The Littlewood conjecture in simultaneous approximation and variants of it

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Seminar on homogeneous dynamics and applications

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Table of contents

Littlewood's conjecture in simultaneous approximation

Littlewood's conjecture (1930s)

For every $\mathbf{x} \in \mathbb{R}^2$ *and every* $c > 0$ *there exist* $\frac{\mathbf{m}}{n} \in \mathbb{Q}^2$ *such that*

$$\left|x_1 - \frac{m_1}{n}\right| \left|x_2 - \frac{m_2}{n}\right| \leq \frac{c^2}{n^3}.$$

Equivalent formulation of the Littlewood conjecture

For every $x \in \mathbb{R}$ denote $|\langle x \rangle| = \mathrm{d}(x, \mathbb{Z})$. Upon multiplying by n^3 the Littlewood conjecture is more commonly phrased as follows:

Littlewood's conjecture (equivalent form.)

Every $\mathbf{x} \in \mathbb{R}^2$ *satisfies*

$$\inf_{n \in \mathbb{N}} n \left|\langle n x_1 \rangle\right| \left|\langle n x_2 \rangle\right| = 0.$$

Rational vectors surrounded by hyperbolas

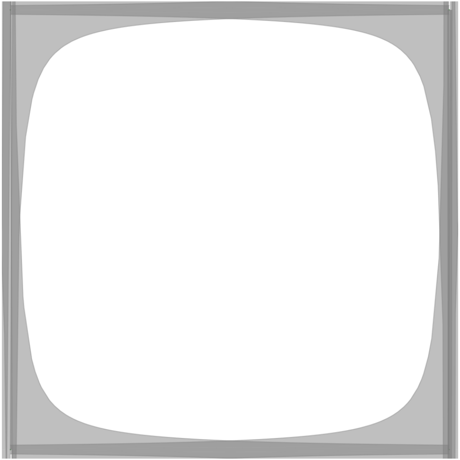

Figure: Rational vectors in the unit square with $n \leq 1$ surrounded by a hyperbola $\left|x_1 - \frac{m_1}{n}\right| \cdot \left|x_2 - \frac{m_2}{n}\right| \leq \frac{c^2}{n^3}$ with $c = \frac{1}{10}$.

Rational vectors surrounded by hyperbolas

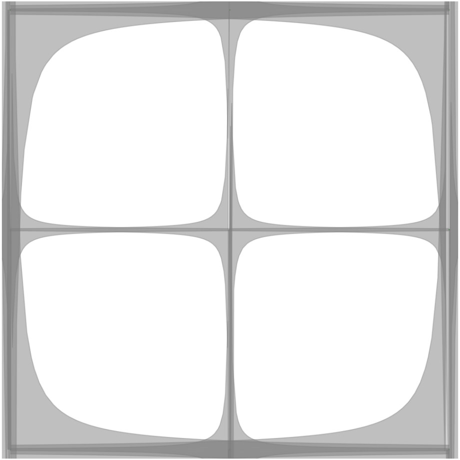

Figure: Rational vectors in the unit square with $n \leq 2$ surrounded by a hyperbola $\left|x_1 - \frac{m_1}{n}\right| \cdot \left|x_2 - \frac{m_2}{n}\right| \leq \frac{c^2}{n^3}$ with $c = \frac{1}{10}$.

Rational vectors surrounded by hyperbolas

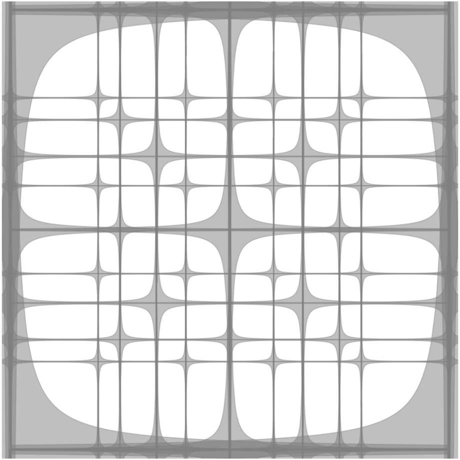

Figure: Rational vectors in the unit square with $n \leq 5$ surrounded by a hyperbola $\left|x_1 - \frac{m_1}{n}\right| \cdot \left|x_2 - \frac{m_2}{n}\right| \leq \frac{c^2}{n^3}$ with $c = \frac{1}{10}$.

Rational vectors surrounded by hyperbolas

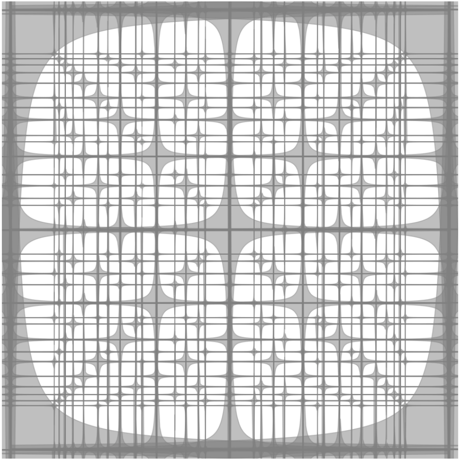

Figure: Rational vectors in the unit square with $n \leq 10$ surrounded by a hyperbola $\left|x_1 - \frac{m_1}{n}\right| \cdot \left|x_2 - \frac{m_2}{n}\right| \leq \frac{c^2}{n^3}$ with $c = \frac{1}{10}$.

