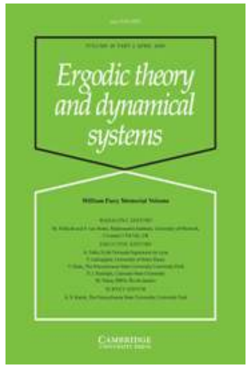

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Ergodic fractal measures and dimension conservation

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Abstract. A linear map from one Euclidean space to another may map a compact set bijectively to a set of smaller Hausdorff dimension. For 'homogeneous' fractals (to be defined), there is a phenomenon of 'dimension conservation'. In proving this we shall introduce dynamical systems whose states represent compactly supported measures in which progression in time corresponds to progressively increasing magnification. Application of the ergodic theorem will show that, generically, dimension conservation is valid. This 'almost everywhere' result implies a non-probabilistic statement for homogeneous fractals.

0. *Introduction*

If $f : X \to Y$ is a Lipschitz map of the metric space X to the metric space Y, then the Hausdorff dimension of $f(X)$ never exceeds that of X: $\dim f(X) \leqslant \dim X$. In the classical situation where f is a linear map of one vector space to another, $\dim X - \dim f(X) = \dim(\ker f)$, the latter being also the dimension of $f^{-1}(y)$ for each $y \in f(X)$. Thus the discrepancy between the dimension of the image and that of the domain of f is accounted for by the size of $f^{-1}(y)$ for any y in the image. In the general context of metric spaces and Hausdorff dimension, a Lipschitz map may decrease dimension without the loss being compensated for by $\dim f^{-1}(y)$ for any $y \in f(X)$. Thus $f : X \to f(X)$ may be a one-to-one map with $\dim f(X) < \dim X$. This can happen even when f is the restriction of a linear map from one Euclidean space to another with domain X, a compact subset of the first Euclidean space. We shall use the term '*fractal*' for compact subsets of Euclidean space.

A map f taking a fractal to another will be termed '*dimension conserving*' when loss of dimension (if there is any) is accounted for by the dimension of fibers $f^{-1}(y)$ in a manner to be made precise in §1. Our main result states that for a certain class of fractals—we call them *homogeneous fractals*—at least linear maps are dimension conserving.

The principal tool in our proof will be the introduction of a dynamical system we call a 'CP-shift system', in which progression in 'time' corresponds to progressively increasing magnification about a point in the support of a compactly supported measure in Euclidean space. This will make available tools from ergodic theory both for the construction and study of fractal sets. More precisely, states in this dynamical system will correspond to 'ergodic fractal measures', and the ergodic theorem will be invoked to deduce almost everywhere information regarding the supports of these measures.

In §1 we introduce the notion of a *micro-set* of a fractal and in our final section we show that, for any set of positive dimension, there exist micro-sets supporting ergodic fractal measures. Thus, for any set A, the information available for ergodic fractal measures is applicable to some microsets of A. Now, for homogeneous fractals micro-sets are homothetic to subsets, so that the information available for ergodic fractal measures can be applied directly to homogeneous fractals. This is a brief sketch of the argument used in the proof of our main result. We expect that the ergodic fractal measures appearing here can be regarded as objects of independent interest, and part of our aim here is to call attention to these. Other applications of the ideas developed here—albeit in a different form—can be found in [**F**] and [**FW**].

1. *Preliminaries*

1.1. *Fractals and their micro-sets.* As indicated, the term 'fractal' will be applied to an arbitrary compact subset in Euclidean space. Hausdorff dimension is defined for any metric space and is finite for fractals, for which it can take any value between zero and the dimension of the ambient space.

Definition 1.1. If $f : A \to \mathbb{R}^n$ is a Lipschitz function from $A \subset \mathbb{R}^m$ to $\mathbb{R}^n$, we say that f is *dimension conserving* (DC) if, for some $\delta \geq 0$,

$$\delta + \dim\{y \mid \dim f^{-1}(y) \geqslant \delta\} \geqslant \dim A \tag{1.1}$$

(there may be more than one such δ). We adopt here the convention that the dimension of the empty set is $-\infty$, so that (1.1) cannot hold if δ is chosen too large, and the set of y in question is empty.

We give two examples of mappings that are not DC.

(a) Let $\varphi : [0, 1] \to \mathbb{R}$ be a continuous function whose graph has dimension greater than one. This will be true, for example, if φ is a typical one-dimensional Brownian motion. If A is the graph of φ and $f : \mathbb{R}^2 \to \mathbb{R}^1$ is given by $f(x, y) = x$, then clearly $f|_A$ is not DC.

(b) For two sets B', B'', $\dim(B' \times B'') \geqslant \dim B' + \dim B''$, where we may have strict inequality. If this happens, then setting $A = B' \times B''$, the projection of A to B' (or B'') clearly cannot be DC.

In what follows we will focus on fractals lying inside the unit cube $Q^{(m)} = [0, 1]^m$ of $\mathbb{R}^m$ for some m. For fixed m we simply denote $Q^{(m)}$ by Q. The family of closed subsets of Q will be denoted by 2^Q. The *Hausdorff metric* in 2^Q is defined by

$$D(A, B) = \inf\{d \mid A \subset B_d \text{ and } B \subset A_d\},$$

where, for any set $S \subset \mathbb{R}^m$, S_d is the union of all open balls of radius d centered in S. Endowed with this metric, 2^Q is a compact metric space. The following notions generalize the notion of a subset of a given set A. We suppose that $A \in 2^Q$.

Definition 1.2. A set $A' \in 2^Q$ is a *mini-set* of A if, for some scalar $\lambda \geqslant 1$ and $u \in \mathbb{R}^m$, $A' \subset (\lambda A + u) \cap Q$.

Definition 1.3. A set $A'' \in 2^Q$ is a *micro-set* of A if is there is a sequence A'_n of mini-sets of A with $A'_n \to A''$ in the Hausdorff metric on 2^Q.

Definition 1.4. A fractal A is *homogeneous* if every micro-set of A is a mini-set.

Definition 1.5. A family $\mathcal{G} \subset 2^Q$ is called a *gallery* if it is closed in 2^Q and, with each $A \in \mathcal{G}$, every mini-set of A is also in $\mathcal{G}$.

Remark. The set of micro-sets of a given fractal A forms a gallery. We denote this gallery by $\mathcal{G}_A$.

Definition 1.6. For any gallery $\mathcal{G}$, $\dim^* \mathcal{G} = \sup\{\dim A \mid A \in \mathcal{G}\}$.

It will be shown later that the supremum in the foregoing definition is always attained by a member of the gallery.

Finally, we have the following.

Definition 1.7. For a fractal A, $\dim^* A$ will denote $\dim^* \mathcal{G}_A$.

We always have $\dim A \leqslant \dim^* A$. In fact, $\dim A \leqslant \overline{\dim}_B A \leqslant \dim^* A$, where $\overline{\dim}_B A$ is the upper 'box dimension' or 'Minkowski dimension' of A. If A is homogeneous, then $\dim A = \dim^* A$ so the various notions of dimension coincide.

The classical Cantor middle-third set is homogeneous. More generally, if $A = \bigcup \varphi_i(A)$ where the φ_i are contracting homotheties and the $\varphi_i(A)$ are disjoint, then A is homogeneous. A further example generalizing that of the Cantor set is that of a closed set $A \subset Q^{(m)}$ invariant under τ_p where

$$\tau_p(x_1, x_2, \ldots, x_m) = (px_1 - [px_1],\ px_2 - [px_2], \ldots, px_m - [px_m]).$$

1.2. *p-ary decomposition of Q.* Fix a dimension m, and let $Q = Q^{(m)} = [0, 1]^m$ denote the unit cube in $\mathbb{R}^m$. Fix an integer $p \geqslant 2$ and consider the partition

$$[0, 1] = \left[0, \frac{1}{p}\right) \cup \left[\frac{1}{p}, \frac{2}{p}\right) \cup \cdots \cup \left[\frac{p-1}{p}, 1\right] = J_0 \cup J_1 \cup \cdots \cup J_{p-1},$$

where all but the last interval are half-open and half-closed. We denote the product set $\{0, 1, \ldots, p-1\}^m$ by Λ. For $\lambda = (i_1, i_2, \ldots, i_m) \in \Lambda$ we define $Q_\lambda = J_{i_1} \times J_{i_2} \times \cdots \times J_{i_m}$. We then obtain a partition

$$Q = \bigcup_{\lambda \in \Lambda} Q_\lambda$$

to disjoint cubes. For any $x \in Q$ there is a unique λ with $x \in Q_\lambda$. We write $\lambda = \lambda(x)$ and $Q_\lambda = Q_1(x)$. Let ρ_λ denote the restriction to Q_λ of the map $t \mapsto pt - \lambda$. In general $\rho_\lambda(Q_\lambda)$ is a partially open subcube of Q, unless $\lambda = (p-1, p-1, \ldots, p-1)$, where $\rho_\lambda(Q_\lambda) = Q$. We can now define subcubes $Q_{\lambda_1,\lambda_2,\ldots,\lambda_\ell}$ for any ℓ and $\lambda_i \in \Lambda$ inductively by

$$Q_{\lambda_1,\lambda_2} = \rho_{\lambda_1}^{-1} Q_{\lambda_2}, \quad Q_{\lambda_1,\lambda_2,\lambda_3} = \rho_{\lambda_1}^{-1} Q_{\lambda_2,\lambda_3}, \ldots, Q_{\lambda_1,\lambda_2,\ldots,\lambda_\ell} = \rho_{\lambda_1}^{-1} Q_{\lambda_2,\lambda_3,\ldots,\lambda_\ell}, \text{ etc.}$$

