroup corresponding to g_1 and let R be the normalizer of N in G. Then there exist finitely many elements $\omega_1, \omega_2, \ldots, \omega_m$ such that for $g \in G$, $\{g_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if $g \in R\omega_i PJ\Gamma$ for some $i=1, 2, \ldots, m$, where J is the set of cusp elements as in Theorem 5. 2. For each $i=1, 2, \ldots, m$ there exists an admissible chart (ρ_i, V_i, v_i) at ∞ for G/Γ such that the corresponding projective isotropy subgroup is a maximal Γ-rational parabolic subgroup and $\rho_i(g_t g)(v_i) \to 0$ for all g in the closure of $R\omega_i P$. The set $X = \bigcup_{i=1}^{m} R\omega_i P$ is a proper closed subset of G. Further given an $x_0 \in X$ and a compact set F of G/Γ there exists a neighbourhood Ω of x_0 in X and a $T \geqq 0$ such that $g_t x\Gamma \notin F$ for any $x \in \Omega$ and $t \geqq T$.*

Proof. Let (ρ, V, v) be an admissible chart at ∞ for G/Γ and let Q be the corresponding projective isotropy subgroup. Suppose that Q contains P. Let

$$Y = \{g \in G \mid \rho(g_t g)(v) \to 0 \quad \text{as} \quad t \to \infty\}.$$

Since $P \subset Q$, Y is invariant under the right action of P on G; that is, $YP = Y$. We now show that it is also invariant under the left action of R on G, that is, $RY = Y$. Let $E = \{u \in V \mid \rho(g_t)(u) \to 0 \text{ as } t \to \infty\}$. Let $g_1 = \delta \cdot \nu$ be the Jordan decomposition of g_1; that is, $\delta, \nu \in G$ are two commuting elements such that $\operatorname{Ad}\delta$ is semisimple (diagonalizable over $\mathbb{C}$), $\operatorname{Ad}\nu$ is unipotent $\big((\operatorname{Ad}\nu - \mathrm{Id})^l = 0$ for some $l\big)$. Then $\rho(g_1) = \rho(\delta)\rho(\nu)$ is the Jordan decomposition of $\rho(g_1)$. Evidently E is the largest $\rho(g_1)$-invariant subspace of V such that all eigenvalues of $\rho(g_1)$ on E are of absolute value <1. Every $\rho(g_1)$-invariant subspace is $\rho(\delta)$-invariant and the sets of eigenvalues on the subspace are the same for $\rho(g_1)$ and $\rho(\delta)$. We conclude that $E = \{u \in V \mid \rho(\delta^i)(u) \to 0 \text{ as } i \to \infty\}$. The normalizer R of N is a parabolic subgroup and in view of the semisimplicity of $\operatorname{Ad}\delta$ for each $x \in R$ $\{\delta^i x \delta^{-i} \mid i = 1, 2, \ldots\}$ is bounded. The above characterization of E in terms of δ now implies that the set E is $\rho(R)$-invariant. Hence $RY = Y$. Thus Y is a union of double cosets of the form $R\omega P$, $\omega \in G$. Since P and R are parabolic subgroups, by Bruhat decomposition (cf. [3], Theorem 5. 15), there are only finitely many distinct double cosets of this form.

We conclude from this that there exist $\omega_1, \omega_2, \ldots, \omega_m$, $m \geqq 0$ such that for $g \in G$, $\rho(g_t g)(v) \to 0$ for some admissible chart (ρ, V, v) in which $\rho(P)$ leaves invariant the subspace spanned by v if and only if $g \in R\omega_i P$ for some $i = 1, 2, \ldots, m$. Since any Γ-rational parabolic subgroup contains $\gamma^{-1} j^{-1} P j \gamma$ for some $j \in J$ and $\gamma \in \Gamma$ (cf. condition e) of Theorem 5. 2) it follows that $\{g_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if $g \in R\omega_i PJ\Gamma$ for some $i = 1, \ldots, m$. By assertion a) in Proposition 5. 6 we can find for each i an admissible chart (ρ_i, V_i, v_i) satisfying the contention of the proposition.

Since for each $i = 1, 2, \ldots, m$ the corresponding set $Y_i = \{g \in G \mid \rho_i(g_t g)(v_i) \to 0\}$, which is a union of cosets of the form $R\omega P$, is closed it follows that $X = \bigcup_{i=1}^{m} R\omega_i P = \bigcup_{i+1}^{m} Y_i$ is closed. Assertion c) of Proposition 5. 6 implies that for any $x_0 \in Y_i$ the last assertion of the present proposition holds; since each Y_i is closed it follows for all $x_0 \in X$. The assertion that X is proper can be deduced either by observing that each Y_i as above is lower dimensional (as all eigenvalues of $\rho(g_t)$ on $\rho(G)$-invariant space cannot be of absolute value <1) or directly from Poincaré recurrence lemma.

5. 10. Remark. Let the notation be as in Proposition 5. 9. Let $C(\Gamma)$ be the commensurator of Γ (cf. § 1). If (ρ, V, v) is an admissible chart at ∞ for G/Γ and $\theta \in C(\Gamma)$ then clearly $(\rho, V, \rho(\theta)(v))$ is an admissible chart at ∞. It follows that for $g \in G$ $\{g_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if $\{g_t g\theta\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory for all $\theta \in C(\Gamma)$. Thus by Proposition 5. 9, $XJ\Gamma = XJC(\Gamma)$. If Γ is an arithmetic lattice then $C(\Gamma) = \mathbf{G}_{\mathbb{Q}}$ (cf. [2], Lemma 15. 11). Since J is contained in $\mathbf{G}_{\mathbb{Q}}$, $XJ\Gamma = X\mathbf{G}_{\mathbb{Q}}$.

5. 11. Remark. A part of the proof of Proposition 5. 9 also implies the following assertion. Let $\{x_t\}$ and $\{y_t\}$ be two one-parameter subgroups such that x_t, $t \in \mathbb{R}$ and y_t, $t \in \mathbb{R}$ commute with each other and let $g_t = x_t y_t$. Suppose that all eigenvalues of $\operatorname{Ad} y_t$ are of absolute value 1. Then for $g \in G$, $\{g_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if $\{x_t g\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory.