Rational vectors surrounded by hyperbolas

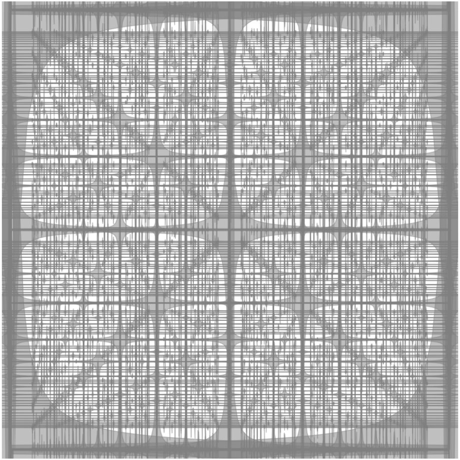

Figure: Rational vectors in the unit square with $n \leq 20$ surrounded by a hyperbola $\left|x_1 - \frac{m_1}{n}\right| \cdot \left|x_2 - \frac{m_2}{n}\right| \leq \frac{c^2}{n^3}$ with $c = \frac{1}{10}$.

Geometric interpretation of Littlewood's conjecture

For every $\mathbf{x} \in \mathbb{R}^2$ denote

$$u(\mathbf{x}) = \begin{pmatrix} 1 & & x_1 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}$$

and note that

$$u(\mathbf{x}) \begin{pmatrix} m_1 \\ m_2 \\ n \end{pmatrix} = \begin{pmatrix} nx_1 + m_1 \\ nx_2 + m_2 \\ n \end{pmatrix}.$$

Geometric interpretation of Littlewood's conjecture (cntd.)

The Littlewood conjecture says that for every $\mathbf{x} \in \mathbb{R}^2$ and every $c > 0$ there are $0 \neq n \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{Z}^2$ such that

$$n \left| nx_1 + m_1 \right| \left| nx_2 + m_2 \right| \leq c \,.$$

Equivalently, the lattice $u(\mathbf{x})\mathbb{Z}^3$ has vectors whose last coordinate is nonzero in the hyperbola given by

$$\left\{ \mathbf{y} \in \mathbb{R}^3 \,:\, |y_1 y_2 y_3| \leq c \right\}.$$

This in turn is equivalent to

$$u(\mathbf{x})\mathbb{Z}^3 \cap \left\{ \mathbf{y} \in \mathbb{R}^3 \,:\, |y_1 y_2 y_3| \leq c,\ |y_1|, |y_2| < 1 \right\} \neq \{\mathbf{0}\}\,.$$

Lattice and hyperbola in $\mathbb{R}^2$

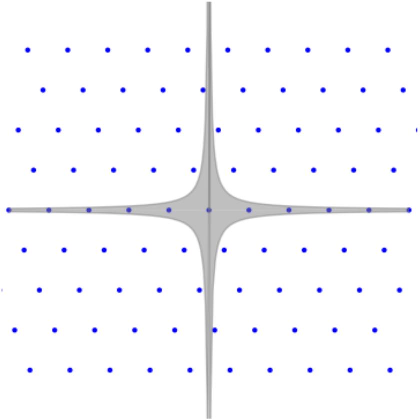

Figure: A lattice $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mathbb{Z}^2$ with no nonzero vectors $\mathbf{y}$ satisfying $|y_1 y_2| \leq c$ and $|y_1| < 1$.

Lattice and hyperbola in $\mathbb{R}^2$

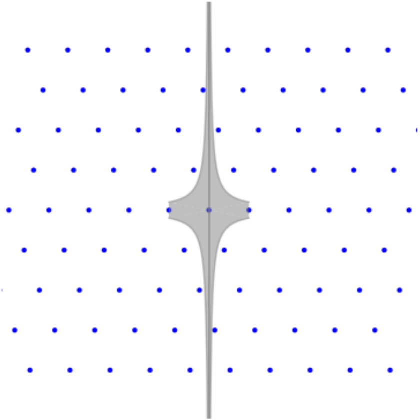

Figure: A lattice $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mathbb{Z}^2$ with no nonzero vectors $\mathbf{y}$ satisfying $|y_1 y_2| \leq c$ and $|y_1| < 1$.

Dynamical formulation of Littlewood's conjecture

For every $\mathbf{t} \in \mathbb{R}^2$ denote

$$a(\mathbf{t}) = \begin{pmatrix} e^{t_1} & & \\ & e^{t_2} & \\ & & e^{-(t_1+t_2)} \end{pmatrix}.$$

For $0 < c < 1$, if

$$u(\mathbf{x})\mathbb{Z}^3 \cap \left\{ \mathbf{y} \in \mathbb{R}^3 \,:\, |y_1 y_2 y_3| \leq c^3,\ |y_1|, |y_2| < 1 \right\} = \{\mathbf{0}\}.$$

then as long as $t_1, t_2 \geq 0$ the lattice

$$a(\mathbf{t})u(\mathbf{x})\mathbb{Z}^3$$

has no nonzero vectors whose supremum norm is smaller than c.

Dynamical formulation of Littlewood's conjecture (cntd.)

On the other hand, if $u(\mathbf{x})\mathbb{Z}^3$ has a nonzero vector in

$$Y = \left\{ \mathbf{y} \in \mathbb{R}^3 \,:\, |y_1 y_2 y_3| \le c^5,\ |y_1|, |y_2| < 1 \right\}$$

then it must also have a nonzero vector in

$$Y' = \left\{ \mathbf{y} \in \mathbb{R}^3 \,:\, |y_1 y_2 y_3| \le c^3,\ |y_1|, |y_2| \le c \right\}.$$

Dynamical formulation of Littlewood's conjecture (cntd.)