The cube $Q_{\lambda_1,\lambda_2,\dots,\lambda_\ell}$ has side $p^{-\ell}$, $Q_{\lambda_1,\lambda_2,\dots,\lambda_{\ell+1}} \subset Q_{\lambda_1,\lambda_2,\dots,\lambda_\ell}$, and if $x \in \bigcap_{\ell\geqslant 1} Q_{\lambda_1,\lambda_2,\dots,\lambda_\ell}$ then $x = \sum_1^\infty \lambda_n/p^n$ where we regard the λ_i as vectors in $\mathbb{R}^m$. The 'p-ary' expansion of points $x \in Q$ is not always unique, whereas the λ_n with $x \in Q_{\lambda_1,\lambda_2,\dots,\lambda_n}$ are unique as a result of the intervals J_i being non-overlapping.

For $w = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \in \Lambda^\ell$ we shall write Q_w for $Q_{\lambda_1,\dots,\lambda_\ell}$ and the length ℓ of w will be denoted by $\ell(w)$. For any $x \in Q$ and $\ell \in \mathbb{N}$, $Q_\ell(x)$ will denote Q_w for the unique word w of length ℓ for which $x \in Q_w$. Finally, Q itself can be denoted by $Q_\emptyset$ where $\emptyset$ is the 'empty' word.

If $A \subset \bigcup_{i=1}^k Q_{w_i}$ with disjoint cubes $\{Q_{w_i}\}$ we shall say that $\{Q_{w_i}\}_1^k$ is a *p-cover* of A. The usual definition of Hausdorff dimension involves covering a set A with countably many balls. When A is compact this can be replaced by a finite set of balls, and each ball can be replaced by the union of cubes whose sides are the same order of magnitude as the radius of the ball. We can thus relate the dimension of a closed subset of Q to its p-covers, and we obtain the following.

LEMMA 1.1. *If A is a closed subset of Q with* $\dim A < \gamma$, *then, for any* $\varepsilon > 0$, *there exists a finite p-cover* $A \subset \bigcup Q_{w_i}$ *with* $\sum p^{-\gamma \ell(w_i)} < \varepsilon$.

We now come to a result which will play an important role in what follows. Here again $Q = Q^{(m)} \subset \mathbb{R}^m$ for any $m \geqslant 1$. Below, we denote the set of probability measures on Q by $\mathcal{P}(Q)$.

THEOREM 1.1. *Let $\theta \in \mathcal{P}(Q)$ and assume that, for every $x \in Q$ outside of a set of θ-measure* 0,

$$\liminf \left\{ -\frac{1}{n} \log_p \theta(Q_n(x)) \right\} \geqslant \beta. \tag{1.2}$$

Then, denoting the support of θ by $|\theta|$, we have $\dim |\theta| \geqslant \beta$.

Proof. If $\dim |\theta| < \beta$, let $\beta' < \beta$ and $\dim |\theta| < \beta'$. By (1.2), we can find a compact set $A \subset |\theta|$ with $\theta(A) > \frac{1}{2}$ and a large N so that for $x \in A$, $n \geqslant N$, $\theta(Q_n(x)) < p^{-\beta' n}$. By Lemma 1.1, since $\dim A < \beta'$, for any $\varepsilon > 0$, there exists a p-cover of A, $\{Q_{w_i}\}$, with $\sum p^{-\beta' \ell(w_i)} < \varepsilon$. If $\varepsilon < p^{-\beta' N}$ all the w_i appearing here will have $\ell(w_i) \geqslant N$. We can assume that $Q_{w_i} \cap A \neq \emptyset$. This implies that $\theta(Q_{w_i}) < p^{-\beta' \ell(w_i)}$. But then $\sum \theta(Q_{w_i}) < \varepsilon$. If ε is now chosen with $\varepsilon < \frac{1}{2}$ then $\theta(A) \leqslant \theta(\bigcup Q_{w_i}) < \frac{1}{2}$ which is a contradiction. Hence $\dim |\theta| \geqslant \beta$. □

2. *Ergodic CP-shift systems*

In this section we introduce a family of dynamical systems that will play a central role in our discussion. Again $Q = Q^{(m)}$ will denote the unit cube in $\mathbb{R}^m$ for a fixed $m \geqslant 1$, and $\mathcal{P}(Q)$ will denote the set of probability measures on Q. The set $\Phi \subset \mathcal{P}(Q) \times Q$ consisting of all pairs (θ, x) such that, for every n, $\theta(Q_n(x)) > 0$, is a Borel set in $\mathcal{P}(Q) \times Q$. We now define a measurable map $T : \Phi \to \Phi$, namely we let T be the 'rescaling map' where for $(\theta, x) \in \Phi$ and $x \in Q_{\lambda_1} = Q_1(x)$,

$$T(\theta, x) = \left(\frac{\rho_{\lambda_1}(\theta|_{Q_{\lambda_1}})}{\theta(Q_{\lambda_1})}, px - \lambda_1 \right). \tag{2.1}$$

Writing $Q_{n+1}(x) = Q_{\lambda_1 w}$ with $\ell(w) = n$, we have $Q_n(px - \lambda_1) = Q_w$ and $\rho_{\lambda_1}(\theta)\,(Q_w) = \theta(\rho_{\lambda_1}^{-1} Q_w) = \theta(Q_{\lambda_1 w}) = \theta(Q_{n+1}(x)) > 0$; hence $T(\theta, x) \in \Phi$.

Definition 2.1. A measure μ on $\mathcal{P}(Q) \times Q$ will be said to be *adapted* if there is a measure ν on $\mathcal{P}(Q)$ such that $d\mu(\theta, x) = d\theta(x)\, d\nu(\theta)$, i.e.

$$\int f(\theta, x)\, d\mu(\theta, x) = \int \left(\int f(\theta, x)\, d\theta(x) \right) d\nu(\theta). \tag{2.2}$$

We now come to the central notion of this section—that of a CP-shift system. Here CP stands for 'conditional probability', which describes the probability measure appearing in $T(\theta, x)$ as the image in Q of the measure on $Q_1(x)$, given by θ conditional on being in $Q_1(x)$.

Definition 2.2. (Φ, T, μ) is an *ergodic CP-shift system* (ECPS) if μ is an adapted T-invariant measure on Φ such that the corresponding measure-preserving system is ergodic.

We present two examples of ECPS systems, both derived from stationary Λ-valued processes. Suppose $\{X_n\}_{n\in\mathbb{Z}}$ is an ergodic, stationary Λ-valued process defined on a probability space Ω. Writing $X_n = (X_n^{(1)}, \ldots, X_n^{(m)})$ with $X_n^{(i)}(\omega) \in \{0, 1, \ldots, p-1\}$, we assume that the event $X_1^{(i)} = X_2^{(i)} = \cdots = X_n^{(i)} = \cdots = p-1$ occurs with probability 0 for each i. Set

$$Z_n(\omega) = \sum_{k=1}^{\infty} \frac{X_{n+k}(\omega)}{p^k}$$

with values in Q. The restriction placed on $\{X_n\}$ implies that, with probability 1, $Z_n \in Q_{X_{n+1}, X_{n+2}, \ldots, X_{n+k}}$ for each k. (Note that, for $m = 1$, the point

$$x = \frac{p-1}{p^2} + \frac{p-1}{p^3} + \cdots = \frac{1}{p}$$

belongs to Q_1 and not to Q_0.) We have $\delta_{Z_0}(Q_{X_1, X_2, \ldots, X_k}) = 1$ so that $(\delta_{Z_0}, Z_0) \in \Phi$ with probability 1. Moreover, $T(\delta_{Z_0}, Z_0) = (\delta_{Z_1}, Z_1)$. It follows that if μ is the distribution of (δ_{Z_0}, Z_0) in Φ, then it is invariant under T, and since $\{X_n\}$ is ergodic, so is the system (Φ, T, μ). It is clear that the distribution on Φ of (δ_{Z_0}, Z_0) is an adapted measure, since Z_0 is exactly the support of δ_{Z_0}. Thus (Φ, T, μ) is an example of an ergodic CP-shift system.