§ 6. Homogeneous spaces of rank 1

Let G be a connected linear semisimple Lie group and let Γ be an irreducible lattice of rank 1. We shall show that in this case all the divergent trajectories of flows are degenerate.

We follow the notations introduced in § 5. 1 as involved in the statement of Theorem 5. 2. Let $\mathfrak{G}$ be the Lie algebra of G and $\mathfrak{U}$ the subalgebra associated to U. Let $V = \bigwedge^l \mathfrak{G}$, the lth exterior power of $\mathfrak{G}$ as a vector space, where l is the dimension of U. Let $\rho: G \to GL(V)$ be the lth exterior power of the adjoint representation of G on $\mathfrak{G}$. Let $v \in V$ be a non-zero element of the line $\bigwedge^l \mathfrak{U}$ in V. Observe that in view of condition d) of Theorem 5. 2 and the fact that $j^{-1}Mj \cap \Gamma$ is a uniform lattice in $j^{-1}Mj$ (cf. § 5. 3) for all $j \in J$ and $\gamma \in \Gamma$, $(\rho, V, \rho(\gamma^{-1}j^{-1})\, v)$ is an admissible chart at ∞ for G/Γ.

6. 1. Theorem. *Let $\{g_t\}$ be a one-parameter subgroup of G and let $g \in G$. Then $g_t g\Gamma \to \infty$ if and only if $g \in XJ\Gamma$, where*

$$X = \{g \in G \mid \rho(g_t g)\, v \to 0 \quad \text{as} \quad t \to \infty\}.$$

In particular, all the divergent trajectories are degenerate.

Proof. If $g \in XJ\Gamma$, say $g = xj\gamma$, where $x \in X$, $j \in J$ and $\gamma \in \Gamma$ then

$$\rho(g_t g)\, \rho(\gamma^{-1}j^{-1})\,(v) = \rho(g_t x)\,(v) \to 0.$$

Since $(\rho, V, \rho(\gamma^{-1}j^{-1})\,(v))$ is an admissible chart at ∞, by Proposition 5. 6 it follows that $g_t g\Gamma \to \infty$ as $t \to \infty$.

Conversely suppose that $g_t g\Gamma \to \infty$ as $t \to \infty$. We shall show that $g \in XJ\Gamma$. Let P_0 be the subgroup $\{x \in P \mid \rho(x)\,(v) = \pm v\}$. Then P_0 contains MU as a subgroup of finite index. In view of condition d) in Theorem 5. 2 and the remark about M at the end of § 5. 3, for all $j \in J$, $j^{-1}P_0 j \cap \Gamma$ is a uniform lattice in $j^{-1}P_0 j$. Clearly P_0 is a normal subgroup of P and P/P_0 is isomorphic to $\mathbb{R}^+$. Since P does not admit any lattice (it is not unimodular) and since for any $j \in J$, $P_0 \cap j\Gamma j^{-1}$ is a lattice in P_0 the last assertion implies that $j\Gamma j^{-1} \cap P_0$ is contained in P_0.

Now for any $j \in J$, $\gamma \in \Gamma$ and $\sigma > 0$ put

$$\Omega_\sigma(j, \gamma) = K S_\sigma P_0 j \gamma.$$

Also for $j \in J$ let $\Gamma_j = j^{-1} P_0 j \cap \Gamma$. Since Γ_j is a uniform lattice in $j^{-1} P_0 j$ there exists a compact subset D of P_0 such that for all $j \in J$, $j^{-1} P_0 j \subset j^{-1} D j \Gamma_j$. Thus

$$P_0 j \gamma = j(j^{-1} P_0 j)\, \gamma \subset D j \Gamma_j \gamma.$$

Let $\sigma > 0$ be small enough so that condition c) of Theorem 5. 2 holds for the set D as above. Now suppose that $\Omega_\sigma(j, \gamma) \cap \Omega_\sigma(j', \gamma')$ is non-empty for some $j, j' \in J$ and $\gamma, \gamma' \in \Gamma$. Then $K S_\sigma D j \gamma_1 \cap K S_\sigma D j' \gamma_2$ is non-empty for some $\gamma_1 \in \Gamma_j \gamma$ and $\gamma_2 \in \Gamma_{j'} \gamma'$. By condition c) of Theorem 5. 2 we get that $j = j'$ and $j \gamma_2 \gamma_1^{-1} j^{-1} \in P$. Since $j \Gamma j^{-1} \cap P$ is contained in P_0, this means that $j \gamma' \gamma^{-1} j^{-1} \in P_0$. Thus $j' \gamma' = j \gamma' = p j \gamma$ for some $p \in P_0$ and consequently $\Omega_\sigma(j', \gamma') = \Omega_\sigma(j, \gamma)$. Thus the sets $\Omega_\sigma(j, \gamma)$, $j \in J$, $\gamma \in \Gamma$ are disjoint whenever they are distinct.

It is easy to see that for all $\sigma > 0$, $G - \bigcup \{\Omega_\sigma(j, \gamma) : j \in J,\ \gamma \in \Gamma\}$ is contained in a set of the form $F\Gamma$ where $F \subset G$ is compact. Since $g_t g \Gamma \to \infty$ it follows that for any $\sigma > 0$ there exists $T \geqq 0$ such that for all $t \geqq T$, $g_t g \in \bigcup \{\Omega_\sigma(j, \gamma);\ j \in J \text{ and } \gamma \in \Gamma\}$. Since $\{g_t g \mid t \geqq T\}$ is connected while for all sufficiently small σ the sets from the union are disjoint open sets, we conclude that there exists $j \in J$ and $\gamma \in \Gamma$ such that for all $t \geqq T$, $g_t g \in \Omega_\sigma(j, \gamma)$. It is also obvious that j and γ can be chosen to be the same for all sufficiently small σ. Therefore there exist $j \in J$ and $\gamma \in \Gamma$ such that for all sufficiently small $\sigma > 0$ there exists $T \geqq 0$ such that $g_t g \gamma^{-1} j^{-1} \in K S_\sigma P_0$.

