$G = SL_2(\mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R}) = \{ m \in \mathrm{Mat}_2(\mathbb{R}) : \mathrm{tr}(m) = 0 \}$

$g_i = \begin{pmatrix} \varrho^{-1} & x_i \\ & \varrho \end{pmatrix}$ for some $\varrho \in]0,1[$, $(x_1, \dots, x_r) \in \mathbb{R}^r - \{0\}$.

$E = \{g_1, \dots, g_r\}$, $B = E^{\mathbb{N}}$, μ on E of full support and $\beta = \mu^{\otimes \mathbb{N}}$.

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}) = \mathfrak{g}$$

is a basis of $\mathfrak{g}$, if $a \in \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, then

$\mathrm{Ad}_a(X) = e^{2t} X$, $\mathrm{Ad}_a(H) = H$, $\mathrm{Ad}_a(Y) = e^{-2t} Y$

$\mathfrak{g} \underset{\Phi}{\cong} \Lambda^2 \mathfrak{g}$ as a G-representation.

$$\Phi(X) = -\frac{1}{2} X \wedge H, \quad \Phi(H) = \underbrace{X \wedge Y}_{\text{eigenv. for EV } 1}, \quad \Phi(Y) = -\frac{1}{2} H \wedge Y$$

$e^{2t} X \wedge e^{-2t} Y = X \wedge Y$

$W^{\wedge 1} \leq \mathfrak{g}$, $W^{\wedge 2} \leq \Lambda^2 \mathfrak{g}$,

<u>Corollary:</u> Given E, μ, B, β as above, then for Thm 2.1 in SW we can choose $W^{\wedge 1} = \mathbb{R} X$, $W^{\wedge 2} = \mathbb{R}(X \wedge H)$.

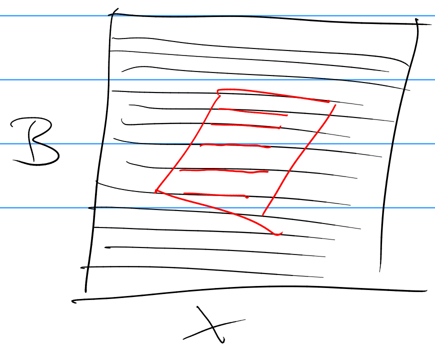
(Lemma 6.1 in SW) (Question: What are the Oseledec subspaces $x_i > 0$).

<u>Florent's talk:</u> $\forall\, b \in B$ $\lim_{n \to \infty} (b_1 \cdots b_n)_* \nu = \nu_b$ exist (for a μ-stationary measure ν on G/Λ (e.g. $\Lambda = SL_2(\mathbb{Z})$)). $X = G/\Lambda$.

$$B^X = B \times X, \quad \beta^X = \int_B \delta_b \otimes \nu_b \, d\beta(b), \quad \nu = \int \nu_b \, d\beta(b).$$

<u>Lemma:</u> Let $\pi_B : B^X \to B$ the canonical projection, $\mathcal{B}_B$ the Borel σ-alg. on B, by $\mathcal{A}_B = \pi_B^{-1} \mathcal{B}_B$. Then $\mathcal{A}_B$ is countably generated, for β^X-a.e. $(b,x) \in B^X$ $[(b,x)]_{\mathcal{A}_B} = \{b\} \times X$ and

$$(\beta^X)^{\mathcal{A}_B}_{(b,x)} = \delta_b \otimes \nu_b.$$



$W^{\wedge 1}$ acts on B^X by $\Phi_\omega(b,x) := (b, \exp(\omega)x)$. Therefore, there exists a β^X-a.e. well-defined, unique family of leafwise measures $\{\sigma_z : z \in B^X\}$ on $W^{\wedge 1}$ for β^X.

