# Plan

Part I :

a) $\beta^X$ and heuristic.
b) finding the formula of $\beta^X$
c) Furstenberg measures

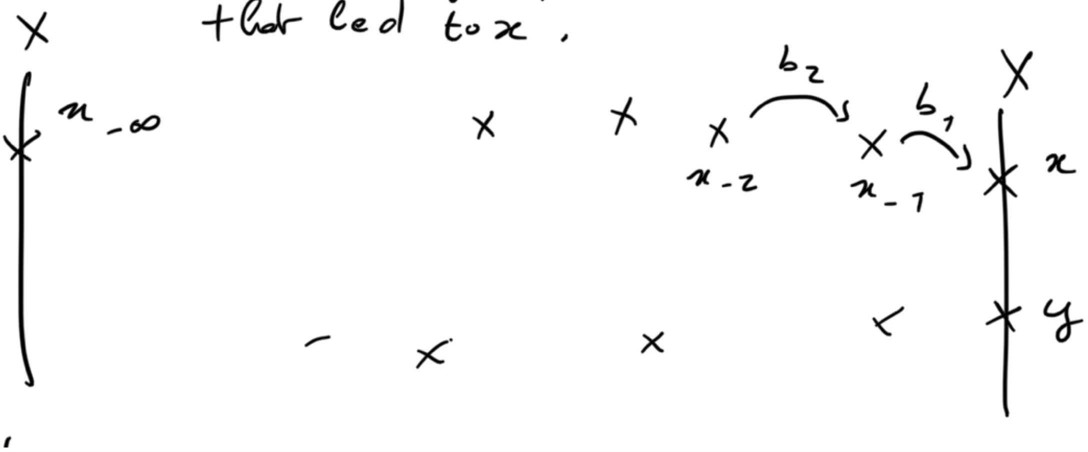
Part II : show that $\beta^X$ a.s. $\underline{\nu_b(W_b(x))} = 0$

a) $\nu$ non atomic
b) $\nu$ non atomic $\Rightarrow$ $\nu_b$ is a.s. non atomic
c) $\nu_b$ is a.s. non atomic $\Rightarrow$ $\nu_b(W_b(x)) = 0$

- $G$ Lie group, $\mu \in \mathcal{P}(G)$ compactly supp.
- $X = G/\Lambda$
- $G \curvearrowright X$
- $\nu \in \mathcal{P}(X)$ $\qquad \mu * \nu = \nu$
- $B = G^{\mathbb{N}^*} = \{(b_1, \dots, b_n, \dots)\}$. $\beta = \mu^{\otimes \mathbb{N}}$.
- $B^X := B \times X$ the space of past trajectories.
  $= \{ (b, x) \quad b \in B,\ x \in X \}$

where i am at $t=0$.

sequence of steps that led to $x$.



$t = -\infty$ $\qquad\qquad$ $t = 0$

$\nu$

$B^X$: the space of possible trajec for the RW from $t = -\infty$ to $t = 0$.

$\beta^X \in \mathcal{P}(B^X)$.

$\beta^X(A)$ is the probability that $A$ is realized by the trajectories of the RW from $t = -\infty$ to $t = 0$.

$$\begin{array}{ccc} B \times X & \xrightarrow{\pi_B} & B \\ \pi_X \downarrow & & \\ X & & \end{array}$$

$\pi_B, \pi_X$ Random variables.

1) $(\pi_B)_* \beta^X = \beta$.

we disregard when we ended up at $t = 0$

2) $[\beta^X(\pi_X \in A \mid \pi_B = b) \overset{=}{} := \nu_b(A)$ desintegration of $\beta^X$ on $\pi_B^{-1}(b)$

I pick at $t = -\infty$ point $x_{-\infty}$ with law $\nu$

I wonder the probability to be in $A$ at $t = 0$ If I apply $b$.

