Lecture notes for seminar on Schmidt games and approximation

Guy Lachman

December 2021

1 Introduction

One of the most efficient and beneficial tools in the study of Diophantine approximations is the tool of dynamics on homogeneous spaces: the study of the dynamics of the action of a Lie group G on a space X on which it acts transitively.
In this lecture we will focus on the special group $SL_d(\mathbb{R})$, acting on the space of lattices in $\mathbb{R}^d$. Our main goal will be to prove some fundamental results regarding this action and show some exciting ways of applying them to problems that originate from the the study of Diophantine approximations.

We first recall from the previous lecture that lattices are discrete additive subgroups of $\mathbb{R}^d$, and that they have the following useful characterization: all lattices in $\mathbb{R}^d$ are images of the lattice $\mathbb{Z}^d$ under a linear transformation. Now, restricting ourselves to the space of lattices of covolume 1, we obtain a transitive action of $SL_n(\mathbb{R})$ on the space of such lattices.

Now, naturally since we have a group acting on a space transitively we would like to consider its quotient by the stabilizer of a certain element (in our case this element is the lattice $\mathbb{Z}^d$). This quotient, would be canonically isomorphic to the original space, by identifying an equivalence class with the image of the special element we chose in our original space ($\mathbb{Z}^d$). So, if we have our space of lattices: X_d, then we have: $SL_d(\mathbb{R})/SL_d(\mathbb{Z}) \cong X_d$.
So in order to study the space of lattices it's enough to consider this quotient space.

Some crucial tools we need to develop in order to properly study the space of lattices, are a natural topology, and measure we can endow it with.

For the topology:
notice first that $SL_d(\mathbb{R})$ can naturally be endowed with a metric, as it can be thought of as a closed manifold in d^2 dimensional space. Then, since $SL_d(\mathbb{Z}) \subseteq SL_d(\mathbb{R})$ is a discrete subgroup, one can just endow $SL_d(\mathbb{R})/SL_d(\mathbb{Z}) = X_d$ with the quotient topology.
Another way of constructing a topology on X_d is the following:
we construct a metric D called the Chaubuty-Fell metric, on the set of closed sets in $\mathbb{R}^d$, $Cl(\mathbb{R}^d)$.
For two closed sets X, Y define: $D(X,Y) = inf(\{1\} \cup \{0 < \epsilon < 1 : \forall x \in X \cap B(0, \frac{1}{\epsilon}) \exists y \in Y s.t \|x - y\| < \epsilon \text{and vice versa}\})$

Theorem 1.1. *Let $\{L_j\}$ be a sequence of lattices in $\mathbb{R}^d$. Then the following are equivalent:*
a. The sequence $\{L_j\}$ converges to a lattice L with respect to D.
b. i. for all $l \in L$ there exists a sequence $l_j \in L_j$ s.t $l_j \to l$.
ii. If $l_{j_k} \in L_{j_k}$ converges to some l_∞ then $l_\infty \in L$.
c. For a basis $v_1, ..., v_d$ of L there exist bases $v_1^{(j)}, ...v_d^{(j)}$ such that for all i, $v_i^{(j)} \to v_i$.
d. The L_js converge to L in the quotient topology on X_d induced on it by the identification: $SL_n(\mathbb{R})/SL_n(\mathbb{Z}) \cong X_d$.

One extremely useful and fundamental tool in the theory of homogeneous dynamics on lattices is Mahler's compactness criterion. Which provides a simple way to verify weather a certain sequence of lattices has a convergent subsequence.

2 Mahler's compactness theorem

Theorem 2.1. *Suppose $A \subseteq G = SL_n(\mathbb{R})$, and let $B = \pi(A)$ be its projection to the quotient. Then the following are equivalent:*
a. $\bar{B}$ is compact
b. There exists some $\delta > 0$ s.t for all the lattices $L \in B$, $\lambda_1(L) > \delta$.
c. here exists some $\delta > 0$ s.t for every $g \in A$, and $v \in \mathbb{Z}^n$, we have: $\|gv\| \geq \delta$.

An equivalent formulation of this claim would be the following: given a sequence of lattices $L_n = \pi(g_n) \longrightarrow \infty$ in the space of lattices if and only if $\lambda_1(L_n) \longrightarrow 0$
Reminder: Recall from Alon's lecture that we defined the $\lambda_i(L)$ to be the smallest radius for which the ball of radius r around the origin contains k linearly independent vectors from the lattice.
in order to prove Mahler's compactness theorem we will also need the following result from the previos lecture:

Theorem 2.2. *:There exist constants c_1, c_2 depending only on the dimension and a basis $v_1, ..., v_d$ such that $c_1\lambda_i(L) \leq \|v_i\| \leq c_2\lambda_{i+1}(L)$ and: $c_1 \leq \frac{\lambda_1(L)...\lambda_d(L)}{covol(L)} \leq c_2$.*

We now have all the tools we need, in order to prove Mahler's theorem:

Proof. There is an obvious equivalence between parts (b) and (c): the $_1$s are bounded uniformly in B from below if and only if, for some uniform constant in B, all the vectors in all the latices have norm bigger than this constant, and the lattices in B are of the form $g\mathbb{Z}^d$ for $g \in A$.

