§ Singular pairs / vectors / matrices:

- Definition: An m x n matrix A is singular if ∀ x > 0, ∃ y ∈ R such that ∀ x_0 ∈ R^n, ∃ v ∈ R^m, t ∈ R^m \{0\} s.t.
  \[ ||Ax_0 - tv|| ≤ c \quad \text{and} \quad ||v|| ≤ q.\]
  (for m=1 these are singular vectors; for n=2 additionally these are singular pairs.)
  Notation: \( \sigma (m,n) = \text{singular m x n matrices}. \)

- Fact: there are no (irrational) singular numbers (i.e. \( m = n = 1 \)). If \( a ∈ R \setminus Q \), then a has an infinite continued fraction expansion

  \[ a = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}, \]

  (assuming \( 0 < a < 1, a_0 = 0 \)). If we truncate this expansion at level \( k \), we obtain the convergents

  \[ \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_k}}}}} = \frac{p_k}{q_k}. \]

  These satisfy the inequalities \( |qa - p| ≥ |q_k a - p_k| \) for all \( p, q ∈ Z \) with \( 0 < q < q_{k+1} \), and

  \[ |q_k a - p_k| > \frac{1}{2q_{k+1}}. \]

  Hence \( |qa - p| > \frac{1}{2q_{k+1}} \) for all \( p, q \) with \( 0 < q < q_{k+1} \), which contradicts the definition above of a singular number.
Fact: The set of singular pairs has zero Lebesgue measure.

For a fixed \( q_2 \), the measure of the set of \( q_1 \) satisfying

\[
|q_1 q_1 + q_2 q_2 - p| \leq \epsilon Q^{-2}
\]

for a fixed \( q_1, q_2, p \in \mathbb{Z} \) (with \( q_1 \neq 0 \)) is \( \leq 2 \epsilon Q^{-2} \).

The number of vectors \( (q_1, q_2) \in \mathbb{Z}^2 \) with \( \|q\| \leq Q \) is \( \leq (2Q+1)^2 - 1 < (3Q)^2 \).

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Then for a given \( Q \), the measure of

\[
A_{Q, \epsilon} = \left\{ q_1, q_2 : |q_1 q_1 + q_2 q_2 - p| \leq \epsilon Q^{-2} \text{ for some } p \in \mathbb{Z}, \quad \frac{q}{Q} = (q_1, q_2) \in \mathbb{Z}^2 \text{ with } \|q\| \leq Q \right\}
\]

is \( \leq 2 \cdot 3^2 \epsilon \). Since the set of singular pairs

\[
\text{Sing} (2, 1) = \bigcap_{Q \geq 2} \bigcap_{Q \geq Q_0} A_{Q, \epsilon},
\]

the measure of \( \text{Sing} (2, 1) \) is zero.
Review of Dani correspondence for singular matrices

Given an $m \times n$ matrix $A$, define

$$g_t = \begin{pmatrix} e^{tv_1} & e^{tv_2} & \cdots & e^{tv_n} \\ e^{-tv_1} & e^{-tv_2} & \cdots & e^{-tv_n} \end{pmatrix}$$

$$U_A = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \\ A \end{pmatrix}$$

Then $A$ is singular $\iff$ the orbit $\left( g_t U_A Z^d \right)_{t \geq 0}$ is divergent in the space of unimodular lattices $X_d := SL_d(\mathbb{R})/SL_d(\mathbb{Z})$, where $d := m+n$.

Recall Minkowski successor minimum of a lattice $\Lambda \subset \mathbb{R}^d$:

$$\lambda_i(\Lambda) = \min \left\{ \lambda : \exists \text{ linearly indep. vectors } u \in \Lambda \right\}$$

s.t. $\|u\| \leq \lambda$

Then Mahler compactness implies that $\left( g_t U_A Z^d \right)_{t \geq 0}$ diverges $\iff \lambda_i(\left( g_t U_A Z^d \right)_{t \geq 0}) \to 0$ as $t \to \infty$. 
Progress toward calculating Hausdorff dimension of singular matrices

- Cheung (2011): \( \dim_H(S(2,1)) = \frac{4}{3} \)
- Cheung / Chevallier (2016): \( \dim_H(S(\mathbb{R}, 1)) = \frac{m^2}{m+1} \)
- Kadyrov / Heidenreich / Lindenstrauss / Margulis (2017):

\[
\dim_H(S(\mathbb{R}, [m,n])) \leq m \left( 1 - \frac{1}{m^n} \right)
\]

The authors of this paper conjectured that this holds here.

- Bugeaud / Cheung / Chevallier (2017): What is the packing dimension \( \dim_p(S(\mathbb{R}, [m,n])) \)?

