The Product Structure

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Seminar on homogeneous dynamics and applications

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Basic setting

- G is a semisimple connected linear Lie group and $\Gamma \subset G$ is a lattice;
- $X = \Gamma \backslash G$ equipped with a left-invariant metric induced from left-invariant Riemannian metric on G;
- $a \in G$: an element of class A;
- μ is Borel a-invariant probability measure on X;
- $U_{-} \subset G^{-}$ be a-normalized and contracted by a;
- $T \subset C_G(a)$ and assume T normalizes U_{-};
- $H = T \ltimes U_{-}$;

where a is class A if (i) a is $\mathbb{R}$-semisimple; (ii) 1 is the only eigenvalue of absolute value 1 for the adjoint action Ad_a; (iii) No two different eigenvalues of Ad_a have the same absolute value.

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Equivalence classes

- $[\cdot]$ be the equivalence classes by portionality of the leaf-wise measure;
- Theorem 6.29 guarantees that there is a function $\rho > 0$ such that $\int \rho d\mu_x^T < \infty$ a.e.;
- Picking a sequence $\{f_i \leqslant \rho\} \subset C_c(T)$ spanning a dense subset.

Then for any two equivalence classes of Radon measure with $\int \rho d\nu_i < \infty$, define $d([\nu_1],[\nu_2]) = \sum_{i=1}^{\infty} 2^{-i} \left| \frac{\int f_i d\nu_1}{\int \rho d\nu_1} - \frac{\int f_i d\nu_2}{\int \rho d\nu_2} \right|$.Assume $\int \rho d\nu_i = 1$, then this metric corresponds to weak* in the space of Radon measures and thus we can interpret μ_x^T as a measurable function with values in a compact metric space.

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Proposition 8.5

There exists $X' \subset X$ of full measure such that for every $x \in X'$ and $h \in H$ with $h.x \in X'$, we have

$$[\mu_x^T] = [(\mu_{h.x}^T)t]$$

where $h = tu' = u''t$ for some $u', u'' \in U_-$ and $t \in T$.

Let $t = e$, then we have

Corollary 8.6

Let $u \in U_-$, then $x, u.x \in X'$ implies $[\mu_x^T] = [\mu_{u.x}^T]$.

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Product structure, Corollary 8.8

There exists $X' \subset X$ of full measure such that for every $x \in X'$, we have

$$\mu_x^H \propto \iota(\mu_x^T \times \mu_x^{U_-}),$$

where $\iota : (t, u) \in T \times U_- \to tu \in H$.

A natural corollary of Corollary 8.8 is a property similar to Corollary 8.6:

Corollary 8.13

Let $t \in T$, then $x, t.x \in X'$ implies $[\mu_x^{U_-}] = [\mu_{t.x}^{U_-}]$.

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Proof of Corollary 8.13

By Corollary 8.8 for x and $t.x$, we obtain

$$\mu_x^H \propto \iota(\mu_x^T \times \mu_x^{U_-}), \quad \mu_{t.x}^H \propto \iota(\mu_{t.x}^T \times \mu_{t.x}^{U_-}).$$

Then applying t from right to the second equation, since Theorem 6.3 implies $\mu_x^H \propto \mu_{t.x}^H t$, we have

$$\mu_x^H \propto \mu_{t.x}^H t \propto \iota(\mu_{t.x}^T \times \mu_{t.x}^{U_-})t.$$

Recall that $H = T \ltimes U^-$, thus $(t_1, u_1) \cdot (t_2, u_2) = (t_1 t_2, u_1 t_1 u_2 t_1^{-1})$, thus we know $\iota(\mu_{t.x}^T \times \mu_{t.x}^{U_-})t = \iota(\mu_{t.x}^T t \times \mu_{t.x}^{U_-})$, which together with $\mu_{t.x}^T t \propto \mu_x^T$ and above equations guarantee that

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Example

The following example is useful to keep in mind among all proofs:

- $G = \mathrm{SL}(3, \mathbb{R})$;
- $a = \begin{pmatrix} e^{-2} & & \\ & e & \\ & & e \end{pmatrix}$;
- $U_- = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\}$, $T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$;
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- By Lusin theorem, for any $\varepsilon > 0$, there exists a compact $K_\varepsilon \subset \Gamma\backslash G$ such that
  - $x \mapsto [\mu_x^T]$ is continuous for $x \in K_\varepsilon$;
  - $\mu(K_\varepsilon) > 1 - \varepsilon$;
  - $[\mu_x^T] = [\mu_{t.x}^T t]$ for $x, t.x \in K_\varepsilon$.
- Let $X_\varepsilon = \{x \in K_\varepsilon : \sup \frac{1}{n} \sum_{i=0}^{n-1} 1_{K_\varepsilon^c}(a^i.x) < \frac{1}{2}\}$, then by maximal ergodic theorem, we have $\int 1_{K_\varepsilon^c} d\mu \geqslant \frac{1}{2}\mu(X_\varepsilon^c)$, i.e. $\mu(X_\varepsilon) > 1 - 2\varepsilon$.
- Suppose $x, h.x \in X_\varepsilon$, the definition of $X_\varepsilon \Rightarrow n_i \to \infty$ such that $a^{n_i}.x, a^{n_i}h.x \in K_\varepsilon$. Then by passing to a subsequence if necessary, we assume that $a^{n_i}.x \to x_0 \in K_\varepsilon$. Together with continuity of $x \mapsto [\mu_x^T]$ on K_ε, we obtain