A second example makes use of the 'past' $\{X_n\}_{n\leqslant 0}$. For almost every ω, the 'prediction' measure $\tilde{\theta}(\omega)$ on $\Lambda^{\mathbb{N}}$ is defined, whereby, for a continuous $f(\xi)$ on $\Lambda^{\mathbb{N}}$,

$$\int f(\xi)\, d\tilde{\theta}(\omega)\,(\xi) = \mathbb{E}(f(X_1, X_2, \ldots) | X_0, X_{-1}, X_{-2}, \ldots)\,(\omega), \tag{2.3}$$

where $\tilde{\theta}(\omega)$ depends on the past: $\tilde{\theta}(\omega) = \tilde{\theta}(X_0(\omega), X_{-1}(\omega), \ldots)$. Define $\pi : \Lambda^{\mathbb{N}} \to Q$ by $\pi(\xi) = \sum_1^{\infty} \xi_k / p^k$ for $\xi = (\xi_1, \xi_2, \ldots)$ and put $\theta(\omega) = \pi(\tilde{\theta}(\omega))$ so that $\theta(\omega) \in \mathcal{P}(Q)$. We can verify that

$$T(\theta(X_0, X_{-1}, \ldots), \pi(X_1, X_2, \ldots)) = (\theta(X_1, X_0, \ldots), \pi(X_2, X_3, \ldots))$$

with probability 1. Moreover with probability 1, for each k,

$$\theta(X_0, X_{-1}, \ldots)(Q_{X_1, X_2, \ldots, X_k}) > 0,$$

and hence $(\theta(X_o, X_{-1}, \ldots),\ \pi(X_1, X_2, \ldots)) \in \Phi$ with probability 1. Denote by $\mathcal{P}(\Lambda^{\mathbb{N}})$ the set of probability measures on $\Lambda^{\mathbb{N}}$. Let $\tilde{\nu}$ be the distribution on $\mathcal{P}(\Lambda^{\mathbb{N}})$ of $\tilde{\theta}(X_0, X_{-1}, X_{-2}, \ldots)$ and $\tilde{\mu}$ the distribution on $\Lambda^{\mathbb{N}}$ of $\{X_n\}_{-\infty}^{\infty}$. Then $d\tilde{\mu} = d\tilde{\theta}\, d\tilde{\nu}(\tilde{\theta})$. It follows that the distribution μ on Φ of the variable $(\theta(X_0, X_{-1}, \ldots),\ \pi(X_1, X_2, \ldots))$ is an adapted measure. Since the process $\{X_n\}$ is ergodic, μ is an ergodic measure for T and we conclude that (Φ, T, μ) is an ergodic CP-shift system.

The ergodic theorem can be applied to any ECPS system with implications for almost every $(\theta, x) \in \Phi$. We will refer informally to the typical measure appearing here as 'ergodic fractal measures'. We will not give a precise definition; rather such measures are those reflecting properties enjoyed by almost every θ for some ECPS system. An example of such a property will follow as a corollary from the next theorem.

THEOREM 2.1. *Let (Φ, T, μ) be an ECPS system with ν the associated measure on $\mathcal{P}(Q)$, and set*

$$h = h_\mu = -\int_\Phi \log_p \theta(Q_1(x))\, d\mu(\theta, x).$$

Then for almost every θ with respect to ν, $\dim|\theta| \geqslant h$. *(Thus h is at most the dimension of the ambient space, and so the integral in question is finite.)*

Proof. We set $I(\theta, x) = -\log_p \theta(Q_1(x))$ and we proceed to evaluate

$$\frac{1}{n}\sum_{k=0}^{n-1} I(T^k(\theta, x)).$$

Let $x \in Q_w$ for $w = \lambda_1\lambda_2 \ldots \lambda_k$. Iterating (2.1) we find

$$T^k(\theta, x) = \left(\frac{\rho_{\lambda_k} \circ \rho_{\lambda_{k-1}} \circ \cdots \circ \rho_{\lambda_1}(\theta|_{Q_{\lambda_1 \ldots \lambda_k}})}{\theta(Q_{\lambda_1 \ldots \lambda_k})},\ p^k x - [p^k x]\right), \tag{2.4}$$

where $[p^k x] = \lambda_k + \lambda_{k-1}p + \cdots + \lambda_1 p^{k-1}$. Also,

$$Q_1(p^k x - [p^k x]) = Q_{\lambda_{k+1}}$$

and

$$\rho_{\lambda_1}^{-1}\rho_{\lambda_2}^{-1} \ldots \rho_{\lambda_k}^{-1}(Q_{\lambda_{k+1}}) = Q_{\lambda_1\lambda_2 \ldots \lambda_{k+1}}.$$

This gives

$$I(T^k(\theta, x)) = -\log_p\left(\frac{\theta(Q_{\lambda_1 \ldots \lambda_{k+1}})}{\theta(Q_{\lambda_1 \ldots \lambda_k})}\right). \tag{2.5}$$

Consequently

$$\frac{1}{n}\sum_{k=0}^{n-1} I(T^k(\theta, x)) = -\frac{1}{n}\log_p \theta(Q_n(x)). \tag{2.6}$$

The ergodic theorem applies to non-negative measurable functions giving a limiting value ∞ if the function is non-integrable. Thus for μ-almost every (θ, x) the limit in (2.6) is h_μ, which *a priori* might be $+\infty$. Inasmuch as μ is an adapted measure, this conclusion holds for ν-almost every θ and θ-almost every x. We now apply Theorem 1.1 to conclude that, for almost every θ, $\dim|\theta| \geqslant h_\mu$. Since θ is a measure on $Q^{(m)}$, $\dim|\theta| \leqslant m$, so that h_μ is finite, and *a posteriori* we find that $I(x)$ is integrable. This concludes the proof of the theorem. □

As a corollary we conclude that 'ergodic fractal measures' have the property that

$$\lim_{n\to\infty}(\theta(Q_n(x))^{1/n}) \tag{2.7}$$

exists for almost every x with respect to θ, and is independent of x. We denote the above limit by $\mathcal{H}(\theta)$. More generally we will say that a measure on Q is *p-regular* if (2.7) exists almost everywhere and is independent of x. For a p-regular θ we will speak of $-\log\mathcal{H}(\theta)$ as the *p-dimension* of θ, denoted by $\dim_p \theta$.

3. *Dimension conservation for ergodic fractal measures*

We will see in §6 that the phenomenon of dimension conservation for arbitrary linear maps restricted to fractals can be studied by looking at the special case of the map $(x, y) \mapsto x$ where $x \in \mathbb{R}^{m_1}$ and $y \in \mathbb{R}^{m_2}$ so that the function in question takes $Q^{(m_1+m_2)}$ to $Q^{(m_1)}$. Having defined p-dimension for measures we define a notion analogous to dimension conservation for measures on $Q^{(m_1+m_2)}$ relative to the foregoing map $Q^{(m_1+m_2)} \to Q^{(m_1)}$. We denote $Q^{(m_1+m_2)}$ by Q, $Q^{(m_1)}$ by Q' and $Q^{(m_2)}$ by Q''. A probability measure $\theta \in \mathcal{P}(Q)$ has a 'Fubini' decomposition

$$\theta = \int_{Q'} \delta_x \times \theta_x \, d\bar{\theta}(x), \tag{3.1}$$

where $\theta_x \in \mathcal{P}(Q'')$ is well defined for almost all $x \in Q'$ relative to $\bar{\theta}$, the projection of θ to Q'.

Definition 3.1. $\theta \in \mathcal{P}(Q)$ satisfies *dimension conservation* (DC) if θ and $\bar{\theta}$ are p-regular, and almost all θ_x are p-regular with the same p-dimension, and for almost all x

$$\dim_p \theta = \dim_p \bar{\theta} + \dim_p \theta_x. \tag{3.2}$$

Note that (3.2) can also be written as

$$-\log\mathcal{H}(\theta) = -\log\mathcal{H}(\bar{\theta}) - \log\mathcal{H}(\theta_x).$$

The main result in this section is the assertion (Theorem 3.1) that, for any ergodic CP-shift system (Φ, T, μ) on $Q = Q^{(m_1+m_2)}$, almost every measure $\theta \in \mathcal{P}(Q)$ satisfies DC. We shall need the following lemma.

LEMMA 3.1. *Let* $(\Omega, \mathcal{B}, \mathbb{P})$ *be a probability space and* $A \in \mathcal{B}$, *and let* $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots$ *be an increasing subsequence of finite subfields of* $\mathcal{B}$. *The conditional probabilities* $\mathbb{P}(A \mid \mathcal{F}_n) = \mathbb{E}(\mathbf{1}_A \mid \mathcal{F}_n)$ *are almost everywhere defined, and setting*

$$f(\omega) = \mathbf{1}_A(\omega) \sup_n(-\log_p \mathbb{P}(A \mid \mathcal{F}_n)(\omega)),$$

then $f(\omega)$ *is integrable:* $\int f(\omega)\, d\mathbb{P}(\omega) < \infty$.

