PERES SCHLAG METHOD

1. Abstract

I will present a paper by Yuval Peres and Wilhelm Schlag. They prove that for every $\varepsilon > 0$ and $\Lambda = \{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ satisfy $\frac{\lambda_{n+1}}{\lambda_n} > 1 + \varepsilon$ for every $n \in \mathbb{N}$, then there exists $\alpha \in \mathbb{R}$ such that

$$\inf_{\lambda \in \Lambda} |\langle \lambda\alpha \rangle| > \frac{1}{240 \log \frac{1}{\varepsilon}} \varepsilon,$$

where $|\langle \cdot \rangle|$ denotes the distance to the integers. The linear dependence in ε cannot be improved, as may be seen by exercise 1 (c), but nothing is known regarding the necessity of the logarthmic factor. As was noted by Yitzhak Katznelson earlier, this provides a coloring of the graph

$$G = (\mathbb{Z}, \{(m,n) \ : \ n - m \in \Lambda\})$$

with $\left\lceil \left(240 \log \frac{1}{\varepsilon}\right) \frac{1}{\varepsilon} \right\rceil$ colors, which is the most efficient coloring of G known to exist, asymptotically as $\varepsilon \to 0$. The proof uses a simple and clever application of Lovász local lemma.

2. exercises

(1) Let $\varepsilon > 0$, $\lambda = 1 + \varepsilon$.
 (a) If $\varepsilon \in \mathbb{N}$, show that $\alpha = \frac{1}{q}$ satisfy $\inf_{n \geq 0} |\langle \lambda^n \alpha \rangle| = \alpha$, whenever $q \nmid \lambda^n$ for all $n \geq 0$.
 (b) For $\Lambda = \{\lambda^n \ : \ n \geq 0\}$, show that $\alpha = \frac{1}{\varepsilon}$ satisfies
 $$\inf_{n \geq 0} |\langle \lambda^n \alpha \rangle| = \left| \left\langle \frac{1}{\varepsilon} \right\rangle \right|.$$
 (c) For
 $$\Lambda = \left\{1, \ldots, \left\lfloor \frac{1}{\varepsilon} \right\rfloor \right\} \cup \left\{ \lceil \lambda \rceil^n \left\lfloor \frac{1}{\varepsilon} \right\rfloor \ : \ n \in \mathbb{N} \right\},$$
 show that every $\alpha \in \mathbb{R}$ satisfy
 $$\inf_{\lambda \in \Lambda} |\langle \lambda\alpha \rangle| \leq \varepsilon.$$

(2) Prove Lovász local lemma: Assume $A_1, \ldots, A_N$ are events in a probability space, and let
$$E = \{(m,n) \ : \ \mathrm{P}(A_m \cap A_n) = \mathrm{P}(A_m)\,\mathrm{P}(A_n)\}.$$
If there are $0 \leq x_1, \ldots, x_N < 1$ that satisfy
$$\mathrm{P}(A_n) \leq x_n \prod_{(m,n) \in E} (1 - x_m),$$
for every $1 \leq n \leq N$, then
$$\mathrm{P}(A_1^c \cap \ldots \cap A_N^c) \geq \prod_{n=1}^{N} (1 - x_n).$$
Hint: prove that every $S \subseteq \{1, \ldots, N\}$ and $n \notin S$ satisfy
$$\mathrm{P}\left(A_n \mid \bigcap_{m \in S} A_m^c\right) \leq x_n,$$
by induction on $|S|$, and by using the formula $\mathrm{P}(A \mid B \cap C) = \frac{\mathrm{P}(A \cap B \mid C)}{\mathrm{P}(B \mid C)} \leq \frac{\mathrm{P}(A \mid C)}{\mathrm{P}(B \mid C)}$.

References

[PS] Yuval Peres and Wilhelm Schlag, Two Erdős problems on lacunary sequences: Chromatic number and Diophantine approximation, Bull. London Math. Soc. (2010) 42 (2): 295-300.