Now let $\|\cdot\|$ be a $\rho(K)$-invariant norm on V such that $\|v\| = 1$. Recall that now S is a one-parameter subgroup and Δ (notation as before) consists of a unique element, say β. As in the proof of Proposition 5. 6 we see that if $\chi : S \to \mathbb{R}^+$ is the character defined by the relation $\rho(s)(v) = \chi(s)\, v$ for all $s \in S$ then there exists $m > 0$ such that $\chi(s) = \beta(s)^m$. Recall also that $\rho(y)(v) = \pm v$ for all $y \in P_0$. Hence if $x \in K S_\sigma P_0$, say $x = k s y$, where $k \in K$, $s \in S_\sigma$ and $y \in P_0$ then

$$\begin{aligned} \|\rho(x)(v)\| &= \|\rho(k)\, \rho(s)\, \rho(y)(v)\| = \|\rho(s)(v)\| \\ &= \chi(s) = \beta(s)^m < \sigma^m. \end{aligned}$$

Thus we have $\|\rho(g_t g \gamma^{-1} j^{-1})\, v\| < \sigma^m$ for all $t \geqq T$ with T as above. In other words, $\rho(g_t g \gamma^{-1} j^{-1})\, v \to 0$. Thus $g \gamma^{-1} j^{-1} \in X$ and hence $g \in X J \Gamma$.

6. 2. Corollary. *Let G be a connected simple Lie group of $\mathbb{R}$-rank 1 and Γ be a non-uniform lattice in G. Let P be a parabolic subgroup of G and K be a maximal compact subgroup of G. Let J be a sufficient set of cusp elements relative to (K, P). Let $\{g_t\}$ be a one-parameter subgroup such that the horospherical subgroup corresponding to g_1 is non-trivial and contained in P. Then*

$$\{x \in G/\Gamma \mid g x \to \infty \quad as \quad t \to \infty\} = P J \Gamma / \Gamma.$$

Proof. Since the $\mathbb{R}$-rank of G is 1, under the above hypothesis the unipotent radical of P is the horospherical subgroup corresponding to g_1. In the notation as above this implies that $\rho(g_t)(v) \to 0$ as $t \to \infty$ or equivalently that $P \subset X$. Since P is the normalizer of the horospherical subgroup corresponding to g_1, as in the proof of Proposition 5.9 we see that X is a union of double cosets of the form $P\omega P$ with $\omega \in G$. Since $\mathbb{R}$-rank of G is 1, by Bruhat decomposition (cf. [3], Theorem 5.15) apart from P itself there is only one other double coset of that form. Since X is proper the latter cannot be contained in X. Hence $X = P$ and hence by Theorem 6.1, $PJ\Gamma/\Gamma$ is precisely the set of points with divergent trajectories under $\{g_t\}$.

6.3. Corollary. *Let G be a connected simple Lie group of $\mathbb{R}$-rank 1 and Γ be a non-uniform lattice in G. Let U be a (non-trivial) horospherical subgroup of G and P be the normalizer of U. Let J be a sufficient set of cusp elements (relative to (K, P) where K is a maximal compact subgroup). Then*

i) *if $g \in PJ\Gamma$ then the U-orbit $Ug\Gamma/\Gamma$ is compact;*

ii) *if $g \notin PJ\Gamma$ then $Ug\Gamma/\Gamma$ is dense in G/Γ.*

Proof. Assertion i) follows from condition d) of Theorem 5.2. Assertion ii) follows from Theorem 1.6 and Corollary 6.2.

It may be observed that the classical result of G. A. Hedlund asserting that for the horocycle flow associated to a non-compact surface of constant negative curvature and finite area, every orbit is either dense or periodic (cf. [16] for a stronger result) follows as a particular case of Corollary 6.3 when we choose $G = SL(2, \mathbb{R})$.

§ 7. Non-degenerate divergent trajectories

As before let G be a connected linear semisimple Lie group and let Γ be an irreducible lattice in G. In this section we shall show that when the rank of Γ is $\geqq 2$ then there exist non-degenerate divergent trajectories on G/Γ (cf. Theorem 7.3 and Corollary 7.4 below).

Let $\{g_t\}$ be a one-parameter subgroup of G. Let X be a subset of G such that for all $x \in X$, $g_t x\Gamma \to \infty$. We say that the divergence is *locally uniform* over X (or, $g_t x\Gamma \to \infty$ locally uniformly for $x \in X$) if for any compact subset D of G/Γ and $x_0 \in X$ there exist a neighbourhood Ω of x_0 in X and $T \geqq 0$ such that $g_t x\Gamma \in G/\Gamma - D$ for all $t \geqq T$ and $x \in \Omega$.

Recall that, by Proposition 5.9, the set of points on degenerate divergent trajectories can be expressed as $XJ\Gamma$ (where X is the finite union of the cosets $R\omega_i P$, $i = 1, 2, \ldots, m$) such that X is closed and the divergence is locally uniform over X. To conclude the existence of non-degenerate trajectories we first prove the following result. The proof is motivated by the proof of Theorem 14 of [7].

Two closed subsets A and B of G are said to be *transversal* to each other if $A \cap B$ is a nowhere dense subset of both A and B in their respective subspace topologies.

7.1. Theorem. *Let G and Γ be as above. Let $C(\Gamma)$ be the commensurator of Γ (cf. § 1). Let $\{g_t\}$ be a one-parameter subgroup of G. Let Y be a closed subset of G such that $g_t x\Gamma \to \infty$ locally uniformly for $x \in Y$. Suppose that there exist closed subsets $E_1, E_2, \ldots, E_l$ of Y, where $l \geqq 1$, such that the following conditions are satisfied:*

a) $Y = E_1 \cup E_2 \cup \cdots \cup E_l$.

b) *For any p and q between 1 and l and $\theta \in C(\Gamma)$, E_p and $E_q\theta$ are transversal to each other unless $E_p = E_q\theta$.*

c) *For any p, $1 \leqq p \leqq l$, the subset $\{x \in E_p \mid x \in E_q\theta$ for some θ such that $E_q\theta \neq E_p\}$ is dense in E_p.*

Then there exists $z \in G - YC(\Gamma)$ such that $g_t z\Gamma \to \infty$ as $t \to \infty$.

