Recall from previous sessions:

- G Lie group, Λ lattice in G, $\mu \in \mathcal{P}(G)$ compactly supported, $X = G/\Lambda$, $\nu \in \mathcal{P}(X)$ st. $\mu * \nu = \nu$.

- We assume (I), (II), (III) are satisfied.

- $V := \mathcal{L}ie(G)$ & $\rho_1 : G \to SL^{\pm}(V)$ adjoint representation.

- β-ae $b \in B$ $\quad \nu_b = \lim (b_1 \dots b_n)_* \nu$.

- $\forall (b,x) \in B^X := B \times X$

$$W_b(x) := \{ y \in X \mid d(b_n^{-1} \dots b_1^{-1} \cdot x,\ b_n^{-1} \dots b_1^{-1} \cdot y) \xrightarrow[n \to \infty]{} 0 \}.$$

We want: $\underline{\beta^X\text{-ae}(b,x) \in B^X \quad \nu_b(W_b(x)) = 0}$

plan of the proof:

1) ν is non atomic ✓

→ 2) property (HC) is verified

3) (1) + (2) $\Rightarrow$ β-ae $b \in B$ ν_b is non atomic. ✓

→ 4) β ae $b \in B$ ν_b is non atomic $\Rightarrow$ what we want.

Let's start with 4.

<u>Technical lemma first.</u> Let (W, λ) be a probabity space and $T : W \to W$, $T_* \lambda = \lambda$. Let $\varphi : X \to [0, +\infty[$ if $\lambda(\varphi^{-1}\{0\}) = 0$ then $\lambda \{ \omega \in W \mid \varphi \circ T^p(\omega) \xrightarrow[p \to \infty]{} 0 \} = 0$.

Proof: Set $Z = \{ \omega \in W \mid \varphi(T^p(\omega)) \underset{p\to\infty}{\longrightarrow} 0 \}$

Assume $\lambda(Z) > 0$. then $\lambda(Z \cap {}^c\underbrace{\varphi^{-1}\{0\}}_{\text{null set}}) > 0$

But $Z \cap {}^c\varphi^{-1}\{0\} = \bigcup Z \cap \varphi^{-1}([\frac{1}{n}, \infty[)$

$\Rightarrow \exists n > 0$ st $\lambda(Z \cap \varphi^{-1}([\frac{1}{n}, \infty[)) > 0$

Poincaré-Recurrence theorem to $\uparrow$: $\exists \xi \in Z$ and $(n_p)_{p\in\mathbb{N}}$

$n_p \underset{p\to\infty}{\longrightarrow} \infty$ st $T^{n_p}(\xi) \in Z \cap \varphi^{-1}([\frac{1}{n}, \infty[)$

$$\underbrace{T^{n_p}(\xi)}_{\underset{p\to\infty}{\to} 0} > \frac{1}{n} \qquad ⚡.$$

So $\lambda(Z) = 0$. $\boxtimes$

Proof of 4: Idea apply Technical lemma to a (W, λ, T).

$$W := B \times X \times X.$$

$$\lambda := \int_B \delta_b \otimes \nu_b \otimes \nu_b \, d\beta(b).$$

$$T_0(b, x, y) := (T(b), b_0^{-1} \cdot x, b_0^{-1} \cdot y)$$

Remark: $T_0 * \lambda = \lambda$.

$$\varphi(b, x, y) = d(x, y).$$

$\rightarrow$ $\varphi \circ T_0^p(b, x, y) = d(b_n^{-1} \cdots b_1^{-1} \cdot x, b_n^{-1} \cdots b_1^{-1} \cdot y)$.

- $\lambda(\varphi^{-1}\{0\}) = \lambda\{(b, x, y) \mid d(x, y) = 0\}$.

$$= \lambda\{B \times \Delta_X\}$$

$$= \int \underbrace{\nu_b \otimes \nu_b}_{\substack{=0 \text{ a.s because } \nu_b \text{ non atomic.}}} (\Delta_X) \, d\beta(b) = 0.$$

$$\bullet \ \lambda \{ (b,x,y) \mid \varphi \circ T_0^p (b,x,y) \to 0 \}.$$

$$= \lambda \{ (b,x,y) \mid d(b_n^{-1} \dots b_1^{-1} x, \ b_n^{-1} \dots b_1^{-1} y) \to 0 \}.$$

$$= \lambda \{ (b,x,y) \mid y \in W_b(x) \}.$$

$$= \iint_{B \ X} \nu_b(W_b(x)) \, d\nu(b)(x) \, d\beta(b)$$

$$= \int_{B^X} (b,x) \mapsto \nu_b(W_b(x)) \, d\beta^X(b,x) \underset{\substack{\text{Technical} \\ \text{lemma}}}{=} 0$$

thus: for β^X-ae $(b,x) \in B^X$ $\nu_b(W_b(x)) = 0$.

$\blacksquare$

Proof of 2: Show that property (HC) holds, ie:

$\exists v : X \times X - \Delta_X \to [0,\infty[$ st $\forall K \in X$:

1) $v_{| K^2 - \Delta_K}$ is proper.