Indeed, if $u(\mathbf{x})\begin{pmatrix}\mathbf{m}\\ n\end{pmatrix} \in Y$ then

$$n\,|nx_1+m_1|\,|nx_2+m_2| \le c^5.$$

Assume WLOG that $|nx_2+m_2| > c$. Then $n\,|nx_1+m_1| < c^4$.
By Dirichlet's theorem there exist integers k, m_2' such that

$$\begin{aligned}&|k(nx_2)+m_2'| < c\\ &0 < k < 1/c\end{aligned}$$

So $n' = kn$ and $m_1' = km_1$ satisfy

$$n'\left|n'x_1+m_1'\right| = k^2\left(n\,|nx_1+m_1|\right) < c^2\,.$$

So $|n'x_1+m_1'| < c$ and $n'\,|n'x_1+m_1'|\,|n'x_2+m_2'| < c^3$, which verifies that $u(\mathbf{x})\begin{pmatrix}\mathbf{m}'\\ n'\end{pmatrix} \in Y'$.

Therefore, there exist $t_1, t_2 \geq 0$ such that $a(\mathbf{t})u(\mathbf{x})\mathbb{Z}^3$ has a nonzero vector in

$$[-c, c]^3 = \left\{\mathbf{y} \in \mathbb{R}^3 \,:\, |y_1|, |y_2|, |y_3| \leq c\right\}.$$

Indeed, if $\mathbf{0} \neq \mathbf{y} \in Y' \cap u(\mathbf{x})\mathbb{Z}^3$ choose $t_1, t_2 \geq 0$ so that $e^{t_1}|y_1| = e^{t_2}|y_2| = c$. Then:

$$e^{-(t_1+t_2)}|y_3| = \frac{1}{c^2}|y_1 y_2 y_3| \leq c\,,$$

so $\mathbf{0} \neq a(\mathbf{t})\mathbf{y} \in [-c, c]^3$.

Dynamical formulation of Littlewood's conjecture (cntd.)

$\mathbf{x} \in \mathbb{R}^2$ *satisfies Littlewood if and only if* $a(\mathbf{t})u(\mathbf{x})\mathbb{Z}^3$ *has short nonzero vectors as* $(t_1, t_2) \to +\infty$.

The topology on the space of lattices

Let $G := \mathrm{SL}_3(\mathbb{R})$, $\Gamma := \mathrm{SL}_3(\mathbb{Z})$, $X := G/\Gamma$. The action $G \curvearrowright X$ is identified with the linear action of G on the space of lattices via the map

$$[g] \to g\mathbb{Z}^3 \,.$$

For every $\varepsilon > 0$ define the set

$$K_\varepsilon := \left\{[g] \in X \;:\; \|\mathbf{v}\| \geq \varepsilon \text{ for any } \mathbf{v} \in g\mathbb{Z}^3 \setminus \{\mathbf{0}\}\right\} \,.$$

Mahler's compactness criterion

$S \subseteq X$ is unbounded if and only if $S \not\subseteq K_\varepsilon$ for every $\varepsilon > 0$.

A Dani correspondence for Littlewood

$\mathbf{x} \in \mathbb{R}^2$ *satisfies Littlewood if and only if* $\{a(\mathbf{t})\,[u(\mathbf{x})] \,:\, t_1, t_2 \geq 0\}$ *is unbounded.*

Conjecture (Littlewood for cones)

For any $\mathbf{t}, \mathbf{t}' \geq 0$ *satisfying* $\operatorname{span} \mathbf{t} \neq \operatorname{span} \mathbf{t}'$, *every* $\mathbf{x} \in \mathbb{R}^2$ *satisfies that the orbit* $\left\{a(\mathbf{t})^k a(\mathbf{t}')^l [u(\mathbf{x})] \,:\, k, l \geq 0\right\}$ *is unbounded.*

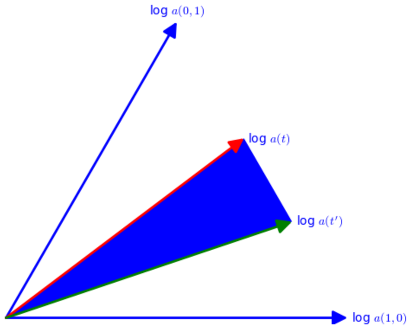

Figure: A two parameter subgroup generated by two diagonal matrices

Rational vectors surrounded by trimmed hyperbolas

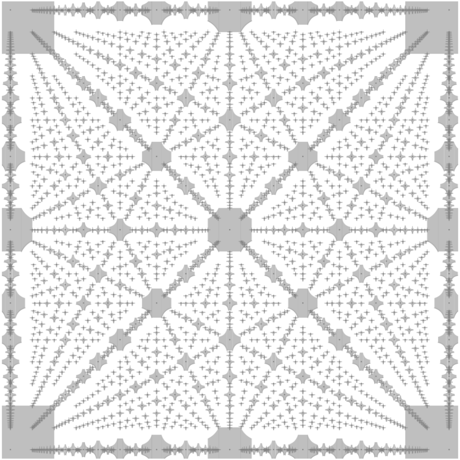

Figure: Rational vectors in the unit square with $n \leq 20$ surrounded by a hyperbola $\left|x_1 - \frac{m_1}{n}\right| \cdot \left|x_2 - \frac{m_2}{n}\right| \leq \frac{c^2}{n^3}$ with $c = \frac{1}{10}$.

Well approximable numbers satisfy Littlewood

Definition

$x \in \mathbb{R}$ is well approximable if $\inf_{n\in\mathbb{N}} n\,|\langle nx\rangle| = 0$. Otherwise, x is badly approximable.

Recall that x is well approximable if and only if upon writing

$$x = [a_0; a_1, a_2, \ldots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2+\cdots}\cdots}$$

the integers a_0 and $a_1, a_2, a_3, \ldots \geq 1$ satisfy

$$\sup_{n\in\mathbb{N}} a_n = \infty\,.$$

If $x_1 \in \mathbb{R}$ is well approximable then (x_1, x_2) satisfies Littlewood for every $x_2 \in \mathbb{R}$.