Proof. We show that $\sum_{N=0}^{\infty} \mathbb{P}(\omega : f(\omega) \geqslant N) < \infty$. $\mathbb{P}(A \mid \mathcal{F}_n)$ is constant on the atoms of $\mathcal{F}_n$, and, for each i, we consider all those atoms $\{A_{i,j}\}_j$ with $A_{i,j} \in \mathcal{F}_i$ for which $-\log_p \mathbb{P}(A \mid \mathcal{F}_i) \geqslant N$. We arrange these by increasing i, counting only those $A_{i,j}$ not contained in a previous $A_{i',j'}$ with $i' < i$. Let this sequence of disjoint sets be denoted by $B_1, B_2, \ldots, B_k, \ldots$, so that $\{\omega \mid f(\omega) \geqslant N\} = \bigcup A \cap B_k$. B_k is an atom of some $\mathcal{F}_n$, and so $\mathbb{P}(A \cap B_k) \leqslant p^{-N}\mathbb{P}(B_k)$. Summing over k, we obtain $\mathbb{P}(f(\omega) \geqslant N) \leqslant p^{-N}$, which proves the lemma. □

Suppose now that (Φ, T, μ) is an ergodic CP-system on $Q = Q' \times Q''$ as above. We can write $\Lambda = \Lambda' \times \Lambda''$ with $\Lambda' = \{0, 1, \ldots, p-1\}^{m_1}$, $\Lambda'' = \{0, 1, \ldots, p-1\}^{m_2}$. We shall denote elements of Λ' by ξ and those of Λ'' by η. Points of Q' can be expressed as $x = \sum_1^\infty \xi_n/p^n$, and those of Q'' expressed as $y = \sum_1^\infty \eta_n/p^n$. The points of Φ have the form $(\theta; x, y)$, and with x and y as above we can write

$$T(\theta; x, y) = \left(\frac{\rho_{\xi_1,\eta_1}(\theta)}{\theta(Q'_{\xi_1} \times Q''_{\eta_1})};\ px - [px],\ py - [py] \right),$$

and iterating, using the fact that $p^i x - [p^i x] = \sum_1^\infty \xi_{n+i}/p^n$ and $p^i y - [p^i y] = \sum_1^\infty \eta_{n+i}/p^n$,

$$T^k(\theta; x, y) = \left(\frac{\rho_{\xi_k,\eta_k}\rho_{\xi_{k-1},\eta_{k-1}} \cdots \rho_{\xi_1,\eta_1}(\theta)}{\theta(Q'_{\xi_1\ldots\xi_k} \times Q''_{\eta_1\ldots\eta_k})};\ p^k x - [p^k x],\ p^k y - [p^k y] \right). \tag{3.3}$$

We note that the measure θ_x on Q'' is given for almost every $x \in Q'$ and $B \subset Q''$ by

$$\theta_x(B) = \lim_{n\to\infty} \frac{\theta(Q'_n(x) \times B)}{\theta(Q'_n(x) \times Q'')} = \lim_{n\to\infty} \frac{\theta(Q'_n(x) \times B)}{\bar{\theta}(Q'_n(x))}$$

according to the martingale convergence theorem.

Define the functions $J_n(\theta; x, y)$ on Φ by

$$J_n(\theta; x, y) = \frac{\theta(Q'_n(x) \times Q''_1(y))}{\theta(Q'_n(x) \times Q'')}$$

and

$$J_\infty(\theta; x, y) = \theta_x(Q''_1(x)),$$

so that $J_n \to J_\infty$.

We apply (3.3) to calculate $J_n(T^k(\theta; x, y))$, which we also write as $T^k J_n(\theta; x, y)$, where we again write $x = \sum \xi_n/p^n$, $y = \sum \eta_n/p^n$:

$$\begin{aligned} T^k J_n(\theta; x, y) &= \frac{\theta(\rho^{-1}_{\xi_1,\eta_1} \cdots \rho^{-1}_{\xi_k,\eta_k}(Q'_n(p^k x - [p^k x]) \times Q''_1(p^k y - [p^k y])))}{\theta(\rho^{-1}_{\xi_1,\eta_1} \cdots \rho^{-1}_{\xi_k,\eta_k}(Q'_n(p^k x - [p^k x]) \times Q''))} \\ &= \frac{\theta(\rho^{-1}_{\xi_1,\eta_1} \cdots \rho^{-1}_{\xi_k,\eta_k}(Q'_{\xi_{k+1}\xi_{k+2}\cdots\xi_{k+n}} \times Q''_{\eta_{k+1}}))}{\theta(\rho^{-1}_{\xi_1,\eta_1} \cdots \rho^{-1}_{\xi_k,\eta_k}(Q'_{\xi_{k+1}\xi_{k+2}\cdots\xi_{k+n}} \times Q''))} \\ &= \frac{\theta(Q'_{\xi_1\xi_2\ldots\xi_{k+n}} \times Q''_{\eta_1\eta_2\ldots\eta_{k+1}})}{\theta(Q'_{\xi_1\xi_2\ldots\xi_{k+n}} \times Q''_{\eta_1\eta_2\ldots\eta_k})}. \end{aligned} \tag{3.4}$$

Replace n by $n-k$:

$$T^k J_{n-k}(\theta;\, x,\, y) = \frac{\theta(Q'_{\xi_1\ldots\xi_n} \times Q''_{\eta_1\ldots\eta_{k+1}})}{\theta(Q'_{\xi_1\ldots\xi_n} \times Q''_{\eta_1\ldots\eta_k})}. \tag{3.5}$$

If we now define $K_n = -\log_p J_n$ and $K_\infty = -\log_p J_\infty$, we obtain

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} T^k K_{n-k}(\theta;\, x,\, y) &= \frac{1}{n} \cdot -\log_p \frac{\theta(Q'_{\xi_1\ldots\xi_n} \times Q''_{\eta_1\ldots\eta_n})}{\theta(Q'_{\xi_1\ldots\xi_n} \times Q'')} \\ &= -\frac{1}{n}\log_p(\theta(Q_{\xi_1,\eta_1\ldots\xi_n,\eta_n})) + \frac{1}{n}\log_p \bar{\theta}(Q'_{\xi_1\ldots\xi_n}),\end{aligned} \tag{3.6}$$

which we write as $R_n + S_n$.

Fix k and let $n \to \infty$ in (3.5); we get

$$T^k K_\infty(\theta;\, x,\, y) = -\log_p \theta_x(Q''_{\eta_1\ldots\eta_{k+1}}) + \log_p \theta_x(Q''_{\eta_1\ldots\eta_k}),$$

so that

$$\frac{1}{n}\sum_{k=0}^{n-1} T^k K_\infty(\theta;\, x,\, y) = -\frac{1}{n}\log_p \theta_x(Q''_{\eta_1\ldots\eta_n}). \tag{3.7}$$

The limit of (3.7) exists almost everywhere (as a number $\leqslant \infty$) by the ergodic theorem; this shows that almost every θ_x is p-regular, and that the limit is in fact a finite constant δ. We now invoke a theorem of Maker [**M**] to assert that the limit in (3.6) also exists almost everywhere and has the same value δ.

THEOREM. (Maker [**M**]) *If $\{f_n\}$ are integrable functions on $(X, \mathcal{B}, \mu)$ where $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system and if $f_n(x) \to f_\infty(x)$ almost everywhere, and if* $\sup|f_n(x)| = g(x)$ *is integrable, then for almost every x,*

$$\lim \frac{1}{n}(f_n(x) + f_{n-1}(Tx) + \cdots + f_1(T^{n-1}x)) = \bar{f}_\infty(x),$$

where $\bar{f}_\infty(x) = \lim(1/n)\sum_{i=0}^{n-1} f_\infty(T^i x)$.

To use Maker's theorem we have to verify that $\sup_n K_n(\theta;\, x,\, y)$ is integrable. We use Lemma 3.1. Let $\mathcal{F}_n$ be the field generated by the p^{nm_1} atoms $\{Q'_{\lambda_1\lambda_2\ldots\lambda_n} \times Q''\}$. Then $J_n(\theta;\, x,\, y) = \theta(Q' \times Q''(y) \mid \mathcal{F}_n)$ and we can write

$$K_n(\theta;\, x,\, y) = -\sum_{\lambda\in\Lambda''} \mathbf{1}_{Q'\times Q''_\lambda} \log_p \theta(Q' \times Q''_\lambda \mid \mathcal{F}_n).$$

By Lemma 3.1 each of the p^{m_1} summands has an integrable supremum and so Maker's theorem is applicable. By the remark following Theorem 2.1, the measure θ in Q is for almost all $(\theta;\, x,\, y)$ a p-regular measure, and so $R_n \to -\log_p \mathcal{H}(\theta) = \dim_p \theta$. As a consequence $\lim S_n$ also exists almost everywhere showing that $\bar{\theta}$ is p-regular. We have thus proved the following.