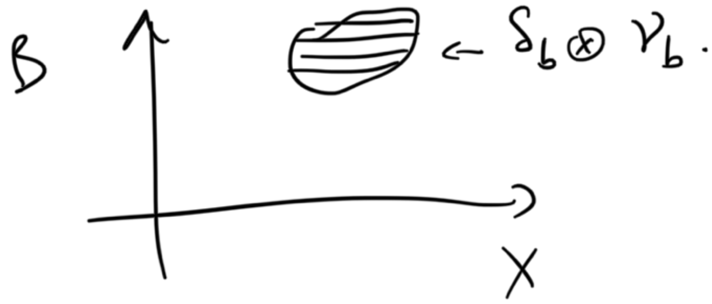
$$x_{\infty} \in (b_1 \dots b_\infty)^{-1} \cdot A$$

$$\nu_b(A) = \nu((b_1 \dots b_\infty)^{-1} A) = (b_1 \dots b_\infty)_* \nu(A)$$

More formally:

$$\nu_b = \lim_{n\to\infty} (b_1 \dots b_n)_* \nu .$$

- $$\beta^X = \int_B \delta_b \otimes \nu_b \, d\beta(b) .$$



Proposition: there is $b \mapsto \nu_b \in \mathcal{P}(X)$ st for $\beta$ ae $b \in B$ $(b_1 \dots b_n)_* \nu \underset{n\to\infty}{\longrightarrow} \nu_b$ (weak* top)

1) construct $\nu_b$.
2) Show the convergence.

for: for $\beta$ ae $b \in B$ $\forall f \in C_c(X)$

$*$ $(b_1 \dots b_n)_* \nu(f)$ converges.

by Riesz $\nu_b$ by duality of $f \mapsto \lim (b_1 \dots b_n)_* \nu(f)$

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$(\mathcal{S}_n)_{n\in\mathbb{N}}$ filtration of $\mathcal{B}$, $\sigma$ algebra of $B$.

- $\mathcal{S}_n$ is the smallest $\sigma$-algebra that makes the $n^{th}$ first projections $B \to G$ measurable,
- $\mathcal{S}_n$ generated by $B_1 \times \dots \times B_n \times B$ $B_i \in \mathcal{B}(G)$

the info contained $\mathcal{F}_n$ is exactly the last $n$ steps of the RW.

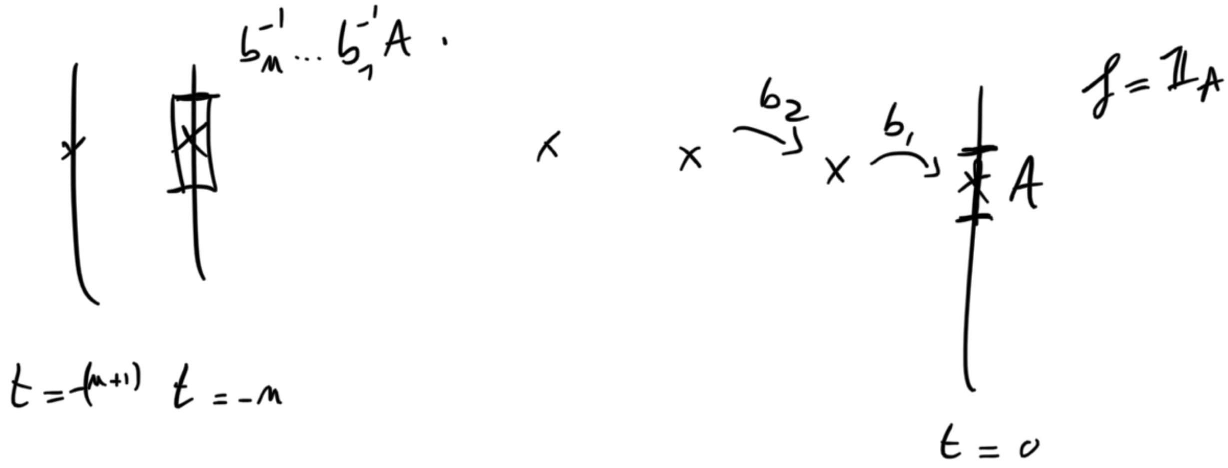
let $f \in C_c(X)$

$$X_n(b) = (b_1 \dots b_n)_* \nu(f).$$

claim. $(X_n)_{n \in \mathbb{N}^*}$ is a Martingale.

1) $X_n$ is $\mathcal{F}_n$-measurable : ok because depends only on the last $n$ steps.

2) a.s. $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n](b) = X_n(b)$.