Proof. Let $N_1, N_2, \ldots$ be open subsets of G/Γ such that $G/\Gamma - N_i$ is compact, $\bar{N}_{i+1} \subset N_i$ for all i and $\bigcap N_i = \emptyset$; in other words, $\{N_i\}$ is a fundamental system of neighbourhoods of ∞. We note that $C(\Gamma)$ is a countable group. We well-order $C(\Gamma)$ such that each element has only finitely many predecessors and the identity is the least element (this is equivalent to fixing an enumeration starting with the identity). We shall find a sequence $\{\Omega_i\}$ of bounded open subsets of G/Γ, a sequence $\{\theta_i\}$ in $C(\Gamma)$ and a monotonically increasing divergent sequence $\{T_i\}$ in $\mathbb{R}^+$ such that the following conditions are satisfied for all $i \geqq 1$:

i) $\bar{\Omega}_i \subset \Omega_{i-1}$.

ii) $g_t x\Gamma \in N_{i-1}$ for all $x \in \Omega_i$ and t such that $T_{i-1} \leqq t \leqq T_i$ ($N_0 = G/\Gamma$ by convention).

iii) There exists (for each i) a unique p, $1 \leqq p \leqq l$ such that $\Omega_i \cap E_p\theta_i$ is non-empty and for this p, $g_t x\Gamma \in N_i$ for all $x \in \Omega_i \cap E_p\theta_i$ and $t \geqq T_i$.

iv) $\Omega_i \cap Y\theta = \emptyset$ if $\theta < \theta_i$.

We first observe that finding this would imply the theorem. Let $z \cap \Omega_i$, which is indeed non-empty in view of condition i). Since $N_{i+1} \subset N_i$ for all i, condition ii) implies that $g_t z\Gamma \in N_i$ for all $t \geqq T_i$, which means that $g_t z\Gamma \to \infty$. In view of conditions iii) and iv), $\{\theta_i\}$ is an infinite sequence of distinct elements. Hence by condition iv), $z \notin YC(\Gamma)$.

For each p, $1 \leqq p \leqq l$ we put $E'_p = E_p - \bigcup_{q \neq p} E_q$. In view of condition b) in the hypothesis E'_p is a (non-empty) open dense subset of E_p.

Let p, $1 \leqq p \leqq l$ be arbitrary and let $y \in E'_p$. There exists a bounded neighbourhood Ω'_1 of y such that $\Omega'_1 \cap Y \subset E'_p$. Also since $g_t x\Gamma \to \infty$ locally uniformly for $x \in Y$ we can find a $T_1 \geqq 0$ and a neighbourhood Ω_1 of y such that $\Omega_1 \subset \Omega'_1$ and $g_t x\Gamma \in N_1$ for all $x \in \Omega_1 \cap E_p$ and $t \geqq T_1$. Choosing Ω_0 to be any bounded open set such that $\bar{\Omega}_1 \subset \Omega_0$, $\theta_1 =$ identity and $T_0 = 0$ we see that conditions i) through iv) are satisfied for $i = 1$.

We now proceed by induction. Suppose that open sets $\Omega_1, \ldots, \Omega_k$, elements $\theta_1, \ldots, \theta_k \in C(\Gamma)$ and $0 \leqq T_1 \leqq T_2 \leqq \cdots \leqq T_k$ have been chosen so that conditions i) through iv) are satisfied for $i = 1, \ldots, k$. We now find Ω_{k+1}, θ_{k+1} and $T_{k+1} \geqq T_k$ such that the conditions are satisfied for $i = k+1$.

By condition iii) for $i = k$ there exists a unique p, $1 \leqq p \leqq l$ such that $\Omega_k \cap E_p\theta_k \neq \emptyset$ and $g_t x\Gamma \in N_k$ for all $x \in \Omega_k \cap E_p\theta_k$ and $t \geqq T_k$. By condition c) of the theorem there exists $\theta \in C(\Gamma)$ such that for some j, $\Omega_k \cap E'_p\theta_k \cap E_j\theta$ is non-empty but $E_p\theta_k \neq E_j\theta$. Let θ_{k+1} be the least possible θ for which the last condition is satisfied and let q, $1 \leqq q \leqq l$ be such that $\Omega_k \cap E'_p\theta_k \cap E_q\theta_{k+1} \neq \emptyset$. This also implies that $\Omega_k \cap E'_p\theta_k \cap E'_q\theta_{k+1} \neq \emptyset$; let y be an element of this set. It is easy to see that since $g_t x\Gamma \to \infty$ locally uniformly for

$x \in Y$ and $\theta_{k+1} \in C(\Gamma)$, $g_t x\Gamma \to \infty$ locally uniformly for $x \in Y\theta_{k+1}$ (cf. proof of Proposition 1. 5 for an idea of the proof). Therefore we can find a $T_{k+1} \geqq T_k$ and a neighbourhood Ω'_{k+1} of y such that $g_t x\Gamma \in N_{k+1}$ for all $x \in \Omega'_{k+1} \cap E_q\theta_{k+1}$ and $t \geqq T_{k+1}$. Since $y \in \Omega_k \cap E_p\theta_k$ by condition iii) for $i=k$ we have $g_t y\Gamma \in N_k$ for all $t \geqq T_k$ and in particular in the interval $T_k \leqq t \leqq T_{k+1}$. By continuity there exists a neighbourhood Ω''_{k+1} of y, which we choose to be such that $\bar{\Omega}''_{k+1} \subset \Omega'_{k+1} \cap \Omega_k$, such that the following holds: $g_t x\Gamma \in N_k$ for all $x \in \Omega''_{k+1}$ and t such that $T_k \leqq t \leqq T_{k+1}$. Now put

$$(7.\ 2) \qquad \Omega_{k+1} = \Omega''_{k+1} - \Big(\bigcup_{\theta < \theta_{k+1}} Y\theta\Big) - \bigcup_{j \neq q} E_j\theta_{k+1}.$$