Gallagher's theorem in simultaneous approximation

Theorem (Spencer 1942)

For any $\alpha > 0$*, almost every* $\mathbf{x} \in \mathbb{R}^2$ *satisfies*

$$\inf_{n \in \mathbb{N}} n^{1+\alpha} |\langle n x_1 \rangle| |\langle n x_2 \rangle| > 0.$$

Theorem (Gallagher 1962)

For any monotonic function $\psi : \mathbb{N} \to (0, \infty)$*, the set*

$$\left\{ \mathbf{x} \in \mathbb{R}^2 \ : \ |\langle n x_1 \rangle| |\langle n x_2 \rangle| < \psi(n) \ \textit{i.o.} \ \right\}.$$

has full or zero Lebesgue measure if the sum

$$\sum_{n \in \mathbb{N}} \psi(n) \log n$$

diverges or converges, respectively.

Gallagher's theorem in simultaneous approximation (cntd.)

Corollary (Gallagher 1962)

Almost every $\mathbf{x} \in \mathbb{R}^2$ *satisfies*

$$\inf_{n\in\mathbb{N}} n \log^2 n \, |\langle nx_1 \rangle| \, |\langle nx_2 \rangle| = 0 \, .$$

Linearly dependent pairs over the rationals

Definition

Two real numbers x_1 and x_2 are linearly dependent over the rationals if there exist $m \in \mathbb{N}$ and $\mathbf{n} \in \mathbb{Z}^2$ such that

$$m + n_1 x_1 + n_2 x_2 = 0\,.$$

If x_1 and x_2 are linearly dependent over the rationals then (x_1, x_2) satisfies Littlewood.

Linearly independent pairs over the rationals

Let $\mathbf{x} \in \mathbb{R}^2$ be such that $[\mathbb{Q}[x_1, x_2] : \mathbb{Q}] = 3$.

Theorem (Cassels – Swinnerton-Dyer 1956)

$\mathbf{x}$ *satisfies Littlewood.*

In fact, a slightly stronger result holds:

Theorem (Peck 1961)

$$\liminf_{n \in \mathbb{N}} n \log n \, |\langle n x_1 \rangle| \, |\langle n x_2 \rangle| < \infty \,.$$

Problem

Is it true that

$$\inf_{n \in \mathbb{N}} n \log n \, |\langle n x_1 \rangle| \, |\langle n x_2 \rangle| > 0 \,.$$

Badly approximable fibres

Theorem (Pollington – Velani 2000)

If $x_1 \in \mathbb{R}$ is badly approximable then the set of all badly approximable x_2 satisfying

$$\liminf_{n\in\mathbb{N}} n\log n\,|\langle nx_1\rangle|\,|\langle nx_2\rangle| \leq 1$$

has full Hausdorff dimension.

Gallagher on badly approximable fibres

Theorem (Chow 2017, Beresnevich–Haynes–Velani 2015)

If $x_1 \in \mathbb{R}$ *is badly approximable,* $\psi : \mathbb{N} \to (0, \infty)$ *monotonic, then the set*

$$\{x_2 \in \mathbb{R} \, : \, |\langle nx_1 \rangle| \, |\langle nx_2 \rangle| < \psi(n) \text{ i.o. } \}$$

has full or zero Lebesgue measure if the series

$$\sum_{n \in \mathbb{N}} \psi(n) \log n$$

diverges or converges, respectively.

Corollary (Chow 2017)

If $x_1 \in \mathbb{R}$ *is badly approximable then almost every* x_2 *satisfies*

$$\inf_{n \in \mathbb{N}} n \log^2 n \, |\langle nx_1 \rangle| \, |\langle nx_2 \rangle| = 0 \, .$$

Combinatorial entropy

Definition

Let S be a finite set. The combinatorial entropy of a sequence over S is the exponential growth rate of the number of different blocks.

Formally, for $\mathbf{a} \in S^{\mathbb{N}}$ and $j \in \mathbb{N}$, a *j-block of* $\mathbf{a}$ is a tuple $\mathbf{b} \in S^j$ such that there exists $i \geq 0$ for which

$$(b_1, \ldots, b_j) = (a_{i+1}, \ldots, a_{i+j}) \; .$$

Let $C_j(\mathbf{a})$ be the number of different n blocks of $\mathbf{a}$. Then the combinatorial entropy of $\mathbf{a}$ is

$$h(\mathbf{a}) := \lim_{j \to \infty} \frac{\log C_j(\mathbf{a})}{j} \, .$$

Fibres with positive combinatorial entropy

Theorem (Einsiedler – Katok – Lindenstrauss 2006)

If $x_1 = [a_1, a_2, a_3, \ldots]$ *is badly approximable and* $h(\mathbf{a}) > 0$ *then for every* x_2 *the pair* (x_1, x_2) *satisfies Littlewood.*

Littlewood with extra logs

Theorem (Moshchevitin, Moshchevitin–Bugeaud 2009)

For every badly approximable number x_1 the set

$$\left\{ x_2 \in \mathbb{R} \ : \ \inf_{n \in \mathbb{N}} n \log^2 n \, |\langle n x_1 \rangle| \, |\langle n x_2 \rangle| > 0 \right\}$$

has full Hausdorff dimension.

Theorem (Badziahin 2012)

For every badly approximable number x_1 the set

$$\left\{ x_2 \in \mathbb{R} \ : \ \inf_{n \in \mathbb{N}} n \log n \log\log n \, |\langle n x_1 \rangle| \, |\langle n x_2 \rangle| > 0 \right\}$$

has full Hausdorff dimension.