$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n](b)$ = Expected mass of the set of points that end up in $A$ at $t=0$ if the RW started $t = -(n+1)$, knowing the last $n$ steps are $b_n, \dots, b_1$.

$$\int_G g_*\nu(b_n^{-1} \dots b_1^{-1} A)\, d\mu(g) = \nu(b_n^{-1} \dots b_1^{-1} A)$$

$$= (b_1 \dots b_n)_* \nu(A) = X_n(b)$$

Show $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$.

for any $A = B_1 \times \dots \times B_n \times B \qquad B_i \in \mathcal{B}(G)$

$$\int_A X_{n+1}(b)\, d\beta(b) = \int_A X_n(b)\, d\beta(b).$$

$(A = B_1 \times \dots \times B_n \times G \times B)$

$$\int_A X_{n+1}(b)\, d\beta(b) = \int_B \int_{B_1 \times \dots B_n} \underbrace{\int_G \underbrace{(b_1 \dots b_{n+1})_* \nu(f)}_{=(b_{n+1})_* \nu(f(b_1 \dots b_n))} d\mu(b_{n+1})}_{\nu(f b_1 \dots b_n)}\, d\mu^{\otimes}(b_1, \dots, b_n)\, d\beta(b)$$

$$= \underbrace{\int_B \int_{B_1 \times \dots \times B_n}}_{=} \nu\underbrace{(f(b_1 \dots b_n \cdot)}_{X_n(b)}\, d\mu^{\otimes}(b_1 \dots b_n)\, d\beta(b)$$

$$= \int_{B_1 \times \dots B_n} X_n(b)\, d\beta(b) = \int_A X_n$$

that proves that $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$.

Doob' martingale convergence theorem

for ae $b \in B$ $(b_1 \dots b_n)_* \nu(f)$ converges $(n \to \infty)$

$\underline{\forall f \in} C_c(X) \Rightarrow$ for ae $b \in B$ convergence holds.

let $(f_p)_{p \in \mathbb{N}} \in C_c(X)$ dense

$\Rightarrow \quad \forall$ ae $b \in B \quad \forall f_p \quad (b_1 \dots b_n)_* \nu(f)$ converges.

1) $(b_1 \dots b_n)_* \gamma(\mu_1 f_{p_1} + \mu_2 f_{p_2})$

$= \mu_1 (b_1 \dots b_n)_* \gamma(f_{p_1}) + \mu_2 (b_1 \dots b_n)_* \gamma f_{p_2})$

2) $\| \nu_b(f) \| \leq \| f \|_\infty$

$$\lambda_b : \begin{array}{ccc} C_c(X) & \to & \mathbb{R} . \\ f & \mapsto & \lim (b_1 \dots b_n)_* \gamma f \end{array}$$

By Riesz: $\lambda_b(f) = \int_X f \, d\gamma_b$.

$\blacksquare$

Proposition: for $\beta$ ae $b \in B$. $T : B \to B$.
$(b_1 \dots b_n \dots) \mapsto (b_2, \dots, b_n, \dots$

1) $(b_1)_* \nu_{T(b)} = \nu_b$.

2) $\int_G \nu_b \, d\beta(b) = \nu$.

Proof:

1) Let $f \in C_c(X)$ $f \circ b_1$.

$\cancel{(b_1 f)_*}$

$(b_1 \dots b_n)_* \gamma(f) = (b_2 \dots, b_n)_* \gamma(f \circ b_1) \to \nu_{T(b)}(f \circ b_1)$

$\hookrightarrow \nu_b(f)$

$\gamma_b(f) = \gamma_{T(b)}(f \circ b_1) = (b_1)_* \gamma_{T(b)}(f)$.

$$\Rightarrow \quad \nu_b = (b_1)_* \nu_{T(b)} .$$

2) $\quad X_n(b) = (b_1 \dots b_n)_* \nu(f)$.

$$\mathbb{E}\left[ X_\infty(b) \mid \mathcal{F}_n \right] = X_n .$$

$n = 1$

$$\int_B \nu_b(f)\, d\beta(b) = \int_B \mathbb{E}\left[ \nu_b(f) \mid \mathcal{F}_1 \right] d\beta(b)$$

$$= \int_B X_1 \, d\beta(b) = \int_B (b_1)_* \nu(f)\, d\beta(b)$$

$$= \int_G (g)_* \nu(f)\, d\mu(g) \overset{\text{stationarity}}{=} \nu(f)$$

$$\Rightarrow \nu = \int_B \nu_b \, d\beta(b).$$

$$\beta^X = \int \delta_b \otimes \nu_b \, d\beta(b).$$

1) $\quad (b_1)_* \nu_{T(b)} = \nu_b$.