Having chosen Ω_{k+1}, θ_{k+1} and T_{k+1} as above, let us verify conditions i) through iv) for $i=k+1$. Condition i) is evident since $\bar{\Omega}_{k+1} \subset \bar{\Omega}''_{k+1} \subset \Omega_k$. Condition ii) holds in view of the choice of Ω''_{k+1} and the fact that $\Omega_{k+1} \subset \Omega''_{k+1}$. Next we show that $\Omega_{k+1} \cap E_q\theta_{k+1}$ is non-empty. In view of (7. 2) it is enough to show that $\Omega''_{k+1} \cap E_j\theta \cap E_q\theta_{k+1}$, where $1 \leqq j \leqq l$ and $\theta \leqq \theta_{k+1}$, is a nowhere dense subset of $\Omega''_{k+1} \cap E_q\theta_{k+1}$ whenever either $\theta < \theta_{k+1}$ or $\theta = \theta_{k+1}$ but $j \neq q$. In view of condition b) in the hypothesis $E_j\theta \cap E_q\theta_{k+1}$ is nowhere dense in $E_q\theta_{k+1}$ unless $E_j\theta = E_q\theta_{k+1}$. The choice θ_{k+1} shows that the latter holds only if $\theta = \theta_{k+1}$ which then implies that $E_j = E_q$. Hence $\Omega_{k+1} \cap E_q\theta_{k+1}$ is non-empty. It is evident that q is the only element between 1 and l for which this holds. Further, since $\Omega_{k+1} \subset \Omega'_{k+1}$ we see that $g_t x\Gamma \in N_{k+1}$ for all $x \in \Omega_{k+1} \cap E_q\theta_{k+1}$ and $t \geqq T_{k+1}$. Thus we have verified condition iii) for $i=k+1$. Condition iv) is evident from (7. 2).

7. 3. Theorem. *Let G be a connected linear semisimple Lie group and Γ be an irreducible lattice in G such that rank of $\Gamma = \mathbb{R}$-rank of $G \geqq 2$. Let $\{g_t\}$ be any one-parameter subgroup of G such that $\mathrm{Ad}\, g_1$ has an eigenvalue of absolute value other than 1. Then $\{g_t\}$ has non-degenerate divergent trajectories on G/Γ; that is, there exists $g \in G$ such that $\{g_t g\Gamma \mid t \geqq 0\}$ is a non-degenerate divergent trajectory.*

Proof. Recall that a lattice as above is arithmetic: there exists an algebraic group $\mathbf{G}$ such that $G = \mathbf{G}^\circ_{\mathbb{R}}$ and Γ is commensurable with $\mathbf{G}_{\mathbb{Z}}$. Further in view of the hypothesis $\mathbb{Q}$-rank of $\mathbf{G} = \mathbb{R}$-rank of $\mathbf{G} \geqq 2$. Let $\mathbf{S}$ be a maximal $\mathbb{Q}$-split torus; then $\mathbf{S}$ is also a maximal $\mathbb{R}$-split torus in view of the condition on the rank. Let $S = \mathbf{S}^\circ_{\mathbb{R}}$. We first show that for any (non-trivial) one-parameter subgroup $\{s_t\}$ of S there exist non-degenerate divergent trajectories. Let Φ be the root system for the adjoint action of $\mathbf{S}$ (cf. § 5. 1). There exists an order Φ such that, denoting by Φ^+ the corresponding set of positive roots, we have $\lambda(s_t) \leqq 1$ for all $\lambda \in \Phi^+$ and $t > 0$ (cf. [5], VI). Let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots corresponding to the order. Then any $\lambda \in \Phi^+$ is of the form $\lambda(s) = \prod_{i=1}^{r} \alpha_i(s)^{m_i}$ for some integers $m_1, \dots, m_r \geqq 0$. Since Γ is an irreducible lattice in G it follows that $\mathbf{G}$ is almost simple as a $\mathbb{Q}$-group. Consequently, the root system is irreducible: thus Δ cannot be expressed as $\Delta_1 \cup \Delta_2$, where Δ_1 and Δ_2 are two disjoint non-empty subsets such that $\Phi^+ \subset \langle \Delta_1 \rangle \cup \langle \Delta_2 \rangle$. (Recall that $\langle \Sigma \rangle$ for $\Sigma \subset \Delta$ denotes the set of elements of Φ contained in the subgroup, of the character group, generated by Σ.)

Let $\mathbf{P}^\alpha$, $\alpha \in \Delta$ denote the maximal standard parabolic subgroup $\mathbf{P}_{\Delta - \{\alpha\}}$ corresponding to the order as above (cf. § 5). Let $P^\alpha = \mathbf{P}^\alpha \cap G$. Let Q be the normalizer of the horospherical subgroup corresponding to s_1. Then $Q = \mathbf{Q}_{\mathbb{R}} \cap G$ where $\mathbf{Q}$ is the standard parabolic subgroup corresponding to the set $\Sigma = \{\alpha \in \Delta \mid \alpha(s_1) = 1\}$. Let X be the set as in Proposition 5. 9. For each $\alpha \in \Delta$ let l be the dimension of U^α, the unipotent radical of P^α. Let $V_\alpha = \bigwedge^{l_\alpha} \mathfrak{G}$ and ρ_α be the l_αth exterior power of the adjoint representation of G. Let v_α be a non-zero vector in the one-dimensional subspace corresponding to the Lie subalgebra of U^α. It is easy to see that for $s \in S$, $\rho_\alpha(s)(v_\alpha) = \chi_\alpha(s)\, v_\alpha$ and that the subspace spanned by v_α is $\rho_\alpha(P^\alpha)$-invariant. If $\alpha \in \Delta$ is such that $\chi_\alpha(s_t) < 1$ for all $t > 0$, then $\chi_\alpha(s_t) \to 0$ as $t \to \infty$ and consequently P^α is contained in X. We show that there are at least two simple roots in Δ for which the condition holds. Since $\{s_t\}$ is nontrivial, there exists $\alpha \in \Delta$ such that $\alpha(s_t) < 1$ for all $t > 0$. Since the root system is irreducible there exists $\beta \in \Delta - \{\alpha\}$ and $\theta \in \Phi^+$ such that $\theta \notin \langle \Delta - \{\alpha\} \rangle \cup \langle \Delta - \{\beta\} \rangle$ (that is, the expression for θ involves both α and β). Since $\lambda(s_t) \leqq 1$ for all $\lambda \in \Phi^+$ and $t > 0$, we get that $\chi_\beta(s_t) \leqq \theta(s_t) \leqq \alpha(s_t) < 1$ for all $t > 0$. Thus X contains both P^α and P^β. Since P^α and P^β generated G, this implies that χ is not contained in any proper subgroup of G.