$$[\mu_{a^{n_i}.x}^T] \to [\mu_{x_0}^T].$$

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- Then by continuity of $x \mapsto [\mu_x^T]$ on K_ε, we have

$$[\mu^T_{a^{n_i} h.x}] \to [\mu^T_{t.x_0}].$$

- Since $T \in C_G(a)$ and μ is a-invariant, a maps an (r, T)-flower $(Y, \mathcal{A})$ to another $\sigma-$algebra $a\mathcal{A}$ of subsets aY, whose atoms are still $T-$platues. As a preserves the measure μ, the conditional measures for $\mathcal{A}$ are mapped to conditional measures for $a\mathcal{A}$. Together with Theorem 6.3 and $T \in C_G(a)$, we have $[\mu_x^T] = [\mu_{a.x}^T]$ a.e..

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- Since $T \in C_G(a)$ and μ is a-invariant, a maps an (r, T)-flower $(Y, \mathcal{A})$ to another $\sigma-$algebra $a\mathcal{A}$ of subsets aY, whose atoms are still $T-$platues. As a preserves the measure μ, the conditional measures for $\mathcal{A}$ are mapped to conditional measures for $a\mathcal{A}$. Together with Theorem 6.3 and $T \in C_G(a)$, we have $[\mu_x^T] = [\mu^T_{a.x}]$ a.e..

Proof of Proposition 8.5

- Since $h = tu'$, $T \in C_G(a)$ and $a^{n_i} u' a^{-n_i} \to e$ as $n_i \to \infty$, we have
$$a^{n_i} h.x = ta^{n_i} u' a^{-n_i} a^{n_i}.x = ta^{n_i}.x \to t.x_0.$$

- Then by continuity of $x \mapsto [\mu_x^T]$ on K_ε, we have
$$[\mu^T_{a^{n_i} h.x}] \to [\mu^T_{t.x_0}].$$

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Proposition 8.5

- Since $[\mu_x^T] = [\mu_{a.x}^T]$ a.e, thus there exists $X'_\varepsilon \subset X_\varepsilon$ with $\mu(X'_\varepsilon) = \mu(X_\varepsilon)$ such that for every $x, h.x \in X'_\varepsilon$, we have

$$[\mu_x^T] = [\mu_{a^{n_i}.x}^T] \to [\mu_{x_0}^T] = [\mu_{t.x_0}^T]t,$$

and

$$[\mu_{h.x}^T]t = [\mu_{a^{n_i}h.x}^T]t \to [\mu_{t.x_0}^T]t,$$

which gives $[\mu_x^T] = [\mu_{h.x}^T]t$.

- Let $\varepsilon_n = \frac{1}{n}$, choose K_{ε_n} increasing and define X' be union of $X'_{\frac{1}{n}}$, we complete the proof.

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Heuristic explanation: μ_x^H is the measure on H such that $\mu_x^H.x$ describes μ along the orbit $H.x$ and $(\mu_x^H)_h^L.h$ describes μ_x^H along the coset Lh, we want $(\mu_x^H)_h^L h.x$ describes μ on the orbit $Lh.x$.

Lemma 8.10

let H be a locally compact second countable group acting on X locally and measure-theoretically free, and let μ be a locally finite Radon measure on X. Assume $H = LM = \iota(L \times M)$ is topologically isomorphic to the product of two closed subgroups $L, M < H$. Then L acts by restriction on X and on H by left translation, and so gives rise to families of leaf-wise measures μ_x^L and $(\mu_x^H)_h^L$ for $x \in X$ and $h \in H$. Then there exists $X' \subset X$ of full measure such that whenever $x \in X'$, we have $[(\mu_x^H)_h^L] = [\mu_{h.x}^L]$ for μ_x^H-a.e. $h \in H$.