2) $\quad \nu = \int \nu_b$,

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Standing Assumption:

$G \curvearrowright G/\Lambda \qquad \langle \operatorname{Supp} \mu \rangle = \Gamma$

$\Gamma$ is not virtually contained in a conjugate of $\Lambda$.

$\rightarrow \quad \forall g \in G \left[\Gamma : \Gamma \cap g \Lambda g^{-1}\right]. \quad > \infty.$

Proposition: There are no atomic $\mu$-stationary measure. $\exists x \in X \quad \nu(\{x\}) > 0$.

Proof: Contradiction. take $\nu$ ergodic atomic $\mu$-stat. measure.

Daniel's talk: $\nu$ is actually finitely supported.

pick $x \in X$ atom of $\nu$. mass $m$

$$\int_G \underbrace{g_* \nu(\{x\})}\, d\mu(g) = \nu(\{x\}) = m.$$

$\Rightarrow$ ae $\forall g \in G \quad \nu(\{g.x\})$ is also an atom of $\nu$.

$\Rightarrow \forall g \in \langle \mathrm{Supp} \rangle \quad g.x$ is an atom

and the $\Gamma.x$ is finite. $\quad x = g\Lambda$.

$\mathrm{Stab}(x) = g \Lambda g^{-1}$.

$\Gamma \cap \mathrm{Stab}(x) = \Gamma \cap g \Lambda g^{-1}$ is finite index in $\Gamma$ $\lightning$

So $\nu$ is atom free.

$\square$

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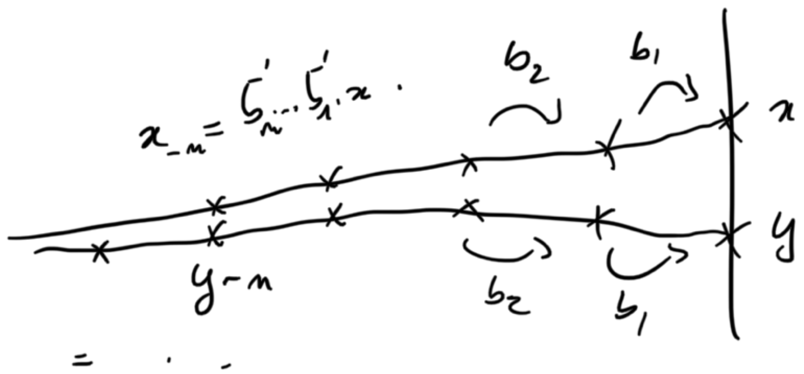
pick a metric $d$ on $X (= G/\Lambda)$

Works for any Continuous action of $G$ on $X$

locally compact, metrisable top. space.

$$W_b(x) = \{ y \in X \mid d(b_n^{-1} \cdots b_1^{-1} \cdot x, b_n^{-1} \cdots b_1^{-1} y) \to 0 \}$$




Showing for $\beta^X$ ae $(b,x) \in B^X$

$$\nu_b(W_b(x)) = 0.$$

[$\nu_b$ are dirac mass $\Rightarrow$ $\nu$ is Dirac mass.

1) $\nu$ non atomic $\Rightarrow$ $\nu_b$ is a.s non atomic

2) $\nu_b$ is as non atomic $\Rightarrow$ $\nu_b(W_b(x))$ as. 0.

Standing assumption (A): $\Delta_X = \{(x,x) \in X \times X\}$

$\exists v : X \times X - \Delta_X \to [0, \infty[$ st for any $K$ compact in $X$:

1) $v_{|K \times K - \Delta_K}$ is proper $\frac{1}{d(x,y)}$

2) $\exists\, 0 < a < 1;\ C > 0 \quad A_\mu(v) < av + C$.

$$A_\mu(v)(x,y) = \int v(g \cdot x, g \cdot y)\, d\mu(g).$$

G

$A_\mu$? Doing RW on $X^2$. $A_\mu(\varphi)$ is the expected value of $\varphi$ after one step of RW on $X^2$.