Recall that X can be expressed as a finite disjoint union of double cosets of the form $Q\omega_i P$, where $\omega_1, \omega_2, \ldots, \omega_m \in G$. For each $i = 1, 2, \ldots, m$ let E_i be the closure of $Q\omega_i P$. Consider the class of subsets I of $\{1, 2, \ldots, m\}$ with the property that X is contained in $\left(\bigcup_{i \in I} E_i\right) \mathbf{G}_Q$. By reindexing if necessary, we may assume $\{1, 2, \ldots, l\}$ to be a minimal element of this class (under inclusion relation).

Let $E_1, E_2, \ldots, E_l$ be as above and let $Y = E_1 \cup E_2 \cup \cdots \cup E_l$. We claim that the conditions of Theorem 7. 1 are satisfied. Condition a) is obvious. Since each E_p is the closure of $Q\omega_p P$, the Zariski closure $\mathbf{E}_p$ of E_p in $\mathbf{G}$ is an irreducible algebraic variety defined over $\mathbb{R}$ and $E_p = \mathbf{E}_p \cap G$ (cf. [4], § 3). Therefore for any $g \in G$, and $1 \leqq p, q \leqq l$, $E_p \cap E_q g$ is either E_p or a lower dimensional subset of E_p. Hence $E_p \cap E_q g$ has no interior point in E_p unless $E_p \subset E_q g$. Recall that $C(\Gamma) = \mathbf{G}_Q$ (cf. [2], § 15). Since by choice of $E_1, E_2, \ldots, E_l$, E_p cannot be contained in $E_q g$ for any $g \in \mathbf{G}_Q$ unless $p = q$, we conclude that for $p \neq q$, $E_p \cap E_q g$ is nowhere dense in E_p. By symmetry we get that E_p is transversal to $E_q g$. This proves condition b) of Theorem 7. 1.

To verify condition c) we proceed as follows. Since $\mathbb{Q}$-rank of $\mathbf{G} = \mathbb{R}$-rank of $\mathbf{G}$ and since $\mathbf{Q}$ and $\mathbf{P}$ are parabolic subgroups defined over $\mathbb{Q}$ each of the double cosets $Q\omega P$, $\omega \in G$ has a rational representative (cf. [3], § 5) and consequently $Q\omega P \cap \mathbf{G}_Q$ is dense in $Q\omega P$ for all $\omega \in G$. Since each E_p is a union of such double cosets $E_p \cap \mathbf{G}_Q$ is dense in E_p for all $p = 1, 2, \ldots, l$. Now suppose first that $l \geqq 2$ and let $1 \leqq p \leqq l$ be given. Let q, $1 \leqq q \leqq l$, be other than p. Let $\theta_0 \in E_q \cap \mathbf{G}_Q$. Then for any $\theta \in E_p \cap \mathbf{G}_Q$ we have $\theta = \theta_0 \cdot (\theta_0^{-1}\theta) \in E_q(\theta_0^{-1}\theta)$. Since $E_q(\theta_0^{-1}\theta)$ is proved to be transversal to E_p this shows that the set defined in the statement of condition c) contains $E_p \cap \mathbf{G}_Q$ and is therefore dense in E_p by our earlier observation. Now suppose that $l = 1$. Let H be the closed subgroup $\{h \in G \mid E_1 h = E_1\}$. Recall that as the closure of a double coset $Q\omega_1 P$, E_1 contains the identity (cf. [4], Corollary 3. 15). Hence $H \subset E_1$. Suppose that H is properly contained in E_1. Clearly H contains P and is therefore itself a finite union of double cosets of the form $P\omega P$. An argument as before therefore implies that H is nowhere dense in E_1. Hence $(E_1 - H) \cap \mathbf{G}_Q$ is dense in E_1. For any $\theta \in (E_1 - H) \cap \mathbf{G}_Q$, $E_1\theta$ is transversal to E_1 and $\theta \in E_1\theta \cap E_1$. Thus the set as in condition c) contains $(E_1 - H) \cap \mathbf{G}_Q$ and is therefore dense in E_1.

Finally, suppose (if possible) that $l = 1$ and $E_1 = H$; in particular, E_1 is a closed subgroup. Recall that $X \subset E_1 \mathbf{G}_Q = H\mathbf{G}_Q$. Since $H\mathbf{G}_Q$ is a countable disjoint union of cosets of H, this is impossible unless $X = H$. But recall that X is a proper subset not contained in any proper subgroup of G. Hence $X = H$. The contradiction shows that the last case does not arise. Thus condition c) is completely verified.

Having verified the conditions we now conclude from Theorem 7. 1 that there exists $z \in G - YC(\Gamma) = G - Y\mathbf{G}_{\mathbb{Q}} = G - X\mathbf{G}_{\mathbb{Q}}$, such that $s_t z\Gamma \to \infty$. On the other hand by Proposition 5. 9 all the points on degenerate divergent trajectories belong to $XJ\Gamma \subset X\mathbf{G}_{\mathbb{Q}}$. (Recall that $J \subset \mathbf{G}_Q$ for the case at hand.) Hence $\{s_t z\Gamma \mid t \geqq 0\}$ is a non-degenerate divergent trajectory.