THEOREM 3.1. *Let (Φ, T, μ) be an ergodic CP-shift system on $Q^{(m_1+m_2)}$. Then for almost every $(\theta,\ x,\ y) \in \Phi$ and with the Fubini decomposition of θ given in (3.1) we have $\bar{\theta}$ p-regular and almost each θ_x is p-regular and*

$$\dim_p \theta = \dim_p \bar{\theta} + \dim_p \theta_x. \tag{3.8}$$

We will be applying this result in the following way. Assume that $\dim_p \theta = \dim |\theta|$. $\dim_p \theta_x$ is a constant δ for almost every x with respect to $\bar{\theta}$. Let $A' \subset Q'$ be a compact set for which this is the case, as well as the convergence of $-(1/n)\log_p \bar{\theta}(Q'_{\xi_1\ldots\xi_n})$ to $\dim_p \bar{\theta}$, and with $\bar{\theta}(A') > 0$. By Theorem 1.1, $\dim A' \geqslant \dim_p \bar{\theta}$. We then have, by (3.8),

$$\delta + \dim\{x \mid \dim\{y \mid (x, y) \in |\theta|\}\} \geqslant \dim|\theta|,$$

and so the map $(x, y) \mapsto x$ taking $Q \to Q'$ is dimension conserving on $|\theta|$. (We have used here the fact that, for almost every x, $|\theta_x| \subset \{y \mid (x, y) \in |\theta|\}$.)

In the following sections we will show that, in any gallery $\mathcal{G}$ of sets, there is a set $A \in \mathcal{G}$ with $\dim A = \dim^* \mathcal{G}$ and supporting a measure θ with $\dim_p \theta = \dim^* \mathcal{G}$ and for which (3.8) is valid, i.e. which satisfies dimension conservation.

4. *Markovian CP shift systems*

We fix m and p and let $Q = Q^{(m)}$, $\Lambda = \{0, 1, \ldots, p-1\}^m$. Points of Q can be represented by sequences $\xi = \{\xi(1), \xi(2), \ldots\} \in \Lambda^{\mathbb{N}}$ where ξ represents the point $\hat{\xi} = \sum_1^\infty \xi(n)/p^n$. This representation is not exactly one-to-one, and there are technical reasons that make it advantageous to replace the connected space Q in $\mathcal{P}(Q) \times Q$ by the totally disconnected $\Xi = \Lambda^{\mathbb{N}}$, and to define a dynamical system on $\mathcal{P}(\Xi) \times \Xi$. The system we will construct for a gallery $\mathcal{G}$ will arise from a Markov process on the space $\mathcal{P}(\Xi) \times \Lambda$, and projecting this to $\mathcal{P}(Q) \times Q$ will give us the desired CPS system.

A probability measure ρ on Ξ determines a function σ on $\Lambda^* = \bigcup_{k \geqslant 0} \Lambda^k$ = finite length words over the alphabet Λ by setting $\sigma(w) = \rho$ (sequences in Ξ with initial segment $= w$). We can also define $\sigma(\emptyset) = 1$ for $\emptyset$ the empty word. We have

$$\begin{aligned} &\text{(a)} \quad \sigma(\emptyset) = 1, \\ &\text{(b)} \quad \sigma(w) \geqslant 0, \\ &\text{(c)} \quad \sigma(w) = \sum_{\lambda \in \Lambda} \sigma(w\lambda). \end{aligned} \tag{4.1}$$

Conversely, a function $\sigma : \Lambda^* \to [0, 1]$ satisfying (4.1) determines a probability measure $\hat{\sigma}$ on Ξ, and we shall identify $\mathcal{P}(\Xi)$ with the space of such functions, which we denote by Σ. Here Σ is a closed subset of $[0, 1]^{\Lambda *}$ in the compact topology of the latter and in our identification the topologies coincide.

If $\sigma \in \Sigma$ and $u \in \Lambda^*$ with $\sigma(u) > 0$, we can define $\sigma^u \in \Sigma$ by

$$\sigma^u(w) = \frac{\sigma(uw)}{\sigma(u)}. \tag{4.2}$$

One sees that the transition $\sigma \mapsto \sigma^u$ corresponds to replacing a measure θ on Q by the conditional probability measure on a subset Q_u with $\theta(Q_u) > 0$. In particular, for $\sigma \in \Sigma$, $\lambda \in \Lambda$ with $\sigma(\lambda) > 0$ and $w \in \Lambda^*$,

$$\widehat{\sigma^\lambda}(Q_w) = \frac{\sigma(\lambda w)}{\sigma(\lambda)} = \frac{\hat{\sigma}(Q_{\lambda w})}{\hat{\sigma}(Q_\lambda)} = \frac{\hat{\sigma}(\rho_\lambda^{-1} Q_w)}{\hat{\sigma}(Q_\lambda)} = \frac{\rho_\lambda(\hat{\sigma}|_{Q_\lambda})}{\hat{\sigma}(Q_\lambda)}(Q_w), \tag{4.3}$$

so that $\widehat{\sigma^\lambda} = \rho_\lambda(\hat{\sigma}|_{Q_\lambda})/\hat{\sigma}(Q_\lambda)$, which is the conditional probability measure on the subcube Q_λ.

We now define 'natural' probabilities on $M = \Sigma \times \Lambda$ as follows. The allowable transitions from a point (σ, λ) are to points of the form $(\sigma^{\lambda'}, \lambda')$ and the probabilities of these transitions are respectively $\sigma(\lambda')$. We need not be concerned that $\sigma^{\lambda'}$may not be defined because in this case the probability of transition to $(\sigma^{\lambda'}, \lambda')$ is 0. Note that λ does not figure in this and so transition probabilities are also induced on Σ directly. The foregoing transitions define a Markov operator $P : \mathcal{C}(M) \to \mathcal{C}(M)$ with

$$Pf(\sigma, \lambda) = \sum \sigma(\lambda') f(\sigma^{\lambda'}, \lambda'), \tag{4.4}$$

and a probability measure $\nu \in \mathcal{P}(M)$ is 'stationary' if, for every $f \in \mathcal{C}(M)$,

$$\int f \, d\nu = \int Pf \, d\nu. \tag{4.5}$$

This can also be written as $P^*\nu = \nu$ for the adjoint operator P^* on $\mathcal{P}(M)$.

We note that the k-step transitions have the form

$$(\sigma, \lambda) \mapsto (\sigma^{\lambda_1}, \lambda_1) \mapsto (\sigma^{\lambda_1\lambda_2}, \lambda_2) \mapsto \cdots \mapsto (\sigma^{\lambda_1\lambda_2\ldots\lambda_k}, \lambda_k),$$

with the probability of this sequence of transitions being

$$\begin{aligned}
&\sigma(\lambda_1)\sigma^{\lambda_1}(\lambda_2)\sigma^{\lambda_1\lambda_2}(\lambda_3) \ldots \sigma^{\lambda_1\lambda_2\ldots\lambda_{k-1}}(\lambda_k) \\
&\quad = \sigma(\lambda_1)\,(\sigma(\lambda_1\lambda_2)/\sigma(\lambda_1))\,(\sigma(\lambda_1\lambda_2\lambda_3)/\sigma(\lambda_1\lambda_2)) \\
&\qquad \ldots (\sigma(\lambda_1\lambda_2\ldots\lambda_k)/\sigma(\lambda_1\lambda_2\ldots\lambda_{k-1})) \\
&\quad = \sigma(\lambda_1\lambda_2\ldots\lambda_k).
\end{aligned}$$

Given a stationary measure ν we obtain a stationary Markov process $\{(X_n, \xi_n)\}_{n\geqslant 0}$ with values in M by setting

$$\begin{aligned}
\mathbb{E}(f_0(X_0)f_1(X_1)\ldots f_k(X_k)) = \int \sum_{\lambda_1,\lambda_2,\ldots,\lambda_k} &\sigma(\lambda_1\lambda_2\ldots\lambda_k) f_0(\sigma, \lambda) f_1(\sigma^{\lambda_1}, \lambda_1) \\
&\ldots f_k(\sigma^{\lambda_1\lambda_2\ldots\lambda_k}, \lambda_k)\, d\nu(\sigma, \lambda).
\end{aligned} \tag{4.6}$$

Definition 4.1. A stationary Markov process on $\Sigma \times \Lambda$ with transition probabilities as prescribed will be called a *natural* Markov process.

The set of stationary measures on M is compact, convex and is spanned by its extremals, which in turn correspond to the ergodic Markov processes with the given transitions.

For $\lambda \in \Lambda$ we write $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ and we consider the event A^i_ℓ that, for all $n \geqslant \ell$, $\xi^{(i)}_n = p - 1$. If T denotes the measure-preserving shift on which $\{(X_n, \omega_n)\}$ is defined, then A^i_ℓ satisfy $T^{-1}A^i_\ell = A^i_{\ell+1} \supset A^i_\ell$. Ergodicity implies $\mathbb{P}(A^i_\ell) = 0$ or 1, and since $\mathbb{P}(A^i_\ell) = \mathbb{P}(A^i_{\ell+1})$, either $\mathbb{P}(\bigcup_\ell A^i_\ell) = 0$ or, with probability 1, $\xi^{(i)}_n = p - 1$ for all n. Since $X_{n+1} = X_n^{\xi_{n+1}}$ (using the notation of (4.2)) the latter possibility imposes a restriction on the values of X_n that occur. Namely $X_n(w) > 0$ only for words $w = (\lambda_1, \ldots, \lambda_\ell)$ having $\lambda^{(i)}_1 = \lambda^{(i)}_2 = \cdots = \lambda^{(i)}_\ell = p - 1$.

