§ Singular pairs / vectors / matrices:

- Definition: An $m \times n$ matrix A is <u>singular</u> if $\forall \epsilon > 0$, $\exists Q_\epsilon$ s.t. $\forall Q \geq Q_\epsilon$, $\exists \vec{p} \in \mathbb{Z}^m$, $\vec{q} \in \mathbb{Z}^n \setminus \{\vec{0}\}$ s.t.

$$\| A\vec{q} - \vec{p} \| \leq \epsilon Q^{-n/m}, \quad \text{and} \quad \|\vec{q}\| \leq Q.$$

(for $m=1$ these are singular <u>vectors</u>; for $n=2$ additionally there are singular <u>pairs</u>.)

Notation: $\mathrm{Sing}(m,n)$ = singular $m \times n$ matrices.

- Fact: there are no (irrational) singular <u>numbers</u> (i.e. $m=n=1$). If $a \in \mathbb{R} \setminus \mathbb{Q}$, then a has an infinite continued fraction expansion

$$a = i_0 + \cfrac{1}{i_1 + \cfrac{1}{i_2 + \cfrac{1}{i_3 + \cdots}}}$$

(assuming $0 < a < 1$, $i_0 = 0$) If we truncate this expansion at level k, we obtain the <u>convergents</u>

$$\cfrac{1}{i_1 + \cfrac{1}{i_2 + \cfrac{1}{\ddots + \cfrac{1}{i_k}}}} =: \frac{p_k}{q_k}.$$

- These satisfy the inequalities $|qa - p| \geq |q_k a - p_k|$ for all $p, q \in \mathbb{Z}$ with $0 < q < q_{k+1}$, and $|q_k a - p_k| > \frac{1}{2 q_{k+1}}$.
- Hence $|qa - p| > \frac{1}{2q_{k+1}}$ for all p, q with $0 < q < q_{k+1}$, which contradicts the definition above of a singular number.

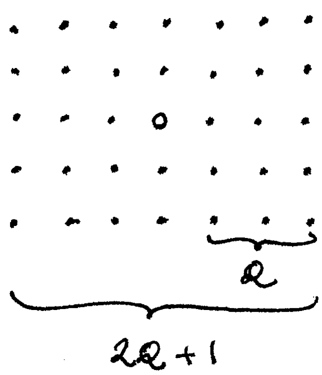
- Fact: The set of singular pairs has zero Lebesgue measure.

  - For a fixed a_2, the measure of the set of a_1 satisfying

$$|q_1 a_1 + q_2 a_2 - p| \leq \epsilon Q^{-2}$$

for a fixed $q_1, q_2, p \in \mathbb{Z}$ (with $q_1 \neq 0$) is $\leq 2\epsilon Q^{-2}$.

  - The number of vectors $(q_1, q_2) \in \mathbb{Z}^2 \setminus \{0\}$ with $\|\vec{q}\| \leq Q$ is $\leq (2Q+1)^2 - 1 < (3Q)^2$.



  - Then for a given Q, the measure of

$$A_{Q,\epsilon} = \{ a_1, a_2 : |q_1 a_1 + q_2 a_2 - p| \leq \epsilon Q^{-2} \text{ for some } p \in \mathbb{Z},\ \vec{q} = (q_1, q_2) \in \mathbb{Z}^2 \text{ with } \|\vec{q}\| \leq Q \}$$

is $\leq 2 \cdot 3^2 \epsilon$. Since the set of singular pairs $\mathrm{Sing}(2,1)$ is $\bigcap_{\epsilon > 0} \bigcup_{Q_\epsilon \geq 1} \bigcap_{Q \geq Q_\epsilon} A_{Q,\epsilon}$, the measure of $\mathrm{Sing}(2,1)$ is zero.