Best upper bound for Littlewood

For any $x \in \mathbb{R}$, every $n \in \mathbb{N}$ satisfies $|\langle nx \rangle| \leq \frac{1}{2}$ and $\inf_{n \in \mathbb{N}} n \, |\langle nx \rangle| < \frac{1}{\sqrt{5}}$.
Therefore every $\mathbf{x} \in \mathbb{R}^2$ satisfies

$$\inf_{n \in \mathbb{N}} n \, |\langle nx_1 \rangle| \, |\langle nx_2 \rangle| < \frac{1}{2\sqrt{5}} \, .$$

Badziahin improved this bound using a computer:

Theorem (Badziahin 2016)

Every $\mathbf{x} \in \mathbb{R}^2$ *satisfies* $\inf_{n \in \mathbb{N}} n \, |\langle nx_1 \rangle| \, |\langle nx_2 \rangle| < \frac{1}{19}$.

The p-adic Littlewood conjecture

The following problem was suggested by De-Mathan and Teulié in 2004 as an analogue of the Littlewood conjecture: Fix a prime p. For a nonzero integer n let its p-adic norm be

$$|n|_p = p^{-\max\left\{k \geq 0 \,:\, p^k | n\right\}}.$$

The p-adic Littlewood conjecture

Does every $x \in \mathbb{R}$ satisfy

$$\inf_{n>0} n \left|\langle nx \rangle\right| |n|_p = 0\ ?$$

An equivalent formulation for p-adic Littlewood

Equivalently, writing $n = kp^l$ gives:

The p-adic Littlewood conjecture (equivalent form.)

Does every $x \in \mathbb{R}$ satisfy

$$\inf_{k \in \mathbb{N},\, l \geq 0} k \left| \left\langle kp^l x \right\rangle \right| = 0 \text{ ?}$$

Examples for $p = 2$

- $\varphi = [1; 1, 1, 1, 1, \dots]$.
- $2\varphi = [3; 4, 4, 4, 4, \dots]$.
- $4\varphi = [6; 2, 8, 2, 8, \dots]$.
- $8\varphi = [12; 1, 16, 1, 16, \dots]$.
- $16\varphi = [25; 1, 7, 1, 34, 1, 7, 1, 34, \dots]$.
- ...

Examples for $p = 2$ (cont.)

- $\sqrt{2} = [1; 2, 2, 2, \ldots]$.
- $2\sqrt{2} = [2; 1, 4, 1, 4, \ldots]$.
- $4\sqrt{2} = [5; 1, 1, 1, 10, \ldots]$.
- $8\sqrt{2} = [11; 3, 5, 3, 22, \ldots]$.
- $16\sqrt{2} = [22; 1, 1, 1, 2, 6, 11, 6, 2, 1, 1, 1, 44, \ldots]$.
- ...

An examples for $p = 3$

- $\varphi = [1; 1, 1, 1, \ldots]$.
- $3\varphi = [4; 1, 5, 1, 5, \ldots]$.
- $9\varphi = [14; 1, 1, 3, 1, 1, 19, \ldots]$.
- $27\varphi = [43; 1, 2, 5, 6, 1, 1, 11, 1, 1, 6, 5, 2, 1, 59 \ldots]$.
- $81\varphi = [131; 16, 2, 5, 1, 3, 5, 1, 1, 2, 1, 1, 8, 1, 19, 4, 2, 1, 2, 1, 1, 1, 1, 35$
  $, 1, 1, 1, 1, 2, 1, 2, 4, 19, 1, 8, 1, 1, 2, 1, 1, 5, 3, 1, 5, 2, 16, 181, \ldots]$.
- ...

Dynamics with two parameters

Let $G : (0,1) \to [0,1)$ be the *Gauss map* defined by

$$G(x) = \left\{ \frac{1}{x} \right\}.$$

The p-adic Littlewood conjecture (equivalent form.)

Every $x \in \mathbb{R} \setminus \mathbb{Q}$ satisfies

$$0 \in \overline{\{G^j p^k x \mod 1 \, : \, j, k \geq 0\}}.$$

Theorem (Furstenberg 1967)

If $p, q \in \mathbb{N}$ satisfy $\frac{\log p}{\log q} \notin \mathbb{Q}$ then every $x \in \mathbb{R} \setminus \mathbb{Q}$ satisfies

$$0 \in \overline{\{p^j q^k x \mod 1 \, : \, j, k \geq 0\}}.$$

Dynamical formulation of p-adic Littlewood

The usual Dani correspondence ties between approximation of a real number x and the orbit

$$\begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mathbb{Z}^2.$$

Is there a homogeneous space $X = G/\Gamma$ with a linear action which is equivalent to passing from

$$\left[\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right]$$

to

$$\left[\begin{pmatrix} 1 & px \\ & 1 \end{pmatrix} \right] ?$$

Take $G := \mathrm{PGL}_2(R) \times \mathrm{PGL}_2(\mathbb{Q}_p)$ and $\Gamma := \mathrm{PGL}_2\left(\mathbb{Z}_p\left[\frac{1}{p}\right]\right)$ embedded diagonally in G. Then

$$\left(\begin{bmatrix} p & \\ & 1 \end{bmatrix}, \begin{bmatrix} p & \\ & 1 \end{bmatrix}\right)\left[\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right)\right] =$$

$$\left[\left(\begin{pmatrix} p & px \\ & 1 \end{pmatrix}, \begin{pmatrix} p & \\ & 1 \end{pmatrix}\right)\right]\left(\begin{bmatrix} 1/p & \\ & 1 \end{bmatrix}, \begin{bmatrix} 1/p & \\ & 1 \end{bmatrix}\right) =$$

$$\left[\left(\begin{pmatrix} 1 & px \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right)\right]$$

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$$\left(\begin{bmatrix} p & \\ & 1 \end{bmatrix}, \begin{bmatrix} p & \\ & 1 \end{bmatrix}\right)\left[\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right)\right] =$$

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$$\left[\left(\begin{pmatrix} 1 & px \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\right)\right]$$

A Dani correspondence for p-adic Littlewood

Theorem (Einsiedler–Kleinbock 2005)

$x \in \mathbb{R}$ satisfies the p-adic Littlewood conjecture if and only if

$$\left\{ \left(\begin{bmatrix} e^t & \\ & 1 \end{bmatrix}, \begin{bmatrix} p^{-k} & \\ & 1 \end{bmatrix} \right) \left[\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) \right] : t \geq 0,\ e^t \geq p^k \right\}$$

is unbounded.