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Proof of Lemma 8.10

- Let $\Xi \subset X$ be R-cross-section for the action of H with positive measure and $\tilde{\mathcal{A}}_H$ be the σ-algebra $B_R^H \times \mathcal{B}(\Xi)$ on $B_R^H \times \Xi$, where $\mathcal{B}(\Xi)$ is the Borel σ-algebra on Ξ.
- By R-cross section definition, $\iota(h, x) = h.x$ is injective on $B_R^H \times \Xi$ and thus $\mathcal{A}_H = \iota(\tilde{\mathcal{A}}_H)$ is a countably generated σ-ring of Borel sets. Moreover, the atom $[x]_{\mathcal{A}_H}$ is an open H-plaque for any $x \in \iota(B_R^H \times \Xi)$.
- Let $\tilde{\mathcal{A}}_L = \{LB \cap B_R^H : B \in \mathcal{B}(M)\}$, where $\mathcal{B}(M)$ is the Borel σ-algebra on M. Recall by assumption, M is a global cross-section of L in H. Then the σ-ring $\mathcal{A}_L = \iota(\tilde{\mathcal{A}}_L \times \mathcal{B}(\Xi))$ is countably generated and $[x]_{\mathcal{A}_L}$ is an open L-plaque for all $x \in \iota(B_R^H \times \Xi)$.
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- Recall Theorem 6.3, the measures μ_x^H and μ_x^L can be defined by the values of conditional measures w.r.t. to a countable collection of σ-rings $\mathcal{A}_H^{(i)}$ and $\mathcal{A}_L^{(i)}$ as above.
- More precisely, recall $\mathcal{A}_H^{(i)} \subset \mathcal{A}_L^{(i)}$, then by Theorem 6.3, we have

$$((\mu_x^H)_h^L|_{U_{h,B_R^H}}).h \propto (\mu_x^H)_h^{B_R^H}$$

$$(\mu_x^H|_{U_{x,\mathcal{A}_H}}).x \propto \mu_x^{\mathcal{A}_H}, \ \ (\mu_{h.x}^L|_{U_{h.x,\mathcal{A}_L}}).(h.x) \propto \mu_{h.x}^{\mathcal{A}_L}$$

$$\mu_{h.x}^{\mathcal{A}_L} \propto \mu_{h.x}^{\mathcal{A}_H}.$$

- Since on each of these σ-rings, we always have above relation, we complete our proof.

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Proof of Corollary 8.8

- Apply Lemma 8.10 with $L = T$, $M = U_-$ and $L = U_-$, $M = T$ and Proposition 8.5, denote the union of the bad null sets as X_0. Then let $x \in X \setminus X_0$ and $Q = B_r^T B_r^{U_-} \subset H$ for some $r > 0$.
- Apply Lemma 8.10 with $L = T$ and $M = U_-$, we obtain that the conditional measures for $\mu_x^H|_Q$ with respect to the σ-algebra $\mathcal{A} = B_r^T \times \mathcal{B}(B_r^{U_-})$ can be obtained from the leaf-wise measures $\mu_{h.x}^T$ for μ_x^H-a.e.$h \in Q$:

$$(\mu_x^H)_h^{\mathcal{A}} \propto (\mu_{h.x}^T h)|_Q.$$

- Recall Proposition 8.5 gives for μ_x^H-a.e. $h = tu \in Q$:

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- Combining above two equations, we obtain that

$$(\mu_x^H)_h^{\mathcal{A}} \propto \mu_x^T|_{B_r^T} \times \delta_u,$$

which is equivalent to $\mu_x^H|_Q$ is a product measure which is proportional to $\iota(\mu_x^T \times \nu_r)$ for some finite measure ν_r on $B_r^{U_-}$.

- Patching these measure ν_r together to obtain a Radon measure ν on U_-, then we have

$$\mu_x^H \propto \iota(\mu_x^T \times \nu).$$

- Restrict to $Q = B_r^T B_r^{U_-} \subset H$ and consider the σ-algebra $\mathcal{A}' = \mathcal{B}(B_r^T) \times B_r^{U_-}$, whose atoms are $tB_r^{U_-}$ for $t \in B_r^T$.

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- Since the conditional measure of $\mu_x^H|_Q$ at $h = tu$ equals $\iota(\delta_t \times \nu_r)$, then for the action of U_- by left multiplication on H, we have

  $$((\mu_x^H)_h^{U_-}|_{U_{h,\mathcal{A}'}})(\{u \in U_- : u.h \in tB_r^{U_-}\}) = ((\mu_x^H)_h^{U_-}|_{U_{h,\mathcal{A}'}})(tB_r^{U_-}h^{-1})$$

  which implies the atom of $h = tu \in Q$ corresponds to the set $V_h = tB_r^{U_-}h^{-1} \subset U_-$ (this due to T normalize U_-).
- Using these σ-rings for all positive integers r, we obtain that $(\mu_x^H)_h^{U_-}$ must be proportional to $t\nu h^{-1}$ for μ_x^H-a.e. h, where $t\nu h^{-1}(A) = \nu(tAh^{-1})$ for $A \subset U_-$.

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- We have already shown for μ_x^H-a.e. $h = tu$:

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- Since we cannot claim $h = e$ for the above formula, we need the following additional steps to establish our corollary:

$$\mu_{h.x}^H \overset{\text{Theorem 6.3}}{\propto} \mu_x^H h^{-1} \propto \iota(\mu_x^T \times \nu)h^{-1} \overset{\text{Proposition 8.5}}{\propto} \iota(\mu_{h.x}^T t \times \nu)h^{-1}$$

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Thanks!