§ Review of Dani correspondence for singular matrices

- Given an $m \times n$ matrix A, define

$$g_t = \begin{pmatrix} e^{t/m} & & & & & \\ & \ddots & & & & \\ & & e^{t/m} & & & \\ & & & e^{-t/n} & & \\ & & & & \ddots & \\ & & & & & e^{-t/n} \end{pmatrix}$$

$$u_A = \begin{pmatrix} 1 & & & & \\ & \ddots & & A & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

Then A is singular $\Longleftrightarrow$ the orbit $(g_t u_A \mathbb{Z}^d)_{t \geq 0}$ is divergent in the space of unimodular lattices $X_d := SL_d(\mathbb{R}) / SL_d(\mathbb{Z})$, where $d := m+n$.

- Recall Minkowski successive minima of a lattice $\Lambda \subset \mathbb{R}^d$:

$$\lambda_i(\Lambda) = \min \left\{ \lambda : \begin{array}{l} \exists\, i \text{ linearly indep. vectors } v \in \Lambda \\ \text{s.t. } \|v\| \leq \lambda \end{array} \right\}$$

Then Mahler compactness implies that $(g_t u_A \mathbb{Z}^d)_{t \geq 0}$ diverges $\Longleftrightarrow$ $\lambda_1 (g_t u_A \mathbb{Z}^d) \xrightarrow[t \to \infty]{} 0$.

§ Progress toward calculating Hausdorff dimension of singular matrices

- Cheung (2011): $\dim_H(\mathrm{Sing}(2,1)) = \frac{4}{3}$
- Cheung / Chevallier (2016): $\dim_H(\mathrm{Sing}(m,1)) = \frac{m^2}{m+1}$
- Kadyrov / Kleinbock / Lindenstrauss / Margulis (2017):

$$\dim_H(\mathrm{Sing}(m,n)) \leq mn\left(1 - \frac{1}{m+n}\right)$$

  The authors of this paper conjectured that $=$ holds here.

- Bugeaud / Cheung / Chevallier (2017): what is the <u>packing dimension</u> $\dim_P(\mathrm{Sing}(m,n))$?

- Das / Fishman / Simmons / Urbański (2017):

$$\dim_H(\mathrm{Sing}(m,n)) = \dim_P(\mathrm{Sing}(m,n)) = mn\left(1 - \frac{1}{m+n}\right).$$

  The authors of this paper used different techniques than those of the previous authors. Specifically, the theorem is a corollary of a "variational principle for templates." This principle also gives results on the dimension theory of "very singular matrices" as well as matrices with a "uniform exponent of irrationality."

§ Properties of the successive minima function:

• defn: if Λ is a unimodular lattice in $\mathbb{R}^d$, define $\vec{h} : [0,\infty) \longrightarrow \mathbb{R}^d$ by

$$h_i(t) := \log \lambda_i(g_t \Lambda)$$

$$\left(\text{where } g_t = \begin{pmatrix} e^{t/m} & & & & \\ & \ddots & & & \\ & & e^{t/m} & & \\ & & & e^{-t/n} & \\ & & & & \ddots \\ & & & & & e^{-t/n} \end{pmatrix}, \; m+n=d \text{ as before.} \right)$$

This "successive minima function" satisfies:

(I) $h_1 \leq \cdots \leq h_d$,

(II) $-\frac{1}{n} \leq h_i'(t) \leq \frac{1}{m}$ $\forall i = 1, \ldots, d$ when h_i differentiable,

(III) "quantized slope condition": for all $j = 1, \ldots, d$ and an interval I on which $f_j < f_{j+1}$, there exists a piecewise linear $F_{j,I}$ with slopes in

$$Z(j) := \left\{ \frac{L_+}{m} - \frac{L_-}{n} : \begin{array}{l} L_+ \in \{0,\ldots,m\} \\ L_- \in \{0,\ldots,n\}, \end{array} \; L_+ + L_- = j \right\}$$

such that $\sum_{i=1}^{j} h_i \asymp F_{j,I}$ on I.

Furthermore $F_{j,I}$ is continuous and convex on I.