Well approximable numbers satisfy p-adic Littlewood

If $x \in \mathbb{R}$ is well approximable then

$$\inf_{n\in\mathbb{N}} n\,|\langle nx\rangle| = 0\,.$$

Since $|n|_p \leq 1$ for every $n \in \mathbb{N}$ this implies that

$$\inf_{n\in\mathbb{N}} n\,|\langle nx\rangle|\,|n|_p = 0\,.$$

Gallagher-type theorem for simultaneously real and p-adic approximation

Theorem (Bugeaud–Haynes–Velani 2011)

For any monotonic function $\psi : \mathbb{N} \to (0, \infty)$*, the set*

$$\left\{ x \in \mathbb{R} \, : \, |\langle nx \rangle| \, |n|_p < \psi(n) \text{ i.o. } \right\}.$$

has full or zero Lebesgue measure if the sum

$$\sum_{n \in \mathbb{N}} \psi(n) \log n$$

diverges or converges, respectively.

Corollary

Almost every $x \in \mathbb{R}$ *satisfies* $\inf_{n \in \mathbb{N}} n \log^2 n \, |\langle nx \rangle| \, |n|_p = 0$.

Quadratic irrationals satisfy p-adic Littlewood

Let x be a quadratic irrational.

Theorem (De-Mathan – Teulié 2004)

x satisfies p-adic Littlewood.

In fact, a slightly stronger result holds:

Theorem (De-Mathan – Teulié 2004, Zorin – Bengoechea 2014)

$$\liminf_{n\in\mathbb{N}} n \log n \, |\langle nx \rangle| \, |n|_p < \infty \, .$$

Problem

Is it true that

$$\inf_{n\in\mathbb{N}} n \log n \, |\langle nx \rangle| \, |n|_p > 0 \; ?$$

Very high or very low complexity imply p-adic Littlwood

Let $x = [a_1, a_2, a_3, \ldots]$ be badly approximable.

Theorem (Einsiedler – Kleinbock 2005)

If $h(\mathbf{a}) > 0$ then x satisfies p-adic Littlewood.

Theorem (Badziahin–Bugeaud–Einsiedler–Kleinbock 2015)

If

$$\sup_{j \geq 1} C_j(\mathbf{a}) - j < \infty$$

then x satisfies p-adic Littlewood.

p-adic Littlewood with extra logs

Theorem (Badziahin–Velani 2011)

The set

$$\left\{ x \in \mathbb{R} \, : \, \inf_{n \in \mathbb{N}} n \log n \log \log n \, |\langle nx \rangle| \, |n|_p > 0 \right\}$$

has full Hausdorff dimension.

Best upper bound for p-adic Littlewood

Any $x \in \mathbb{R}$ satisfies $\inf_{n\in\mathbb{N}} n\,|\langle nx\rangle| < \frac{1}{\sqrt{5}}$, so:

Theorem

Every $x \in \mathbb{R}$ *satisfies* $\inf_{n\in\mathbb{N}} n\,|\langle nx\rangle|\,|n|_p < \frac{1}{\sqrt{5}}$.

Badziahin improved this bound for $p = 2$ using a computer:

Theorem (Badziahin 2016)

Every $x \in \mathbb{R}$ *satisfies* $\inf_{n\in\mathbb{N}} n\,|\langle nx\rangle|\,|n|_2 < \frac{1}{9}$.

Recently, John Blackman (2020) reported on new bounds achieved by a different approach:

Prime p	2	3	5	7	11	13	17	19
$m_{PLC}(p)$	$< \frac{1}{15}$	$< \frac{1}{9}$	$< \frac{1}{9}$	$< \frac{1}{4}$	$< \frac{1}{5}$	$< \frac{1}{6}$	$< \frac{1}{7}$	$< \frac{1}{7}$

Figure: Current records for upper bounds for some small primes

Diophantine approximation over function fields

- q a fixed prime power.
- $\mathbb{F}_q$ the field with q elements.
- $\mathbb{F}_q[t]$ the ring of polynomials with coefficients in $\mathbb{F}_q$.
- $\mathbb{F}_q(t)$ the field of rational functions in variable t.
- $\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ the field of all Laurent series with finitely many nonzero coefficients for positive powers of t

$$\theta = \theta_{-h}t^h + \ldots + \theta_0 + \theta_1 t^{-1} + \theta_2 t^{-2} + \ldots$$

where $\theta_i \in \mathbb{F}_q$ for every $i \geq -h$.

The analogy between $\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ and $\mathbb{R}$

For $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$:

- The *polynomial part* and *fractional part* are

$$\theta = \underbrace{\theta_{-h}t^h + \ldots + \theta_0}_{[\theta]} + \underbrace{\theta_1 t^{-1} + \theta_2 t^{-2} + \ldots}_{\langle\theta\rangle}$$

- The *degree* and *absolute value* are

$$\deg \theta = h$$
$$|\theta| = q^{\deg \theta}.$$

- The absolute value of the fractional part is the distance to the polynomials

$$|\langle\theta\rangle| = \operatorname{dist}\left(\theta, \mathbb{F}_q[t]\right).$$

Getting used to the notation

Let $\theta = \theta_{-h} t^h + \ldots + \theta_0 + \theta_1 t^{-1} + \theta_2 t^{-2} + \ldots \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$.