Note: $W^{\wedge 1}$ acts on X by $\omega . x := \exp(\omega) x$. Any $W^{\wedge 1}$-subordinate σ-algebra $\mathcal{A}$
$\overset{\shortparallel}{\omega}$ on $F \subseteq B^X$ is a refinement of $\mathcal{A}_B|_F$. So the conditional measures w.r.t. $\mathcal{A}$ are of the form $\delta_b \otimes \lambda^{\mathcal{A}}_{(b,x)}$. Therefore for β^X a.e. $(b,x) \in B$ the leafwise ⓦ $\sigma_{(b,x)} = \tilde{\sigma}_x$, where $\tilde{\sigma}_x$ is the leafwise measure for ν_b w.r.t. the action of $W^{\wedge 1}$ on X.

We also denote $\Phi_\omega(x) := \omega . x$.

Definition: Let $F \subseteq B^X$, let $\mathcal{A}$ be a countably generated σ-algebra on F. Then $\mathcal{A}$ is $W^{\wedge 1}$-subordinate if for all $z \in F$ we have

$$[z]_{\mathcal{A}} = V_z . z \qquad \text{for } V_z \subseteq W^{\wedge 1} \text{ open bdd subset}$$

Denote $T_X : B^X \longrightarrow B^X$, $T_X(b,x) = (Tb, b_1^{-1} x)$ and recall that $(T_X)_* \beta^X = \beta^X$.

Lemma: For β^X-a.e. $z \in B^X$ we have

$$\sigma_z = (\mu_{g^{-2}})_* \sigma_{T_X z} \quad , \text{ where } \mu_{g^{-2}} : W^{\wedge 1} \longrightarrow W^{\wedge 1}, \; \omega \longmapsto g^{-2}\omega .$$

(Recall: $g_i = \begin{pmatrix} g^{-1} & x_i \\ & g \end{pmatrix}$, and in particular $Ad_{g_i}|_{W^{\wedge 1}} = \mu_{g^{-2}}$)
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{\mathbb{R}X}$

Proof: Denote $\Phi_\omega(b,x) = (b, \exp(\omega)x)$, then

1) $T_X \circ \Phi_\omega = \Phi_{g^2\omega} \circ T_X = \Phi_{\mu_{g^2}(\omega)} \circ T_X$

2) $(X, \nu_b) \cong (X, \nu_{Tb})$ are isomorphic under $\boxed{\alpha : X \to X, x \mapsto b_1^{-1} x}$
$\Rightarrow$ The leafwise measures for ν_{Tb} agree w/ the leafwise measures for $(b_1^{-1})_* \nu_b$.

3) Using the characterizing property of leafwise measures, show that $(\mu_{g^{-2}})_* \tilde{\sigma}_{\alpha(x)} = \tilde{\sigma}_x$.

$\{\tilde{\sigma}_x : x \in X\}$ are leafwise measures for ν_b

4) Using the "identification" $\underset{\tilde{\sigma}_x}{\sigma_{(b,x)}} = (\mu_{g^{-2}})_* \underset{\tilde{\sigma}_{\alpha(x)}}{\sigma_{T_X(b,x)}}$ $\quad (T_X(b,x) = (Tb, b_1^{-1}x))$

$\omega \propto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $b_1^{-1} = \begin{pmatrix} \varsigma & y \\ & \varsigma^{-1} \end{pmatrix}$ ∴ $Ad_{b_1^{-1}}(\omega) = \varsigma^2 \omega$

<u>Step 1:</u> $(T_x \circ \Phi_\omega)(b,x) = T_x(b, \exp(\omega)x) = (Tb, b_1^{-1}\exp(\omega)x)$

$$= (Tb, \exp(Ad_{b_1^{-1}}\omega)\, b_1^{-1}x)$$

$$= (Tb, \exp(\varsigma^2\omega) b_1^{-1}x) = (\Phi_{\varsigma^2\omega} \circ T_x)(b,x)$$