Definition 4.2. We say that the ergodic process $\{(X_n, \omega_n)\}$ has dimension m' (for $m' < m$) if the foregoing restriction on $X_n(w) > 0$ takes place for $m - m'$ superscripts i. If it happens for no i, then we say that the process has full dimension m.

Clearly a process of dimension m' can be identified with a process of full dimension m' constructed with $\Lambda' = \{0, 1, \ldots, p-1\}^{m'}$.

Suppose now that $\{(X_n, \xi_n)\}$ is an ergodic process of full dimension. This process takes values in $\Sigma \times \Lambda$. We can form another ergodic stationary process $\{(X_n, \omega_n)\}$ with values in $\Sigma \times \Xi$ by setting $\omega_n = (\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+k}, \ldots)$. Consider the map $\Xi \to Q$ given by $\xi = (\xi(1), \xi(2), \ldots) \mapsto \hat{\xi} = \sum \xi(n)/p^n$. This induces a map $\mathcal{P}(\Xi) \to \mathcal{P}(Q)$ and if we identify $\mathcal{P}(\Xi)$ with Σ we obtain a map $\sigma \mapsto \hat{\sigma}$. While the map $\xi \mapsto \hat{\xi}$ is not one-to-one globally, by our assumption that $\{(X_n, \xi_n)\}$ has full dimension we may confine ourselves to ξ for which the event $\xi(n)^{(i)} = p-1$ for some ℓ and all $n \geqslant \ell$ does not occur, and for these $\xi \mapsto \hat{\xi}$ is one-to-one. In this case the inverse map $\hat{\xi} \mapsto \xi$ is defined by the sequence of conditions

$$Q_{w_r} = Q_r(\hat{\xi}), \tag{4.7}$$

where w_r is the initial r-segment of ξ: $w_r = \xi(1)\xi(2)\ldots\xi(r)$. We call these sequences, or points in Ξ, *regular*.

LEMMA 4.1. *For an ergodic process of full dimension $\{(X_n, \omega_n)\}$, if $w \in \Lambda^*$ then, with probability* 1, $\hat{X}_n(Q_w) = X_n(w)$.

Proof. We consider the pair (X_0, ω_0). Since, with probability 1, ω_0 is regular, we have with probability 1 that $\mathbb{P}(\omega_0 \text{ is regular } \mid X_0) = 1$. This means that X_0 as a measure is supported on the set of regular points. Let Ξ_w denote the sequences beginning with w. Then $X_0(\Xi_w) = X_0(w)$, where on the left side X_0 is regarded as a measure. But denoting regular points by R, $\widehat{\Xi_w \cap R} = Q_w$. This proves the lemma. □

We shall need the following lemma regarding conditional probabilities.

LEMMA 4.2. *Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space, $\mathcal{F}$ a sub-σ-field of $\mathcal{B}$ and $\Omega = \bigcup A_i$ a partition to disjoint sets. Then*

$$Z = \sum \mathbf{1}_{A_i} \mathbb{P}(A_i \mid \mathcal{F})$$

is positive with probability 1.

Proof. We have $0 \leqslant Z \leqslant 1$, and so

$$\lim_{n\to\infty} Z^{1/n} = \mathbf{1}_{(Z>0)}$$

and the lemma will follow if we show that $\mathbb{E}(Z^{1/n}) \to 1$. Now

$$Z^{1/n} = \sum \mathbf{1}_{A_i} \mathbb{P}(A_i \mid \mathcal{F})^{1/n}$$

and so $\mathbb{E}(Z^{1/n} \mid \mathcal{F}) = \sum \mathbb{P}(A_i \mid \mathcal{F})^{1+1/n}$. This converges to $\sum \mathbb{P}(A_i \mid \mathcal{F}) = \sum \mathbb{E}(\mathbf{1}_{A_i} \mid \mathcal{F}) = \mathbb{E}(\sum \mathbf{1}_{A_i} \mid \mathcal{F}) = 1$. All the expressions are bounded, so $\mathbb{E}(Z^{1/n}) = \mathbb{E}(\mathbb{E}(Z^{1/n} \mid \mathcal{F})) \to 1$. This proves the lemma. □

We use this in the following lemma.

LEMMA 4.3. *Let $\{(X_n, \omega_n)\}$ be the ergodic process derived from the Markov process $\{(X_n, \xi_n)\}$ of full dimension, and let $\Phi \subset \mathcal{P}(Q) \times Q$ as in §2. Then with probability* 1, $(\hat{X}_n, \hat{\omega}_n) \in \Phi$.

Proof. Taking $n = 0$, the requirement is that $\hat{X}_0(Q_n(\hat{\omega}_0)) > 0$ for all $n \geqslant 1$. Writing $\omega_0 = \lambda_1\lambda_2 \ldots \lambda_n \ldots$, by the regularity of ω_0 it follows that $Q_n(\hat{\omega}_0) = Q_{\lambda_1\lambda_2\ldots\lambda_n}$. By Lemma 4.1, $\hat{X}_0(Q_{\lambda_1\lambda_2\ldots\lambda_n}) = X_0(\lambda_1\lambda_2 \ldots \lambda_n)$. Letting $\mathcal{F}$ be the σ-field spanned by X_0 and the events A_i correspond to the events $\xi_1 = \lambda_1, \ldots, \xi_n = \lambda_n$ for the various choices of $\lambda_1, \ldots, \lambda_n$, the foregoing lemma states that $\mathbb{P}(\xi_1 = \lambda_1, \ldots, \xi_n = \lambda_n \mid X_0) > 0$ on the set for which $\xi_1 = \lambda_1, \ldots, \xi_n = \lambda_n$. But by the Markovian condition the conditional probability in question is precisely $X_0(\lambda_1\lambda_2 \ldots \lambda_n)$. □

LEMMA 4.4. *With $T : \Phi \to \Phi$ as in §2, we have almost everywhere $T(\hat{X}_n, \hat{\omega}_n) = (\hat{X}_{n+1}, \hat{\omega}_{n+1})$.*

Proof. We write $\omega_n = \xi_{n+1}, \xi_{n+2}, \ldots$ so that $X_{n+1} = X_n^{\xi_{n+1}}$ and $\hat{\omega}_{n+1} = p\hat{\omega}_n - \xi_{n+1}$. By (4.3),

$$\widehat{X_n^{\xi_{n+1}}} = \rho_{\xi_{n+1}}(\hat{X}_n|_{Q_{\xi_{n+1}}})/\hat{X}_n(Q_{\xi_{n+1}}),$$

and so, by (2.1), $T(\hat{X}_n, \hat{\omega}_n) = (\hat{X}_{n+1}, \hat{\omega}_{n+1})$. □

As a consequence of this lemma the distribution μ of $(\hat{X}_0, \hat{\omega}_0)$ on $\mathcal{P}(Q) \times Q$ is T-invariant. We have thus obtained an ergodic CPS system from the ergodic Markov process $\{(X_n, \xi_n)\}$ provided that the foregoing measure is adapted.

LEMMA 4.5. *The distribution of $(\hat{X}_0, \hat{\omega}_0)$ is an adapted measure on $\mathcal{P}(Q) \times Q$.*

Proof. We write $\omega_0 = \xi_1\xi_2, \ldots, \xi_n, \ldots$ and we must show that, for any $\lambda_1, \lambda_2, \ldots, \lambda_n$ in Λ, $\mu(x \in Q_{\lambda_1,\ldots,\lambda_n} \mid \theta) = \theta(Q_{\lambda_1,\ldots,\lambda_n})$. Since $\hat{X}_0$ is determined by X_0, the latter equality follows from

$$\begin{aligned}\mathbb{P}(\hat{w}_0 \in Q_{\lambda_1,\ldots,\lambda_n} \mid X_0) &= \mathbb{P}(\xi_1 = \lambda_1, \ldots, \xi_n = \lambda_n \mid X_0) \\ &= X_0(\lambda_1, \ldots, \lambda_n) = \hat{X}_0(Q_{\lambda_1,\ldots,\lambda_n}).\end{aligned}$$ □

We have proved the following.

THEOREM 4.1. *If $\{(X_n, \xi_n)\}$ is an ergodic natural Markov process on $\Sigma \times \Lambda$ with the prescribed transition probabilities and of full dimension, $\omega_n = \xi_{n+1}, \xi_{n+2}, \ldots$, then the distribution of $(\hat{X}_n, \hat{\omega}_n)$ defines an ergodic CPS system.*

We refer to a system constructed in this way as a *Markovian CPS system*. When the process $\{(X_n, \xi_n)\}$ has dimension $m' < m$, we have seen by ergodicity that $m - m'$ of the coordinates of the ξ_n are constant ($= p - 1$) , and we can regard the process as having full dimension as a natural Markov process with Λ replaced by $\Lambda' = \{0, 1, \ldots, p-1\}^{m'}$. The resulting CPS system will effectively be confined to a face $Q' = [0, 1]^{m'}$ of $Q = [0, 1]^m$.