2) $\exists \ 0 < a < 1, \ C > 0$ st $A_\mu v < a v + C$

Recall: $A_\mu(f)(x,y) = \int_G f(g.x, g.y) \, d\mu(g).$

(HC) says : RW on X^2 moves points away from the diagonal equivalently RW on X moves points appart. formal way to say this:

<u>Lemma 1:</u> $\exists\, 0 < a_0 < 1$, $\delta_0 > 0$ and $n_0 \geq 1$ st $\forall \delta < \delta_0$

$$\forall n \geq n_0 ; \forall v \in V \quad \int_G \|\rho(g) \cdot v\|^{-\delta} \, d\mu^{*n}(g) \leq a_0 \|v\|^{-\delta}.$$

So if $y = e^{\omega} . x$ then $g.y = e^{\rho(g).\omega} g.x$. by lemma 1, we expect after n steps $\rho(g).\omega$ to increase and then $g.x$ is further from $g.y$ than x is from y.

Notice that μ verifies (HC) if and only if μ^{*n} verifies (HC) So up to replacing μ by μ^{*n} in lemma we can assume $n_0 = 1$ in lemma 1

Define :

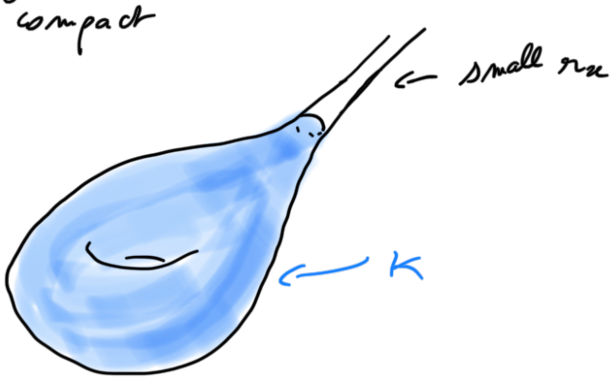
- r_x = injectivity radius at $x = \sup\{r \mid \omega \mapsto e^{\omega}.x$ is injective$\}$.
- $r_{x,y} = \frac{1}{2} \min(r_x, r_y)$
- $d_0(x,y) = \begin{cases} \|\omega\| & \text{if } y = e^{\omega}.x \quad \|\omega\| \leq r_x \\ r_{x,y} & \text{otherwise} \end{cases}$
- $v_0(x,y) = d_0^{-\delta}(x,y)$ with δ as in lemma 1

Let's see if v_0 satisfies (HC). Let K be a compact of X

$v_0(x,y)$ gets large when

1) $e^{\omega} x = y$ with $\|\omega\| \ll 1$ ie (x,y) close to Δ_X

2) r_x or r_y really small ie x or y is off a big compact



2) is not possible in K so $v_0|_{K\times K-\Delta_X}$ is proper.

Define $R = \sup_{g\in \mathrm{supp}\,\mu} \sup(\|\rho(g)\|, \|\rho(g^{-1})\|)$

Now, if • $d_0(x,y) \leq R^{-1} r_{x,y}$ then one can show, using

$$gy = e^{\rho(g).w} g.x \iff y = e^{w} x \quad \text{that}$$

$$v_0(g.x, g.y) = \|\rho(g)w\|$$

and then $\quad A_\mu(v_0)(x,y) \leq a_0 v_0(x,y)$ by lemma 1.

• if $d_0(x,y) > R^{-1} r_{x,y}$ then in that case

$$d_0(g.x, g.y) \geq R^{-2} r_{x,y} \quad \text{thus}$$

$$A_\mu(v_0)(x,y) \leq R^{2s} r_{x,y}^{-s} \leq 2R^{2s} \underbrace{r_x^{-s} + r_y^{-s}}.$$

might get very large if x and y are off a compact.

We need to control the injectivity radii. That is the point of

<u>lemma 2:</u> There is $u : X \to [0,\infty[$ and $a<1$, $c>0$, $\kappa>0$ st:

1) $$\int_G u(g.x)\, d\mu(g) \leq a\, u(x) + C.$$

2) $\quad \mu(x) \geqslant r_x^{-\varkappa}$

Now use lemma 2 to define $v(x,y) = v_0(x,y) + C_0(\mu(x)+\mu(y))$ /

$C_0 = \dfrac{4R^{2\delta}}{1-a_0}$. We can also assume $a = a_0$. Chose $\delta < \varkappa$

We saw $A_\mu v_0(x,y) \leqslant \begin{cases} a_0 v_0(x,y) & \text{if } d(x,y) \leq R\, r_{x,y} \\ 2R^{2\delta}(r_x^{-\delta} + r_y^{-\delta}) \leq 2R^{2\delta}(\mu(x)+\mu(y)) \end{cases}$

lemma 2

In any case $A_\mu v_0(x,y) \leq a_0 v_0(x,y) + 2R^{\delta}(\mu(x)+\mu(y))$.

So $A_\mu(v)(x,y) = A_\mu(v_0)(x,y) + C_0 A_\mu(\mu(x)+\mu(y))$

$\leq a_0 v_0(x,y) + 2R^{\delta}(\mu(x)+\mu(y)) + a_0 C_0(\mu(x)+\mu(y)) + 2C_0 C$

$\leq a_0 v_0(x,y) + (\underbrace{2R^{\delta}+a_0C_0}_{\frac{1+a_0}{2}})(\mu(x)+\mu(y)) + 2C_0C$.

$\leq \dfrac{1+a_0}{2} v(x,y) + 2C_0 C$. $\boxed{}$

A word on the proof of lemma 1 and 2.

1) uses Taylor development for $S \mapsto \|\rho(g)v\|^{\delta}$ and prop 3.3 of [SW]

2) follows from lemma 1. In the case $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$

$\mu = \dfrac{1}{\text{Syst}}$.