• If A is an $m \times n$ matrix, we have an associated successive minima function $\vec{h}_A : [0,\infty) \to \mathbb{R}^d$, $(h_A)_i(t) = \log \lambda_i(g_t u_A \mathbb{Z}^d)$. If A is singular, $(h_A)_1(t) \xrightarrow[t\to\infty]{} -\infty$ by Mahler compactness.

§ Templates

- An $m \times n$ - template is a continuous piecewise-linear map $\vec{f} : [0,\infty) \to \mathbb{R}^d$ satisfying (with $d = m+n$):

(I) $f_1 \leq \cdots \leq f_d$

(II) $-\frac{1}{n} \leq f_i' \leq \frac{1}{m}$ $\forall i$

(III) $\forall j = 1, \ldots, d$ and interval I on which $f_j < f_{j+1}$,

$$\sum_{i=1}^{j} f_i$$

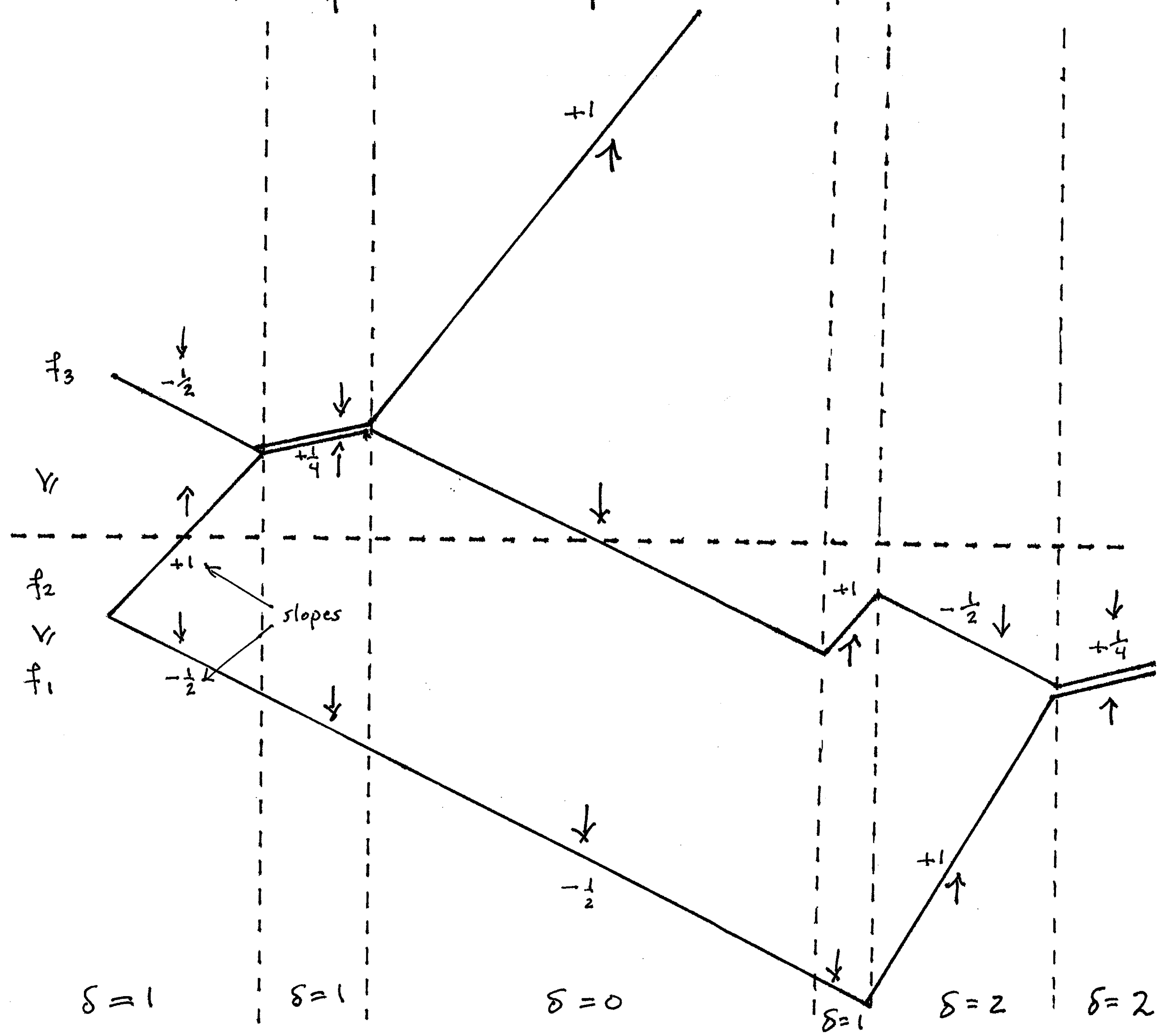
is convex & piecewise linear on I, with slopes in