Let us prove that $(a) \Rightarrow (b)$: Suppose that $\bar{B}$ is compact, and assume on the contrary that there exists a sequence of lattices L_n such that $\lambda_1(L_n) \to 0$ all in B. Since B is cocompact, there we may pass to a convergent subsequence, that converges to some lattice L (w.l.o.g L_n itself).
Now since L is a lattice it has, for some $0 \leq \epsilon \leq \lambda_1(L)$. For all n large enough, L_n contains a vector v_n of length less than ϵ. We may multiply that v_n by an integer, and assume w.l.o.g that $\frac{\epsilon}{2} \leq \|v_n\| \leq \epsilon$. Hence (passing to seubsequence) $v_n \to v$ for some $v \in L$ with $\frac{\epsilon}{2} \leq \|v\| \leq \epsilon$ (here we are using the equivalence theorem for the topologies). Such a vector exists for all $\epsilon \geq 0$ and this is a contradiction.

For $(b) \Rightarrow (a)$: Take a sequence L_n of lattices in B. We show it has a convergent subsequence: the lattices Λ_n are unimodular,so they have covolume 1. From 2.2, it follows that: $c_1 \leq \lambda_1(L_n)...\lambda_d(_n) \leq c_2$ hence since $_1 leq \lambda_2 leq...leq \lambda_d$ and since the λ_1s are bounded from below by δ we obtain: $delta^{d-1}\lambda_d(L_n) \leq \lambda_1(L_n)^{d-1}\lambda_d(L_n) \leq \lambda_1(L_n)...\lambda_d(L_n) \leq c_2$ So we have a uniform bound on the λ_ds.
Take a basis $v_1^n, ..., v_d^n$ of L_n taken such that $c_1\lambda_i(L_n) \leq v_i^n \leq c_2\lambda_i(L_n)$ Hence there is a ball of a uniform radius R ($R = fracc_2{}^2\delta^d$) such that all the lattices in the sequence have a basis to them in the ball of radius R. Take now $g_n = ((v_1^n, ..., v_d^n)) \in SL_n(\mathbb{R})$ to be the matrix with columns v_i^n. Then $\pi(g_n) = L_n$, and since the v_i^ns have bounded norms then the g_ns have bounded entries. Hence g_n has a convergent subsequence g_{nk} and then $\pi(g_{nk}) = L_{nk}$ converges as well. So we showed $\bar{B}$ is compact hence we are done! ☐

3 Dani correspondence

In this section we prove and discuss the following theorem which is due to Dani: (The original theorem was stated and proved in a more general setting but we will only discuss this version of it): $define$ g$_t$ to be the matrix: $g_t = \begin{pmatrix} e^t & & & & & \\ & e^t & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & e^t & \\ & & & & & e^{-dt} \end{pmatrix}$ which is a $(d+1)\times(d+1)$ matrix.

For a vector $\mathbf{t}$ define the matrix $g_{\mathbf{t}} = \begin{pmatrix} e^{t_1} & & & & & \\ & e^{t_2} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & e^{t_d} & \\ & & & & & e^{-[\mathbf{t}]} \end{pmatrix}$ where $[t] = t_1 + ... + t_d$.

For a vector $x \in \mathbb{R}^d$ define: $\tau(x) = \begin{pmatrix} 1 & & & & x_1 \\ & 1 & & & x_2 \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & 1 & x_d \\ & & & & 1 \end{pmatrix}$ We also define, $L_x = \pi(\tau(x))$.

Theorem 3.1. *A vector $x \in \mathbb{R}^d$ is in BA_d if and only if the orbit $\{g_t L_x : t \geq 0\}$ is bounded.*

Proof. the vectors in $g_t L_x$ are of the form $g_t\tau(x)v$ where $v \in \mathbb{Z}^{d+1}$. Let us separate the last coordinate of v from the first d ones and write $v = (-\mathbf{p}, q)$. Now: $g_t\tau(x)v = (e^t(qx_1 - p_1), ..., e^t(qx_d - p_d), e^{-dt}q)$.
Let us first show that if the orbit is bounded then the vector is in BA_d: Using Mahler compactness, obtain that there exits some $\delta > 0$ such that for all v, $\|g_t\tau(x)v\| \geq \delta$ using the sup norm. (The orbit is bounded and hence cocompact). Take δ to be smaller than 1, and choose t such that $e^{-dt}q = \frac{\delta}{2}$. Now $max_i|e^t(qx_i - p_i)| \geq \delta$. Hence we obtain: $\|x - \frac{1}{q}\mathbf{p}\|_\infty \geq \frac{\delta}{qe^t} = \frac{\delta}{q}\frac{\delta^{\frac{1}{d}}}{2^{\frac{1}{d}}q^{\frac{1}{d}}} = \frac{c}{q^{1+\frac{1}{d}}}$ (c depends on both x and the dimension).

