VEECH SURFACES AND VEECH GROUPS – EXERCISES

Exercise 1. Let x be a translation surface, let $V \subset \mathbb{R}^2$ be the collection of developments of saddle connections of x, and let Γ be the Veech group of x. Show that Γ leaves V invariant (in its linear action on $\mathbb{R}^2$).

Exercise 2. Let x be a square tiled surface consists of a finite number of squares T_i with same size and some edge identification to make it connected, and assume that x has singular points. Show that for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $\gamma(x)$ is also a square tiled surface with the same number of squares (this statement was needed in the proof that square tiled surfaces are Veech surfaces). Prove that there are only 3 possible gluings for a square tiled surface consisting of three squares.

Exercise 2.5. Let x be the torus with one marked point. What is the set of saddle connection holonomies for x? Show that the Veech group of x $\mathrm{SL}_2(\mathbb{Z})$ (in the lecture it was shown that $\mathrm{SL}_2(\mathbb{Z})$ is a subgroup of the Veech group, show that there are no additional elements). *Hint:* use Ex. 1.

Exercise 3. Show that any billiard in a rational triangle has at least one closed billiard trajectory. Thus, first show that it suffices to find a closed path not going through singularities in a translation surface, and then find one.

Hint: Use trajectories hitting a side at a right angle.

Exercise 4. Let $a_1, \ldots, a_n$ be positive numbers, $I = [0, \sum a_i]$ and σ a permutation on n symbols. Let (I, T, μ) be an interval exchange transformation corresponding to this data as defined in class where μ is the Lebesgue measure on I. Let x be a translation surface constructed from this IET where $I \subset x$ traverses x horizontally. Let L_t be the straight line flow, $x \mapsto x + (0, t)$ in local coordinates. Let $R : I \to I$ be the first return map with respect to I, that is, for $x \in I$, $R(x) = L_{t_x}(x)$ where $t_x = \min\{t > 0 : L_{t_x}(x) \in I\}$. Show that $R = T$.

Exercise 5. Let x_{2n} be the regular $2n$-gon, with opposite sides identified (so that x_8 is the octagon discussed in the lecture). Compute the genus of x_{2n} and the number and cone angles of singular points.

Exercise 6. Prove the existence of a surface whose Veech group is the cyclic group

$$\left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \cong \mathbb{Z}.$$

Exercise 7. Let x be a translation surface and let $g \in G = \mathrm{SL}_2(\mathbb{R})$. Express gx in terms of the definition of a translation surface using an atlas of charts.

Solution: Suppose $(\varphi_\alpha, U_\alpha)$ are the charts in the atlas for x, where U_α is an open subset of $S \setminus \Sigma$ and $\varphi_\alpha : U_\alpha \to \mathbb{R}^2$ is a homeomorphism onto its image. Then the charts in the atlas for gx are $\{g \circ \varphi_\alpha, U_\alpha)$.