Now let $\{g_t\}$ be any one-parameter subgroup of G. Let $g_t = b_t u_t$ be the Jordan decomposition of g_t, where b_t, $t > 0$ are semisimple elements and u_t, $t > 0$ are unipotent elements commuting with b_t. Let $\mathbf{T}$ be a torus defined over $\mathbb{R}$ containing $\{b_t\}$ and contained in the centralizer of $\{u_t\}$. Using the decomposition of $\mathbf{T}$ into split and anisotropic components (cf. [1], p. 219) we can write b_t as $c_t d_t$, where $\{d_t\}$ is contained in a split torus, $\{c_t\}$ is contained in a compact subgroup of G and the elements d_t, c_t and u_t, $t \geqq 0$ commute with each other. We note that since $\operatorname{Ad} g_1$ has an eigenvalue of absolute value other than 1, $\{d_t\}$ is non-trivial. Since $\mathbf{S}$ as above is also a maximal $\mathbb{R}$-split torus there exists $g \in G$ such that $g d_t g^{-1} \in S$ for all t. Replacing $\{g_t\}$ by $\{g g_t g^{-1}\}$ if necessary, we may assume that $\{d_t\}$ is contained in S. We write s_t for d_t and use the earlier notation. Recall that by Remark 5. 11, for $x \in G$, $\{g_t x\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory if and only if $\{s_t x\Gamma \mid t \geqq 0\}$ is a degenerate divergent trajectory; viz. if and only if $x \in X\mathbf{G}_{\mathbb{Q}}$ in the notation as above. Since the set X together with $E_1, E_2, \ldots, E_l$ as above satisfy the conditions of Theorem 7. 1, applying the theorem to the one-parameter subgroup $\{g_t\}$ we conclude there exists $z \in G - X\mathbf{G}_{\mathbb{Q}}$ such that $g_t z\Gamma \to \infty$. Thus $\{g_t z\Gamma \mid t \geqq 0\}$ is a non-degenerate divergent trajectory of $\{g_t\}$.

7. 4. Corollary. *Let $\{g_t\}$ be a one-parameter subgroup of $SL(n, \mathbb{R})$, $n \geqq 3$ such that g_t has an eigenvalue of absolute value other than* 1. *Then there exists $x \in SL(n, \mathbb{R})/SL(n, \mathbb{Z})$ such that $\{g_t x \mid t \geqq 0\}$ is a non-degenerate divergent trajectory.*

Proof. Rank of $SL(n, \mathbb{Z}) = \mathbb{R}$-rank of $SL(n, \mathbb{R}) = n - 1 \geqq 2$. Further $\operatorname{Ad} g_1$ has an eigenvalue of absolute value other than 1 since g_1 has. Hence the corollary follows from Theorem 7. 3.

Appendix. Orbits of horospherical flows

We now give a proof of Theorem 1. 6. As stated earlier it is motivated by certain ideas of D. S. Ornstein and M. Ratner. We begin by recalling the statement of the theorem.

1. 6. Theorem. *Let G be a connected Lie group and Γ be a lattice in G. Let $\{g_t\}$ be a one-parameter subgroup of G such that* $\operatorname{Ad} g_t$, $t \in \mathbb{R}$ *are diagonalizable over $\mathbb{C}$. Let U be the horospherical subgroup corresponding to g_1* (*or any g_t, $t \geqq 0$*). *Suppose that the U-action on G/Γ is ergodic. Let $g \in G$ be such that $\{g_t g\Gamma \mid t \geqq 0\}$ is not a divergent trajectory. Then $Ug\Gamma/\Gamma$ is dense in G/Γ.*

Proof. Let $\mathfrak{G}$ be the Lie algebra of G. Let $\mathfrak{G}^+$, $\mathfrak{G}^\circ$ and $\mathfrak{G}^-$ respectively be the largest $\operatorname{Ad} g_1$-invariant subspace on which all eigenvalues of $\operatorname{Ad} g_1$ are of absolute value <1, $=1$ and >1 respectively. It is well-known that $\mathfrak{G}^+$, $\mathfrak{G}^\circ$ and $\mathfrak{G}^-$ are Lie subalgebras of $\mathfrak{G}$ and that $\mathfrak{G}^+$ is precisely the Lie subalgebra corresponding to the horospherical subgroup U (cf. [9], § 1, for instance). We denote Z and U^- the analytic (connected Lie) subgroups corresponding to $\mathfrak{G}^\circ$ and $\mathfrak{G}^-$ respectively.

Since $\mathrm{Ad}\, g_t$, $t \in \mathbb{R}$ are diagonalizable over $\mathbb{C}$ and all eigenvalues of $\mathrm{Ad}\, g_t$ on $\mathfrak{G}^\circ$ are of absolute value 1 it follows (cf. [9] for some details) that there exists a norm $\| \ \|$ on $\mathfrak{G}^\circ$ which is $\mathrm{Ad}\, g_t$-invariant; that is

$$\text{(A. 1)} \qquad \|(\mathrm{Ad}\, g_t)(\xi)\| = \|\xi\| \quad \text{for all} \quad \xi \in \mathfrak{G}^\circ \quad \text{and} \quad t \in \mathbb{R}.$$

Similarly the conditions on eigenvalues on $\mathfrak{G}^+$ and $\mathfrak{G}^-$ imply that there exist norms, which also we denote by $\| \ \|$, on $\mathfrak{G}^+$ and $\mathfrak{G}^-$ such that

$$\text{(A. 2)} \qquad \|(\mathrm{Ad}\, g_t)(\xi)\| < C e^{-\mu t} \|\xi\|$$

if either $\xi \in \mathfrak{G}^+$ and $t \geqq 0$ or $\xi \in \mathfrak{G}^-$ and $t \leqq 0$, where C and $\mu > 0$ are suitable constants. We equip $\mathfrak{G}^+$, $\mathfrak{G}^\circ$ and $\mathfrak{G}^-$ with the norms as above.