1) the only way to make $v$ get big is by getting to the diagonal.

2) If $v(x,y)$ is large $C$ becomes irrelevant

$$a v(x,y) + C \sim a v(x,y)$$

$$A_\mu(v)(x,y) < a v(x,y).$$

(1) + (2) says that the RW on $X^2$ pushes points away from the diagonal. (RW on $X$ moves point appart...)

**Proposition:** If for $\beta$ a.e. $b \in B$ $\nu_b$ is a Dirac Mass, then $\nu$ is also a Dirac Mass.

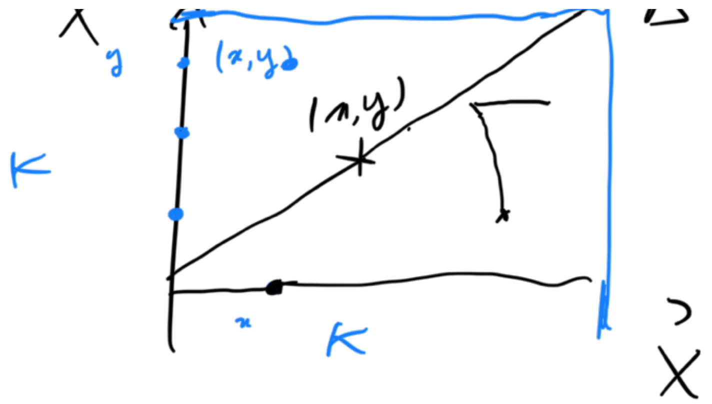
We'll consider RW on $X^2$.

1) $A \to$ RW pushes away points from $\Delta_X$.

2) $\nu_b$ are Dirac Masses $\to$ RW gets close to the diagonal

$\nu_b(A)$ = pick at random $x_{-\infty} \in X$ apply $b$. $\nu_b(A)$ probability to end up in $A$.

" RW on $X$ makes point go close to each other "




$(x,y) \sim \nu \otimes \nu$.

- $\nu \otimes \nu(\Delta_K) = 1 \iff \nu$ is completely atomic. $\nu$ is a Dirac Mass.

Technical lemma: let $K_0 \subset B$. For $\beta$-a.e. $b \in B$

- $$\frac{1}{p} \sum_{k=1}^{p} \mu^{\otimes k}\left(\{(g_1, \dots, g_k) \mid (g_1, \dots, g_k, b) \in K_0\}\right) \xrightarrow[p \to \infty]{} \beta(K_0).$$

Remark: the statement is clear when $K_0 = B_1 \times \dots \times B_m \times B$.

indeed if $p \geq m$ $(g_1, \dots, g_p, b) \in K \iff g_i \in B_i$ $i \leq m$

$$\mu^{\otimes p}\{(g_1, \dots, g_p) \mid (g_1, \dots, g_p, b) \in K\} =$$
$$\mu(B_1) \times \dots \times \mu(B_m) = \beta(B_1 \times \dots \times B_m \times B) = \beta(K).$$

Proof: Use the Chacon-Ornstein ergodic theorem

Let $(W, \lambda)$ probability space.

$L$ operator on $L^1(\lambda)$: 1) $L(\varphi) > 0$ when $\varphi > 0$
2) $\|L\| \leq 1$.

then for any $f$ measurable for $\lambda$ almost $\omega \in W$.

$$\frac{1}{p} \sum' L^k(f)(\omega) \quad \text{converges. as } p \to \infty.$$

( $L = f \mapsto f \circ T \quad T : W \to W \quad T_* \lambda = \lambda.$ )
gives Birkhoff.

use Chacon Ornstein for $f = \mathbb{1}_{K_0}$

$$L_\mu : \begin{cases} L_1(B,\beta) \to L_1(B,\beta) \\ \varphi \longmapsto \left( b \mapsto \int_G \varphi(\underbrace{gb}_{\text{concatenation}}) \, d\mu(g) \right). \end{cases}$$

1) $L_\mu$ positive ✓.