- Das / Fishman / Simmons / Urbanski (2017):

\[
\dim_H(S(\mathbb{R}, [m,n])) = \dim_p(S(\mathbb{R}, [m,n])) = m \left( 1 - \frac{1}{m^n} \right)
\]

The authors of this paper used different techniques than those of the previous authors. Specifically, the theorem is a corollary of a "variational principle for templates." This principle also gives results on the dimension theory of "very singular matrices" as well as matrices with a "uniform exponent of irrationality."
Properties of the successive minima function:

defn: if $\Lambda$ is a unimodular lattice in $\mathbb{R}^d$, define $h: [0, \infty) \rightarrow \mathbb{R}^d$ by

$$h_i(t) := \log \lambda_i(j_t \Lambda)$$

(where $j_t = \left( e^{i\alpha} \cdots e^{i\alpha} \right)$, $0 < \alpha < \pi$, $j_t$ as before)

This "successive minima function" satisfies:

（II） $h_1 \leq \cdots \leq h_d$,

（III） $-\frac{1}{n} \leq h_i'(t) \leq \frac{1}{n}$ $\forall i = 1, \ldots, d$ when $h_i$ differentiable,

（III*） "quantized slope condition": for all $j = 1, \ldots, d$ and an interval $I$ on which $f_j < f_{j+1}$, there exists a piecewise linear $F_{j, I}$ with slopes in

$$E(j) := \left\{ \frac{l_+}{m} - \frac{l_-}{m} : \begin{array}{l} l_+ \in \{0, \ldots, m^2\} \\ l_- \in \{0, \ldots, m^2\} \\ l_+ + l_- = j \end{array} \right\}$$

such that $\sum_{i=1}^d h_i(i) \leq F_{j, I}$ on $I$.

Furthermore, $F_{j, I}$ is continuous and convex on $I$.

If $A$ is an $m \times m$ matrix, we have an associated successive minima function $h_A: [0, \infty) \rightarrow \mathbb{R}^d$,

$(h_A)_i(t) = \log \lambda_i(j_t A^{\mathbb{Z}^d})$. If $A$ is singular,

$(h_A)_i(t) \rightarrow -\infty$ by Mahler compactness.
§ Templates

An \( m \times n \) \text{-} template is a continuous piecewise-linear map \( \overline{f} : [0, \infty) \to \mathbb{R}^d \) satisfying (with \( d = m n \)):

(1) \( \overline{f}_1 \leq \cdots \leq \overline{f}_d \)

(2) \( -\frac{1}{n} \leq \overline{f}_i \leq \frac{1}{m} \quad \forall i \)

(3) \( \forall j = 1, \ldots, d \) and interval \( I \) on which \( \overline{f}_j < \overline{f}_{j+1} \)

\[ \sum_{i=1}^{j} \overline{f}_i \]

is convex \& piecewise-linear on \( I \), with slopes \( \overline{m} \)

\[ Z(j) = \left\{ \frac{l_1}{m} - \frac{l_2}{n} : l_1, l_2 \in \mathbb{Z} \div \{0, \ldots, m\}, \quad l_1 + l_2 = j \right\} \]

To state the "variational principle," we will need to define the "upper and lower contraction rates" \( \delta, \delta \) of a template. We will do this via an example on the next page.
Example of a template

The joint graph of a template $\mathbf{\Gamma}$ is just a graph displaying all its component functions' graphs at the same time.

Below is a picture of the joint graph of a $1 \times 2$ template: recall $-\frac{1}{2} \leq f_1 \leq 1$, $-\frac{1}{2} \leq f_2 \leq \frac{1}{2}$, and the quantized slope condition.

On each interval of linearity, we have a contraction ratio $\delta$ defined as follows: if on each component function we have particles moving along the graph at the same speed, with sticky collisions, $\delta$ = the number of pairs moving towards each other (including pairs traveling together).
Contraction ratios & variational principle

On each interval $I$ of linearity, we have an associated contraction ratio $\delta(\bar{f}, I)$ defined on the previous page. The average contraction ratio on $[0,T]$ is:

$$\Delta(\bar{f}, T) := \frac{1}{T} \int_0^T \delta(\bar{f}, t) dt$$

Then the upper and lower contraction ratios are

$$\delta^*(\bar{f}) = \liminf_{T \to \infty} \Delta(\bar{f}, T), \quad \delta^*(\bar{f}) = \limsup_{T \to \infty} \Delta(\bar{f}, T)$$

Recall that for every $m \times n$ matrix $A$ we have an associated successive umbra function $t_A: [0,\infty) \to \mathbb{R}^d$, $d=m+n$.

Realization theorem:

(I) For every $m \times n$ matrix $A$, there is a $m \times n$ template $\bar{f}$ such that $t_A \preceq \bar{f}$.

(II) For every $m \times n$ template $\bar{f}$ there is an $m \times n$ matrix $A$ such that $t_A \preceq \bar{f}$.

Thus for every template $\bar{f}$, we can study the non-empty set $D(\bar{f}) := \{ A : t_A \preceq \bar{f} \}$, or given a collection $\mathcal{F}$ of templates, $D(\mathcal{F}) := \bigcup_{\bar{f} \in \mathcal{F}} D(\bar{f})$. How big is such a set in terms of dimension?

We say a collection $\mathcal{F}$ is closed under finite perturbations if whenever $\bar{f} \in \mathcal{F}$ and $\bar{f} \preceq g$, then $g \in \mathcal{F}$ as well.

A variational principle: Let $\mathcal{F}$ be a collection of templates closed under finite perturbations. Then

$$\dim_H(\mathcal{F}) = \sup_{\bar{f} \in \mathcal{F}} \delta^*(\bar{f}), \quad \dim_p(\mathcal{F}) = \sup_{\bar{f} \in \mathcal{F}} \delta^*(\bar{f}).$$

The proof uses a new variant of Schmidt games.