- When does

$$|\langle \theta \rangle| < q^{-\ell} \text{ ?}$$

- Precisely when

$$\theta_1 = \ldots = \theta_\ell = 0.$$

Getting used to the notation

Let $\theta = \theta_{-h}t^h + \ldots + \theta_0 + \theta_1 t^{-1} + \theta_2 t^{-2} + \ldots \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$.

- When does
$$|\langle\theta\rangle| < q^{-\ell} \text{ ?}$$
- Precisely when
$$\theta_1 = \ldots = \theta_\ell = 0.$$

The approximation problem

Given $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ and $\ell \geq 0$, are there solutions $0 \neq N \in \mathbb{F}_q[t]$ to the inequality

$$|N|\,|\langle N\theta \rangle| < q^{-\ell}\ ?$$

The approximation problem (cont.)

Recall that $\langle\theta\rangle = \theta_1 t^{-1} + \theta_2 t^{-2} + \ldots$ and let $N = n_0 + n_1 t + \ldots + n_h t^h$. Then:

$$\begin{aligned}\langle N\theta\rangle = &\left(\theta_1 n_0 + \ldots \theta_{h+1} n_h\right) t^{-1} + \\ &\left(\theta_2 n_0 + \ldots \theta_{h+2} n_h\right) t^{-2} + \\ &\ldots\end{aligned}$$

The approximation problem by a linear recurrence

If $n_h \neq 0$ then $|N| = q^h$, so $|N|\,|\langle N\theta\rangle| < q^{-\ell}$ if and only if $|\langle N\theta\rangle| < q^{-(h+\ell)}$, which is equivalent to

$$\sum_{i=0}^{h} \theta_{j+i} n_i = 0 \text{ for every } 1 \leq j \leq h + \ell.$$

The approximation problem in matrix notation

In other words, if $N \in \mathbb{F}_q[t]$ with $\deg N = h$ is a solution to $|N|\,|\langle N\theta \rangle| < q^{-\ell}$, then

$$h+\ell \text{ rows} \left\{ \overbrace{\begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \cdots & \theta_{h+1} \\ \theta_2 & \theta_3 & \ddots & & \vdots \\ \theta_3 & \ddots & & & \\ \vdots & & & & \\ \theta_{h+\ell} & \cdots & & & \theta_{2h+\ell} \end{pmatrix}}^{h+1 \text{ columns}} \right. \begin{pmatrix} n_0 \\ n_1 \\ \vdots \\ n_h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The approximation problem using determinants

If the inequality $|N|\,|\langle N\theta\rangle| < q^{-\ell}$ has a solution with $\deg N = h$ then

$$\begin{vmatrix} \theta_1 & \theta_2 & \cdots & \theta_{h+j} \\ \theta_2 & \iddots & \iddots & \theta_{h+j+1} \\ \vdots & \iddots & \iddots & \vdots \\ \theta_{h+j} & \theta_{h+j+1} & \cdots & \theta_{2h+2j-1} \end{vmatrix} = 0.$$

for every $1 \le j \le \ell$. In fact, the other direction also holds.

The Hankel matrix of a sequence

For a sequence $\theta_1, \theta_2, \ldots \in \mathbb{F}_q$ its *Hankel matrix* is

$$H_\theta = \begin{pmatrix} \theta_1 & \theta_2 & \theta_3 & \ddots \\ \theta_2 & \theta_3 & \ddots & \ddots \\ \theta_3 & \ddots & \ddots & \\ \ddots & \ddots & & \end{pmatrix}.$$

Theorem (Folklore)

Assume $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ and $\ell \geq 0$. Then $|N|\,|\langle N\theta\rangle| < q^{-\ell}$ has a nonzero solution if and only if H_θ has ℓ **consecutive leading principal minors** *that vanish.*

The Littlewood conjecture over function fields

The following problem was raised by Davenport and Lewis in 1963:

Problem (The Littlewood conjecture over function fields)

Is it true that every $\theta, \varphi \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ *satisfy*

$$\inf_{0 \neq N \in \mathbb{F}_q[t]} |N| \, |\langle N\theta \rangle| \, |\langle N\varphi \rangle| = 0 \ ?$$

A Dani correspondence for the Littlewood conjecture over function fields

Let $G = \mathrm{SL}_3\left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)\right)$, $\Gamma = \mathrm{SL}_3\left(\mathbb{F}_q[t]\right)$, $X = G/\Gamma$.

$\boldsymbol{\theta} \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)^2$ *satisfies LCFF if and only if*

$$\left\{ \begin{pmatrix} t^k & & \\ & t^l & \\ & & t^{-(k+l)} \end{pmatrix} \left[\begin{pmatrix} 1 & & \theta_1 \\ & 1 & \theta_2 \\ & & 1 \end{pmatrix} \right] : k, l \geq 0 \right\}$$

is unbounded.

The t-adic Littlewood conjecture

The following problem was suggested by De-Mathan and Teulié in 2004 together with its real counterpart:

Problem (The t-adic Littlewood conjecture (equivalent form.))

Is it true that every $\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ satisfies

$$\inf_{0 \neq N \in \mathbb{F}_q[t],\, k \geq 0} |N| \left| \left\langle N t^k \theta \right\rangle \right| = 0 \text{ ?}$$

A Dani correspondence for t-adic Littlewood

Let $G = \mathrm{SL}_2\left(\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)\right) \times \mathrm{SL}_2\left(\mathbb{F}_q\left((t)\right)\right)$, $\Gamma = \mathrm{SL}_2\left(\mathbb{F}_q\left[t, t^{-1}\right]\right)$, $X = G/\Gamma$.