Now for the second direction: suppose $x \in BA_d$. Let us show the orbit isn't bounded: take $c > 0$, such that $\|x - \frac{1}{q}\mathbf{p}\| \geq \frac{c}{q^{1+\frac{1}{d}}}$ for all $\mathbf{p} \in \mathbb{Z}^d$ and $p \in \mathbb{Z}$. Again write $v = (-\mathbf{p}, q)$. First, if $q = 0$: Then $\|g_t\tau(x)v\| = e^t\|\mathbf{p}\| \geq e^t \geq 1$. Hence in this case the orbit is bounded. In the case when $q \neq 0$: if $e^{-dt}q \geq 1$, then obviously $\|g_t\tau(x)v\| \geq 1$.
Otherwise, we must have $e^{-dt}q < 1$ and hence $q^{-\frac{1}{d}} > e^{-t}$ and hence, since $\|x - \frac{1}{q}\mathbf{p}\| \geq \frac{c}{q^{1+\frac{1}{d}}} \Rightarrow \|qx - \mathbf{p}\|_\infty \geq \frac{c}{q^{\frac{1}{d}}} \geq ce^{-t}$. So there must exist some i such that $e^t(qx_i - p_i) \geq c$. Hence $\|g_t\tau(x)v\|_\infty \geq c$, So the orbit is bounded. $\square$

4 Singular vectors

definition 4.1. *Say a vector $x \in \mathbb{R}^d$ is* singular *if for any $\delta > 0$ there exists some T_0 such that for all $T \geq T_0$, we can find $\mathbf{p} \in \mathbb{Z}^n$ and $q \in \mathbb{Z}^+$ such that $\|qx - \mathbf{p}\| < \frac{\delta}{T^{\frac{1}{d}}}$ and $q < T$*

Singular vectors are often referred to as those for which Dirichlet's theorem can be infinitely improved. Note that the existence of singular vectors is trivial: if $d > 1$: we can take a vector x which is not totally irrational, meaning that its coordinates are not independent over $\mathbb{Q}$. The existence of singular vectors which are totally irrational was proved in the work of Khintchine in the 1920s. However, the set of singular vectors has Lebesgue measure 0.
In this lecture we will mostly be interested in the following generalizations of singular vectors:

definition 4.2. *Given $Y \in M_{m,n}$, we say Y can be ϵ improved, and write $Y \in DI_\epsilon(m,n)$ if for any sufficiently large t we can find $q \in \mathbb{Z}^n$, $\boldsymbol{q} \neq 0$ and $\boldsymbol{p} \in \mathbb{Z}^m$ such that $\|Y\boldsymbol{p} - \boldsymbol{q}\| \leq \epsilon e^{-\frac{t}{m}}$ and $\|\boldsymbol{q}\| \leq \epsilon e^{\frac{t}{n}}$. We call Y singular if $Y \in DI_\epsilon$ for all $\epsilon > 0$.*

Putting the previous definition in context: the Dirichlet theorem for matrices states:

Theorem 4.3. *for any t we can find $q \in \mathbb{Z}^n$, $\boldsymbol{q} \neq 0$ and $\boldsymbol{p} \in \mathbb{Z}^m$ such that: $\|Y\boldsymbol{p} - \boldsymbol{q}\| \leq e^{-\frac{t}{m}}$ and $\|\boldsymbol{q}\| \leq e^{\frac{t}{n}}$*

We have the following dynamical correspondence theorem for singular matrices:
For matrix $Y \in M_{m,n}(\mathbb{R})$. Define the following matrix $\tau(Y) = \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix} \in SL_{m+n}(\mathbb{R})$. Also define $\bar{\tau}$ to be $\bar{\tau} = \pi \circ \tau$. When π stands for the projection from $SL_{m+n}(\mathbb{R})$ to $SL_{m+n}(\mathbb{R})/SL_{m+n}(\mathbb{Z})$.

Theorem 4.4. *A matrix Y is singular if and only if for all t $g_t = diag(e^{\frac{t}{m}}, ..., e^{\frac{t}{m}}, e^{-\frac{t}{n}}, ..., e^{-\frac{t}{n}})$, $g_t\bar{\tau(Y)}$ is a divergent sequence (in the space of lattices).*