The subgroups U and U^- are simply connected nilpotent subgroups and the exponential maps $\exp: \mathfrak{G}^+ \to U$ and $\exp: \mathfrak{G}^- \to U^-$ are analytic isomorphisms. Also there exists a neighbourhood Σ_0 of 0 in $\mathfrak{G}^\circ$ such that the restriction of the exponential map to Σ_0 is an analytic isomorphism onto a neighbourhood of the identity in Z. For $T > 0$, we denote by U_T and U_T^- the images under the respective exponential map of the sets $\{\xi \in \mathfrak{G}^+ \mid \|\xi\| < T\}$ and $\{\xi \in \mathfrak{G}^- \mid \|\xi\| < T\}$ respectively. Similarly if the set

$$\{\xi \in \mathfrak{G}^\circ \mid \|\xi\| < \varepsilon\},$$

where $\varepsilon > 0$, is contained in Σ_0 then we denote its image under the exponential map by Z_ε. Since for any $\xi \in \mathfrak{G}$ $g_t(\exp \xi)\, g_{-t} = \exp(\mathrm{Ad}\, g_t)(\xi)$ conditions (A. 1) and (A. 2) imply the following:

A. 3. Lemma. *For all $t > 0$ we have*

i) $g_t Z_\varepsilon g_{-t} = Z_\varepsilon$, *whenever Z_ε is defined.*

ii) $g_t U_T g_{-t} \subset U_{T'}$ *where* $T' = C T e^{-\mu t}$.

iii) $g_{-t} U_T^- g_t \subset U_{T'}^-$ *where* $T' = C T e^{-\mu t}$.

In the sequel we also need the following.

A. 4. Lemma. *Let $\delta > 0$ and $y \in G/\Gamma$ be given. Then there exist an $\varepsilon > 0$ and a neighbourhood Σ of y such that the following holds: for any $y_1, y_2 \in \Sigma$ and $u_1 \in U_\varepsilon$ there exist $u_2 \in U$ and $p \in U_\delta^- Z_\delta$ such that $u_2 y_2 = p u_1 y_1$.*

Proof. Since Γ is a discrete subgroup and since $\mathfrak{G}$ is the direct sum of $\mathfrak{G}^+$, $\mathfrak{G}^\circ$ and $\mathfrak{G}^-$ it follows that for a sufficiently small $\rho > 0$ the map $\psi: U_\rho^- \times Z_\rho \times U_\rho \to G/\Gamma$ defined by $\psi(u^-, z, u) = u^- z u y$ for all $u^- \in U_\rho^-$, $z \in Z_\rho$ and $u \in U_\rho$ is a diffeomorphism. It is straightforward to deduce the Lemma from this. We omit the details.

We are now ready to prove the theorem. Let $g \in G$ be such that the trajectory $\{g_t g \Gamma \mid t \geqq 0\}$ is not divergent. Then there exists a compact subset K of G/Γ such that $\{t \geqq 0 \mid g_t g \Gamma \in K\}$ is not bounded. Let Ω be any non-empty open subset of G/Γ. We shall show that $\Omega \cap U g \Gamma/\Gamma$ is non-empty. Let $\delta > 0$ be such that $Z_\delta U_\delta^- (G/\Gamma - \Omega)$ is not dense in Ω and let C_δ be an open subset of the complement of $Z_\delta U_\delta^- (G/\Gamma - \Omega)$ in G/Γ.

Since K is compact using Lemma A. 4 we can find finitely many open subsets $\Sigma_1, \Sigma_2, \ldots, \Sigma_l$ of G/Γ and an $\varepsilon > 0$ such that K is covered by $\Sigma_1, \ldots, \Sigma_l$ and the condition of Lemma A. 4 is satisfied for each Σ_i (together with ε) for the choice of δ as above. Let $\alpha = \min_{1 \leqq i \leqq l} m(\Sigma_i)$, where m is the G-invariant probability measure on G/Γ. Let $T \geqq 0$ be such that $m(U_T C_\delta) > 1 - \alpha$. We note that since $m(C_\delta) > 0$ and the action of U is ergodic, such a T exists.

Because of our choice of K, there exists $t \geqq \mu^{-1} \log CT/\varepsilon$ such that $g_t g \Gamma \in K$. Let $1 \leqq i \leqq l$ be such that $g_t g \Gamma \in \Sigma_i$. Thus $g\Gamma \in g_{-t} \Sigma_i$. Since $m(g_{-t}\Sigma_i) = m(\Sigma_i) \geqq \alpha$ and $m(U_T C_\delta) > 1 - \alpha$, $g_{-t}\Sigma_i \cap U_T C_\delta$ is non-empty. Let $y \in g_{-t}\Sigma_i \cap U_T C_\delta$. Since $y \in U_T C_\delta$ there exists $u' \in U_T$ such that $u' y \in C_\delta$. Put $u_1 = g_t u' g_{-t}$. Then by Lemma A. 3 and the relation $t \geqq \mu^{-1} \log CT/\varepsilon$ we get $u_1 \in U_\varepsilon$. Since $g_t g \Gamma$ and $g_t y$ belong to Σ_i by Lemma A. 4 (and the choice of Σ_i) there exist $u_2 \in U$ and $p \in U_\delta^- Z_\delta$ such that

$$u_2 g_t g \Gamma = p u_1 g_t y = p g_t u' y \in p g_t C_\delta.$$

Hence by Lemma A. 3 $(g_{-t} u_2 g_t)(g\Gamma) \in (g_{-t} p g_t) C_\delta \subset g_{-t}(U_\delta^- Z_\delta) g_t C_\delta \subset U_\delta^- Z_\delta C_\delta$. Our choice of C_δ ensures that $U_\delta^- Z_\delta C_\delta \subset \Omega$. Hence $(g_{-t} u_2 g_t)(g\Gamma) \in \Omega$. Since $g_{-t} u_2 g_t \in U$ this implies that $\Omega \cap Ug\Gamma/\Gamma$ is non-empty, thus proving the theorem.

We would like to record here that in a recent preprint entitled "Orbits of horospherical flows", by a different method the author is able to conclude the density of orbits of horospherical flows under a weaker condition than in Theorem 1. 6. The result is used, together with certain other ideas, to deduce that the closure of any orbit of a horospherical subgroup U coincides with an orbit of a closed subgroup H of G containing U and admits a finite measure invariant under the action of H; this generalizes Corollary 6. 3 to the case of groups of $\mathbb{R}$-rank $\geqq 2$.

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Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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