Recall from §2 the definition of $I(\theta, x)$ as $-\log_p \theta(Q_1(x))$. For a Markovian CPS system we can write $I(\hat{X}_n, \hat{\omega}_n) = -\log_p X_n(\xi_{n+1})$. With some abuse of notation we denote the latter expression as well by $I(X_n, \xi_{n+1})$, representing the information obtained knowing that the transition $(X_n, \xi_n) \to (X_n^{\xi_{n+1}}, \xi_{n+1})$ has taken place. More precisely, we define two functions on Σ and $\Sigma \times \Lambda$ respectively.

Definition 4.3. If $\sigma \in \Sigma$ we set

$$\mathcal{E}(\sigma) = -\sum_{\lambda \in \Lambda} \sigma(\lambda) \log_p \sigma(\lambda)$$

taking $t \log t = 0$ for $t = 0$. For $\sigma \in \Sigma$ and $\lambda \in \Sigma$ we set

$$I(\sigma, \lambda) = -\log_p \sigma(\lambda).$$

We now have the following.

PROPOSITION 4.1. *For the stationary Markov process* $\{(X_n, \xi_n)\}$ *we have*

$$\mathbb{E}(\mathcal{E}(X_n)) = \mathbb{E}(I(X_n, \xi_{n+1})). \tag{4.8}$$

Proof. The right-hand side can be written as $\mathbb{E}(\mathbb{E}(I(X_n, \xi_{n+1}) \mid X_n))$ and, by the definition of transition probabilities, the inside expression is $\mathcal{E}(X_n)$. $\square$

We combine the foregoing with Theorem 2.1 to obtain the following.

THEOREM 4.2. *Let* (Φ, T, μ) *be the Markovian CPS system derived from the ergodic Markov process* $\{(X_n, \xi_n)\}$. *Then for almost every* $(\theta, x) \in \mathcal{P}(Q) \times Q$ *we will have* θ *p-regular and*

$$\dim |\theta| \geqslant \dim_p \theta \geqslant \mathbb{E}(\mathcal{E}(X_n)). \tag{4.9}$$

We conclude this section with the following lemma.

LEMMA 4.6. *Regarding* $\mathcal{E}(\sigma)$ *as a function on* $\Sigma \times \Lambda$ *we have*

$$\mathcal{E}(\sigma) + P\mathcal{E}(\sigma) + P^2\mathcal{E}(\sigma) + \cdots + P^{n-1}\mathcal{E}(\sigma) = -\sum_{\ell(w)=n} \sigma(w) \log_p \sigma(w).$$

Proof. Iterating (4.4) we can write

$$P^k f(\sigma, \lambda) = \sum \sigma(\lambda_1\lambda_2 \ldots \lambda_k) f(\sigma^{\lambda_1\lambda_2\ldots\lambda_k}, \lambda_k).$$

With $f(\sigma, \lambda) = \mathcal{E}(\lambda)$ we obtain

$$\begin{aligned} P^k\mathcal{E}(\sigma) &= -\sum_{\lambda_1\lambda_2\ldots\lambda_k} \sigma(\lambda_1\lambda_2 \ldots \lambda_k) \sum_{\lambda} \sigma^{\lambda_1\lambda_2\ldots\lambda_k}(\lambda) \log_p \sigma^{\lambda_1\lambda_2\ldots\lambda_k}(\lambda) \\ &= -\sum_{\lambda_1\lambda_2\ldots\lambda_k,\lambda} \sigma(\lambda_1\lambda_2 \ldots \lambda_k\lambda) \log_p \left(\frac{\sigma(\lambda_1\lambda_2 \ldots \lambda_k\lambda)}{\sigma(\lambda_1\lambda_2 \ldots \lambda_k)}\right) \\ &= -\sum_{\ell(w)=k+1} \sigma(w) \log_p \sigma(w) + \sum_{\ell(w)=k} \sigma(w) \log_p \sigma(w). \end{aligned}$$

Adding these expressions for $k = 0, 1, \ldots, n-1$ we obtain the identity of the theorem. $\square$

5. *Ergodic fractal measures on galleries*

Recalling the notation of §2 for $(\theta, x) \in \Phi$ we consider the orbit $\{T^n(\theta, x) = (\theta_n, x_n)\}$ in $\mathcal{P}(Q) \times Q$. We note that the supports of all the measures θ_n in the orbit are mini-sets of $|\theta|$, and for the entire orbit closure, the measures involved are micro-sets of $|\theta|$. This suggests defining, for any gallery $\mathcal{G}$ of sets in Q, the closed T-invariant subspace $\Phi_{\mathcal{G}} \subset \Phi$ consisting of pairs (θ, x) with supports $|\theta| \in \mathcal{G}$. Our main result in this section is the assertion that, for any gallery $\mathcal{G}$ having sets of positive Hausdorff dimension, there exist non-degenerate ergodic fractal measures supported in the gallery. More precisely, we show that if $\dim^* \mathcal{G} > 0$, then there exists an ergodic CPS system in $\Phi_{\mathcal{G}}$ such that, for almost every (θ, x), $\dim |\theta| \geqslant \dim^* \mathcal{G}$.

Toward this end we shall avail ourselves of the construction in the foregoing section producing a Markovian CPS system. In view of Theorem 4.2, we shall have achieved this goal once we find a natural Markov process $\{(X_n, \xi_n)\}$ with $\mathbb{E}(\mathcal{E}(X_n)) = \dim^* \mathcal{G}$ and with $|\hat{X}_n| \in \mathcal{G}$.

We introduce a new measure of the 'size' of sets in a gallery $\mathcal{G}$. For any set $A \subset Q$ and $\ell \geqslant 1$ we set

$$N_\ell(A) = \#\{w \mid \ell(w) = \ell \text{ and } A \cap Q_w \neq \emptyset\}.$$

Definition 5.1. For a gallery $\mathcal{G}$, $N_\ell(\mathcal{G}) = \max_{A \in \mathcal{G}} N_\ell(A)$.

LEMMA 5.1. $N_{\ell_1+\ell_2}(\mathcal{G}) \leqslant N_{\ell_1}(\mathcal{G}) N_{\ell_2}(\mathcal{G})$.

Proof. For a given set $A \in \mathcal{G}$, each Q_{w_1} meeting A with $\ell(w_1) = \ell_1$ determines a mini-set A' of A and the number of subcubes of degree $\ell_1 + \ell_2$ of Q_{w_1} meeting A is $N_{\ell_2}(A')$. Thus $N_{\ell_1+\ell_2}(A) \leqslant N_{\ell_1}(A) N_{\ell_2}(\mathcal{G})$. □

It follows that

$$\lim_{\ell \to \infty} \frac{\log_p N_\ell(\mathcal{G})}{\ell}$$

exists.

Definition 5.2. For a gallery $\mathcal{G}$,

$$\Delta(\mathcal{G}) = \lim_{\ell \to \infty} \frac{\log_p N_\ell(\mathcal{G})}{\ell}.$$

Clearly the upper box-dimension (or upper Minkowski dimension) of any set in $\mathcal{G}$ is bounded by $\Delta(\mathcal{G})$. It will follow from the main theorem of this section that there exist sets in $\mathcal{G}$ with Hausdorff dimension $\geqslant \Delta(\mathcal{G})$. As a result we must have equality and $\Delta(\mathcal{G}) = \dim^* \mathcal{G}$ as defined in §1.

We fix a gallery $\mathcal{G}$ of sets in $Q = Q^{(m)} \subset \mathbb{R}^m$, and $\Lambda = \{0, 1, \ldots, p-1\}^m$ as usual, and Σ, Ξ are defined as in §4.

Definition 5.3. We define a compact subset $\Sigma_{\mathcal{G}} \subset \Sigma$ by $\Sigma_{\mathcal{G}} = \{\sigma \in \Sigma : |\hat{\sigma}| \in \mathcal{G}\}$.

$\Sigma_{\mathcal{G}}$ is closed and therefore compact, because the support of a weak limit of measures is contained in the limit of the supports. We have seen in (4.3) that if $\sigma(\lambda) > 0$ then $\widehat{\sigma^\lambda}$ is $\hat{\sigma}$ conditioned on Q_λ, and the support of this is again in $\mathcal{G}$ if $|\sigma| \in \mathcal{G}$.

Finally, we set $M_{\mathcal{G}} = \Sigma_{\mathcal{G}} \times \Lambda$, and observe that the 'natural' transitions defined on $M \subset \Sigma \times \Lambda$ take $M_{\mathcal{G}}$ to itself. There will exist stationary measures on $M_{\mathcal{G}}$ for these transition probabilities and these will determine processes $\{(X_n, \xi_n)\}$ with $X_n \in \Sigma_{\mathcal{G}}$. We shall call these '$\mathcal{G}$-restricted processes'. Clearly the ergodic components of a $\mathcal{G}$-restricted process are $\mathcal{G}$-restricted, and we note the following for an ergodic $\mathcal{G}$-restricted natural Markov process.