2) pick $\varphi \in L^1(B,\beta)$

$$\| L_\mu \varphi \|_{L_1} = \int_B | L_\mu \varphi | \, d\beta$$

$$= \int_B \left| \int_G \varphi(gb) \, d\mu(g) \right| d\beta(b)$$

$$\leq \int_B \left( \int_G \left| \varphi(gb) \right| d\mu(g) \right) d\beta(b)$$

$$\leq \int_B | \varphi(b') | \, d\beta(b').$$

$$\begin{pmatrix} G \times B \to B \\ (g,b) \mapsto (gb) \end{pmatrix}_* \mu \otimes \beta = \beta.$$

$\leq \|\varphi\|_{L^1}$

$$\|L_\mu \varphi\| \leq \|\varphi\| \Rightarrow \|L_\mu\| \leq 1.$$

for almost every $b \in B$.

$$\frac{1}{p}\sum_{k=1}^{p} L_\mu^k(\mathbb{1}_{K_0})(w) \xrightarrow[p\to\infty]{} \underline{f(w)}.$$

$$L_\mu f = f.$$

Show that the only eigenfunctions for eigenvalue 1 are the constants.

$L_\mu$ is the adjoint of $T$ (shift map).

$L^2$ . $\langle L_\mu f, g\rangle = \langle f, Tg\rangle$,

they have the same eigenfunctions for eigenvalue $1$.

$\rightsquigarrow$ if $f$ is $L_\mu$ inv, $\Rightarrow$ $T$-invariant $\Rightarrow$ $f$ is constant.
$L^2$.

then for almost every $b \in B$

$$\frac{1}{p}\sum_{k=1}^{p} L^k \mathbb{1}_{K_0}(b) \xrightarrow[p\to\infty]{} \beta(K_0) = \int f$$

$L^k \mathbb{1} \ (b) \quad \int \ldots$ ok

$$\mathbb{1}_{K_0}(b) = \int_{G^k} \mathbb{1}_{K_0}(g_1 \cdots g_k b)\, d\mu(g_1, \ldots, g_k)$$

$$= \mu^{\otimes k}\{(g_1, \ldots, g_k) \mid (g_1, \ldots, g_k, b) \in K_0\}.$$

$$\frac{1}{p}\sum_{k=1}^{p} \mu^{\otimes k}\{(g_1, \ldots, g_k) \mid (g_1, \ldots, g_k, b) \in K_0\} \xrightarrow[k \to \infty]{} \beta(K_0)$$

$\blacksquare$

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By contradiction, assume $\nu$ is not Dirac mass.

$$\kappa : B \to X \qquad \nu_b = \delta_{\kappa(b)}.$$

- $\nu = \kappa_* \beta$.

$$\nu = \int_B \nu_b \, d\beta = \int_B \delta_{\kappa(b)}\, d\beta = \int_X \delta_x \, d\kappa_*\beta.$$
$$= \kappa_*\beta.$$

$$\beta \otimes \beta(\{(b, b') \in B \times B \mid \kappa(b) \neq \kappa(b')\}) = \nu \otimes \nu({}^c\Delta_X) > 0$$

because $\nu$ is not a Dirac mass.

$\Rightarrow$ I can chose $b$ and $b'$ so that $\kappa(b) \neq \kappa(b')$

$$(*) \quad \frac{1}{p}\sum_{n=1}^{p} A_\mu^n \sigma(\kappa(b), \kappa(b'))$$

1) $(*)$ is not bounded above as $p \to \infty$.

2) $(*)$ is bounded above.

let's do $t$:

$$A_\mu(v)(x,y) = \int_G v(g.x, g.y)\, d\mu(g)$$

$$A_\mu^n(v)(x,y) = \int_{G^n} v(g_1 \cdots g_n x, g_1 \cdots g_n . y)\, d\mu^{\otimes n}$$

$$= \int_G v(g.x, g.y)\, d\mu^{*n}(g)$$

$$\mu^{*n} = \begin{pmatrix} G^n \to G \\ (g_1, \dots, g_n) \mapsto g_1 \cdots g_n \end{pmatrix}_* \mu^{\otimes n}.$$

$$A_\mu^n v(K(b), K(b')) = \int_G v(g_1 \cdots g_n K(b), g_1 \cdots g_n \cdot K(b'))\, d\mu^{\otimes n} g_1, \dots, g_n$$

$\bullet$ $g_1 \cdots g_n \cdot K(b) = K(g_1, \dots, g_n, b)$.