$\theta \in \mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$ *satisfies the t-adic Littlewood conjecture if and only if*

$$\left\{ \left(\begin{bmatrix} t^k & \\ & 1 \end{bmatrix}, \begin{bmatrix} t^{-l} & \\ & 1 \end{bmatrix} \right) \left[\left(\begin{pmatrix} 1 & \theta \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right) \right] : k \geq 0,\ k \geq l \right\}$$

is unbounded.

The t-adic Littlewood conjecture via Hankel determinants

Problem (The t-adic Littlewood conjecture)

Is it true that for every sequence $\theta_1, \theta_2, \ldots \in \mathbb{F}_q$ and every $\ell \geq 0$, H_θ has ℓ **consecutive adjacent minors** *that vanish?*

In other words, is it true that for every $\theta_1, \theta_2, \ldots \in \mathbb{F}_q$ and every $\ell \geq 1$ there exist $h \geq 0$ and $k \geq 0$ such that every $1 \leq j \leq \ell$ satisfies

$$\begin{vmatrix} \theta_{k+1} & \theta_{k+2} & \cdots & \theta_{k+h+j} \\ \theta_{k+2} & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & \vdots \\ \theta_{k+h+j} & \cdots & \cdots & \theta_{k+2h+2j-1} \end{vmatrix} = 0 \ ?$$

The t-adic Littlewood conjecture is false

Theorem (Adiceam–Nesharim–Lunnon 2020)

There exists a sequence $\theta_1, \theta_2, \ldots \in \mathbb{F}_3$ such that every adjacent minor of H_θ is either nonzero or becomes nonzero when being added the following row and column.

Array of Hankel determinants with isolated zeros 100×200

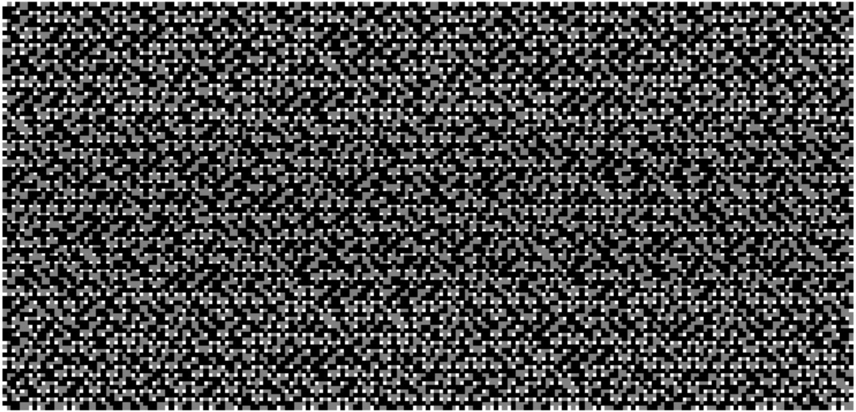

Figure: All $j \times j$ adjacent minors of H_θ with upper-left entry θ_i, $\downarrow i$, $\rightarrow j$. White=0, Gray=1, Black=2.

The paperfolding sequence

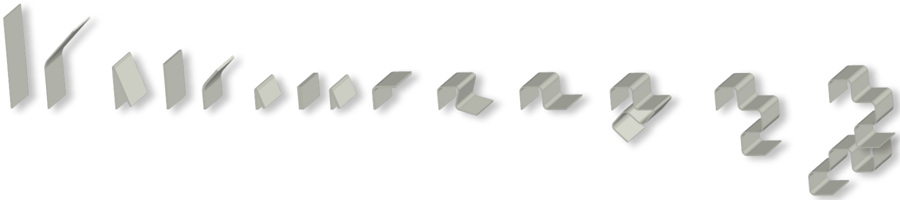

Figure: Folding a piece of paper

Hankel determinants of the paperfolding sequence

Let φ be the *paperfolding sequence*, defined by $\varphi_{2^{j-1}} = 0$ and $\varphi_i = 1 - \varphi_{2^j - i}$ for every $j \geq 1$ and $2^{j-1} < i < 2^j$.

$$\varphi = (0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1 \ldots)$$

Theorem (Adiceam–Nesharim–Lunnon 2020)

H_φ has no four consecutive adjacent minors that vanish mod 3.

The paperfolding Hankel determinants 100×200

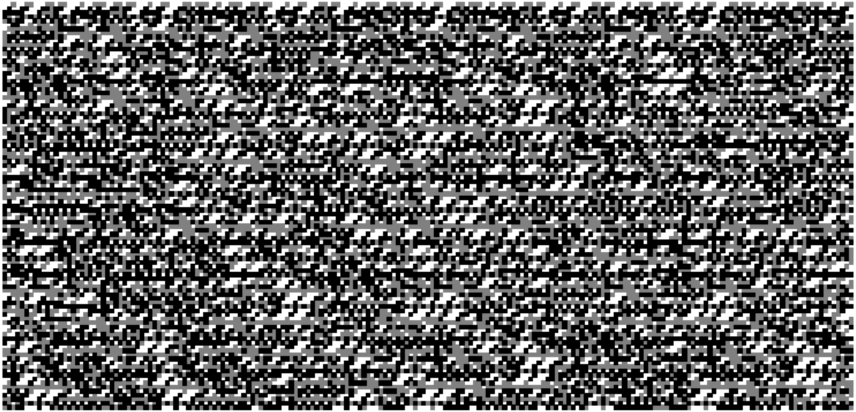

Figure: All $j \times j$ adjacent minors of H_φ with upper-left entry θ_i, $\downarrow i$, $\rightarrow j$.
White=0, Gray=1, Black=2.

Summary

	Littlewood conjecture	p-adic Littlewood conjecture
$\mathbb{R}$	open	open
$\mathbb{F}_q\left(\left(\frac{1}{t}\right)\right)$	open	false for $q = 3$