PROPOSITION 5.1. *If $\{(X_n, \xi_n)\}$ is an ergodic $\mathcal{G}$-restricted natural Markov process, then* $\mathbb{E}(\mathcal{E}(X_n)) \leqslant \Delta(\mathcal{G})$.

Proof. This follows from Theorem 4.2, since if (Φ, T, μ) is the Markovian CPS system derived from $\{(X_n, \xi_n)\}$, then the measures θ appearing for almost every (θ, x) in Φ satisfy $|\theta| \in \mathcal{G}$. □

COROLLARY. *For any $\mathcal{G}$-restricted natural Markov process $\{(X_n, \xi_n)\}$, $\mathbb{E}(\mathcal{E}(X_n)) \leqslant \Delta(\mathcal{G})$.*

We now show how to construct a $\mathcal{G}$-restricted process with $\mathbb{E}(\mathcal{E}(X_n)) = \Delta(\mathcal{G})$. Since this expectation is the average of the same expression for the ergodic components, and for these we have the inequality of Proposition 5.1, this will imply the existence of an ergodic $\mathcal{G}$-restricted process with $\mathbb{E}(\mathcal{E}(X_n)) = \Delta(\mathcal{G})$, and again using Theorem 4.2 this will give us the existence statement described in the introduction to this section.

Remark. In the foregoing analysis we have used implicitly the fact that the ergodic components of a stationary Markov process are Markov processes with the same transition probabilities. This follows from [**D**, p. 460, Theorem 1.1].

PROPOSITION 5.2. *For any gallery $\mathcal{G}$ there exists a $\mathcal{G}$-restricted natural Markov process $\{(X_n, \xi_n)\}$ with* $\mathbb{E}(\mathcal{E}(X_n)) = \Delta(\mathcal{G})$.

Proof. The stationary Markov process in question is determined by a stationary measure ν on $\Sigma_{\mathcal{G}} \times \Lambda$, and $\mathbb{E}(\mathcal{E}(X_n)) = \int \mathcal{E}(\sigma)\, d\nu(\sigma, \lambda) = \int \mathcal{E}(\sigma)\, d\bar{\nu}(\sigma)$, $\bar{\nu}$ being the projection of ν on $\Sigma_{\mathcal{G}}$. For each ℓ, let $A_\ell \in \mathcal{G}$ with $N_\ell(A_\ell) = N_\ell(\mathcal{G})$. Let θ_ℓ be a measure supported on A_ℓ such that $\theta_\ell(Q_w) = N_\ell(\mathcal{G})^{-1}$ for each w with $A_\ell \cap Q_w \neq \emptyset$. Let $\sigma_\ell \in \Sigma_{\mathcal{G}}$ with $\hat{\sigma}_\ell = \theta_\ell$. We fix $\lambda_0 \in \Lambda$ and form the Dirac measure $\delta_{(\sigma_\ell, \lambda_0)}$ on $M_{\mathcal{G}}$. Finally, let

$$\nu_\ell = \frac{1}{\ell}(\delta_{(\sigma_\ell,\lambda_0)} + P^*\delta_{(\sigma_\ell,\lambda_0)} + P^{*2}\delta_{(\sigma_\ell,\lambda_0)} + \cdots + P^{*(\ell-1)}\delta_{(\sigma_\ell,\lambda_0)}).$$

Here, P^* is the operator on $\mathcal{P}(M_{\mathcal{G}})$ adjoint to $P : \mathcal{C}(M_{\mathcal{G}}) \to \mathcal{C}(M_{\mathcal{G}})$. Any weak limit of the $\nu_\ell \in \mathcal{P}(M_{\mathcal{G}})$ will be a stationary measure ν on $\Sigma_{\mathcal{G}} \times \Lambda$. We claim that $\int \mathcal{E}(\sigma)\, d\nu(\sigma, \lambda) = \Delta(\mathcal{G})$. To see this, note that

$$\begin{aligned}\int \mathcal{E}(\sigma)\, d\nu_\ell(\sigma, \lambda) &= \frac{1}{\ell}\int [\mathcal{E}(\sigma) + P\mathcal{E}(\sigma) + P^2\mathcal{E}(\sigma) + \cdots + P^{\ell-1}\mathcal{E}(\sigma)]\, d\delta_{\sigma_\ell} \\ &= \frac{1}{\ell}[\mathcal{E}(\sigma_\ell) + P\mathcal{E}(\sigma_\ell) + P^2\mathcal{E}(\sigma_\ell) + \cdots + P^{\ell-1}\mathcal{E}(\sigma_\ell)].\end{aligned}$$

Referring back to Lemma 4.6 we see that this expression equals

$$-\frac{1}{\ell}\sum_{\ell(w)=\ell} \sigma_\ell(w) \log_p \sigma_\ell(w).$$

Now

$$\sigma_\ell(w) = \hat{\sigma}_\ell(Q_w) = \theta_\ell(Q_w) = N_\ell(\mathcal{G})^{-1},$$

which gives

$$\int \mathcal{E}(\sigma)\, d\nu_\ell(\sigma, \lambda) = \frac{\log_p N_\ell(\mathcal{G})}{\ell}.$$

As $\ell \to \infty$ this converges to $\Delta(\mathcal{G})$ and, since $\mathcal{E}(\sigma)$ is a continuous function, the weak convergence to ν implies

$$\int \mathcal{E}(\sigma)\, d\nu(\sigma, \lambda) = \Delta(\mathcal{G}),$$

which is the statement of the proposition. □

In view of our previous remarks we obtain the following.

THEOREM 5.1. *For any gallery $\mathcal{G}$ of sets in $Q = Q^{(m)} \subset \mathbb{R}^m$, there exists an ergodic Markovian CPS system (Φ, T, μ) so that, for almost every (θ, x), $|\theta| \in \mathcal{G}$, θ is p-regular and* $\dim |\theta| = \dim_p \theta = \Delta(\mathcal{G})$.

In particular, there exists a set $A \in \mathcal{G}$ with Hausdorff dimension equal to $\Delta(\mathcal{G})$, and so $\sup_{A\in\mathcal{G}}\{\dim A\}$ is always attained and equals $\Delta(\mathcal{G})$.

6. *Dimension conservation*

The results of the foregoing section apply in particular to the gallery $\mathcal{G}_A$ of micro-sets of a set $A \in 2^Q$. $\Delta(\mathcal{G}_A) \geqslant \dim A$. We can combine Theorem 5.1 with Theorems 1.1 and 3.1 to show that the projection $(x, y) \mapsto x$ of $Q^{(m_1+m_2)} \to Q^{(m_1)}$ is dimension conserving on some micro-set of A. Namely, let (Φ, T, μ) be the ergodic CPS system with measures θ supported on micro-sets of A with $\dim_p \theta = \Delta(\mathcal{G}_A)$. By Theorem 3.1 we have a decomposition for almost every θ,

$$\theta = \int_{Q^{(m_1)}} \delta_x \times \theta_x \, d\bar{\theta}(x),$$

where $\theta_x \in \mathcal{P}(Q^{(m_2)})$ with $\bar{\theta}$ p-regular and almost every θ_x p-regular with $\dim_p \theta_x = \delta$, and

$$\dim_p \bar{\theta} + \delta = \Delta(\mathcal{G}_A). \tag{6.1}$$

Moreover, $|\theta| \in \mathcal{G}_A$. Since, by Theorem 1.1, for any p-regular measure ρ, $\dim |\rho| \geqslant \dim_p \rho$, (6.1) implies that on the set $|\theta|$ the projection $(x, y) \mapsto x$ is dimension conserving. Since $\dim A \leqslant \Delta(\mathcal{G}_A)$, this gives the following.

PROPOSITION 6.1. *If L is a projection map from one Euclidean space to another, every compact set A in the domain of L has a micro-set A' with* $\dim A' \geqslant \dim A$ *on which L is dimension conserving.*

Now for any linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ we can find a projection map with domain $\mathbb{R}^n$ and with the same kernel and range as L. This allows us to generalize the foregoing proposition to the following.

THEOREM 6.1. *If L is a linear map from one Euclidean space to another, every compact set A in the domain of L has a micro-set of dimension $\geqslant \dim A$ on which L is dimension conserving.*

In the case of a homogeneous fractal, we can replace 'micro-set' in the above statement by mini-set. Now a mini-set is homothetic to a subset and it is easy to see that, for linear maps, conservation of dimension for a set and for a homothetic image are equivalent. Finally, since the subset is question has dimension no less than $\dim A$, dimension conservation for the subset implies dimension conservation for A. This gives the final result.

THEOREM 6.2. *If A is a homogeneous set in Euclidean space, the restriction of a linear map to A is dimension conserving.*

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