$\nu_b = \delta_{K(b)}$ $\quad (b_1 \dots b_n)_* \nu_{T^n(b)} = \nu_b$

$b' = (b_1, \dots, b_n, b)$

$\delta_{K(b')} = (b_1 \dots b_n)_* \nu_b = (b_1 \dots b_n)_* \delta_{K(b)} =$

$\delta_{b_1 \dots b_n K(b)}$ proves the claim.

$$A_\mu^n v(K(b), K(b')) = \int_{G^n} v(K(\underbrace{b_1 \dots b_n}\, b), K(\underbrace{b_1 \dots b_n}\, b')\, d\mu^{\otimes} b_1, \dots$$

distance on $B$ $\quad d_B(b,b') = \sum \frac{1}{2^n} d(b_i, b_i')$.

Let $K_0$ compact set of measure $\geq 1-\varepsilon$.

$\beta(K_0) \geq 1-\varepsilon$. st. $\mathcal{K}_{|K_0}$ is continuous

(Hence uniformly continuous).

$K = \mathcal{K}(K_0)$.

Apply (A) to $K$: $\quad v_{|K \times K - \Delta_K}$ is proper.

- $M > 0 \quad \exists n_M > 0 \quad \forall n \geq n_M \quad \forall g_1 \dots g_n \in G$.

if $(g_1, \dots, g_n b) \in K_0 \quad (g_1, \dots, g_n b') \in K_0$ then

$$v(\mathcal{K}(g_1 \cdots g_n b); \mathcal{K}(g_1 \cdots g_n b')) > M.$$

$v$ is proper: $\quad v^{-1}([M, \infty[)$ is a complementary of a compact in $K \times K - \Delta_K$.

$$\frac{1}{p} \sum_{k=1}^{p} A_\mu^k (v)(\mathcal{K}(b), \mathcal{K}(b')) = \frac{1}{p} \sum_{k=1}^{p} \int_{G^k} v(\mathcal{K}(g_1 \cdots g_k b), \mathcal{K}(g_1 \cdots g_k b'))$$

Technical lemma to $K_0$.

- $$\frac{1}{p} \sum_{k=1}^{p} \mu^{\otimes k} \{ (g_1, \dots, g_n) \mid g_1 \cdots g_n b \in K_0 \} \geq 1 - 2\varepsilon.$$

If $p_0$ large enough. $\forall p \geq p_0$

$$\frac{1}{p}\sum_{k=1}^{p}\mu^{\otimes k}\Big(\underbrace{\{(g_1\cdots g_k) \mid (g_1\cdots g_k b)\in K_0\ (g_1\cdots g_k b')\in K_0}_{K_M^k}\Big) \geq 1-4\varepsilon.$$

$$\frac{1}{p}\sum_{k=1}^{p}\int_{G^k} v(\kappa(g_1\cdots g_k b), \kappa(g_1\cdots g_k b'))\,d\mu^{\otimes}(g_1\cdots g_k)$$

$$\geq \frac{1}{p}\sum_{k=m_M}^{p}\int_{K_M^k} \underbrace{v(g_1\cdots g_k b,\ g_1\cdots g_k}_{\geq M} b')\,d\mu^{\otimes}$$

$$\geq \frac{1}{p}\sum_{k=m_M}^{p} M\,\mu^{\otimes k}\{(g_1\cdots g_k) \mid (g_1\cdots g_k b)\in K_0\ (g_1,\cdots,g_k b')\in K_0\}$$

$$\geq \frac{1}{p}\sum_{k=m_M}^{p} M(1-4\varepsilon) \geq M\Big(1-4\varepsilon-\frac{m_M-p}{p}\Big).$$

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Show $\frac{1}{p}\sum_{k=1}^{p} A_\mu^k(\kappa(b),\kappa(b'))$ .

$$A_\mu^n v(\kappa(b),\kappa(b')) < a^n v(\kappa(b),\kappa(b')) + C(1+\dots+a^n).$$

$$\frac{1}{p}\sum_{k=1}^{p} A_\mu^k(\kappa(b),\kappa(b')) \leq \frac{1}{1-a} v(\kappa(b),\kappa(b')) + \frac{C}{1-a}$$