$$Z(j) = \left\{ \frac{L_+}{m} - \frac{L_-}{n} : \begin{array}{l} L_+ \in \{0,\ldots,m\} \\ L_- \in \{0,\ldots,n\} \end{array}, L_+ + L_- = j \right\}$$

- To state the "variational principle," we will need to define the "upper and lower contraction rates" $\underline{\delta}, \overline{\delta}$ of a template. We will do this via an example on the next page.

§ Example of a template

- The *joint graph* of a template $\vec{f}$ is just a graph displaying all its component functions' graphs at the same time.
- Below is a picture of the joint graph of a 1×2-template: recall $-\frac{1}{2} \leq f_i' \leq 1$, $f_1 \leq f_2 \leq f_3$, and the quantized slope condition.



On each interval of linearity, we have a contraction ratio δ defined as follows: if on each component function we have particles moving along the graph at the same speed, with sticky collisions, $\delta =$ the number of pairs moving towards each other (including pairs traveling together).

§ Contraction ratios & variational principle

- On each interval I of linearity, we have an associated contraction ratio $\delta(\vec{f}, I)$ defined on the previous page. The average contraction ratio on $[0,T]$ is:

$$\Delta(\vec{f}, T) := \frac{1}{T}\int_0^T \delta(\vec{f}, t)\,dt$$

Then the <u>upper</u> and <u>lower</u> <u>contraction</u> <u>ratios</u> are

$$\underline{\delta}(\vec{f}) = \liminf_{T\to\infty} \Delta(\vec{f}, T), \qquad \overline{\delta}(\vec{f}) = \limsup_{T\to\infty} \Delta(\vec{f}, T)$$

- Recall that for every $m\times n$ matrix A we have an associated successive minima function $\vec{h}_A : [0,\infty) \to \mathbb{R}^d$, $d = m+n$.

- Realization theorem:
  - (I) For every $m\times n$ matrix A, there is a $m\times n$ template $\vec{f}$ such that $\vec{h}_A \asymp \vec{f}$,
  - (II) For every $m\times n$ template $\vec{f}$ there is an $m\times n$ matrix A such that $\vec{h}_A \asymp \vec{f}$.

- Thus for every template $\vec{f}$, we can study the non-empty set $\mathcal{D}(\vec{f}) := \{A : \vec{h}_A \asymp \vec{f}\}$, or given a collection $\mathcal{F}$ of templates, $\mathcal{D}(\mathcal{F}) := \bigcup_{\vec{f}\in\mathcal{F}} \mathcal{D}(\vec{f})$. How big is such a set in terms of dimension?

- We say a collection $\mathcal{F}$ is <u>closed</u> <u>under</u> <u>finite</u> <u>perturbations</u> if whenever $f \in \mathcal{F}$ and $f \asymp g$, then $g \in \mathcal{F}$ as well.

☆ Variational principle: Let $\mathcal{F}$ be a collection of templates closed under finite perturbations. Then

$$\dim_H(\mathcal{D}(\mathcal{F})) = \sup_{\vec{f}\in\mathcal{F}} \underline{\delta}(\vec{f}), \qquad \dim_P(\mathcal{D}(\mathcal{F})) = \sup_{\vec{f}\in\mathcal{F}} \overline{\delta}(\vec{f}).$$

The proof uses a new variant of Schmidt games.