<u>Step 2, step 4:</u> ✓

<u>Step 3:</u> Recall the charact. property: Let $F \subseteq X$ (ω)-able, let $\mathcal{A}$ be a $W^{\wedge 1}$-subordinate σ-algebra on F, let λ_F denote the normalized restriction of ν_b to F. Then $\forall f \in C_c(X)\ \forall x \in F$

Characterize lw measures for β^X by lw measures for $\{\nu_b : b \in B\}$!

$$\int_F f \, d(\lambda_F)_x^{\mathcal{A}} = \frac{1}{\tilde{\sigma}_x(V_x)} \int_{V_x} (f \circ \Phi_\omega)(x) \, d\tilde{\sigma}_x(\omega)$$

$\alpha = \alpha(b) : X \longrightarrow X$, $\alpha(x) = b_1^{-1}x$. Then

$$(F, \mathcal{A}, \lambda_F) \underset{\alpha}{\cong} (\alpha^{-1}F, \alpha^{-1}\mathcal{A}, \lambda_{\alpha^{-1}F})$$

where $\alpha^{-1}\mathcal{A}$ is again a $W^{\wedge 1}$-subordinate σ-algebra (on $\alpha^{-1}F$).

This only works on X. If $\tilde{\mathcal{A}}$ is $W^{\wedge 1}$-subordinate on $\tilde{F} \subseteq B^X$, then $T_x^{-1}\tilde{\mathcal{A}}$ is <u>not</u> $W^{\wedge 1}$-subordinate.

Let $z \in T_x^{-1}\tilde{F}$, then

$$[z]_{T_x^{-1}\tilde{\mathcal{A}}} = T_x^{-1}([T_x z]_{\tilde{\mathcal{A}}}) \qquad (z = (b,x))$$

$$= T_x^{-1}(\underbrace{\{Tb\} \times V_{T_x z} \cdot b_1^{-1}x}_{\text{b/c } \tilde{\mathcal{A}} \text{ is } W^{\wedge 1}\text{-subordinate}})$$

$E = \operatorname{supp}\mu = \{g \in G : \mu(U_g) > 0\ \forall U \subseteq G \text{ open nghd. of } 1\}$

$$\overset{E \text{ is finite}}{=} \bigsqcup_{g \in E} \{gTb\} \times \underbrace{g V_{T_x z} \cdot b_1^{-1}x}_{\varsigma^{-2} V_{T_x z} \cdot g b_1^{-1} x}$$

Therefore every atom of $T_x^{-1}\tilde{\mathcal{A}}$ is a finite union of pieces of ~~orbits~~.

Hence probably can be fixed using the tower prop. for conditional measures (Einsiedler, Ward: Prop. 5.20)

Note also: Given $x \in \alpha^{-1}F$, we have

$$[x]_{\alpha^{-1}\mathcal{A}} = \alpha^{-1}([\alpha(x)]_{\mathcal{A}}) = \alpha^{-1}(V_{\alpha(x)}.\alpha(x))$$

$$= (b_1 V_{\alpha(x)} b_1^{-1}).x = U_x . x$$

In particular $U_x = \eta_{\varrho^{?}}(V_{\alpha(x)})$.

$$\frac{1}{\tilde{\sigma}_{\alpha(x)}(V_{\alpha(x)})} \int_{V_{\alpha(x)}} (f \circ \Phi_\omega)(\alpha(x))\, d\tilde{\sigma}_{\alpha(x)}(\omega) = \int_F f\, d(\lambda_F)^{\mathcal{A}}_{\alpha(x)}$$

$$= \int_{\alpha^{-1}F} (f \circ \alpha)\, d(\lambda_{\alpha^{-1}F})^{\alpha^{-1}\mathcal{A}}_x$$

(Einsiedler, Ward: Cor. 5.24)

$$= \frac{1}{\tilde{\sigma}_x(U_x)} \int_{U_x} (f \circ \alpha \circ \Phi_\omega)(x)\, d\tilde{\sigma}_x(\omega)$$

$$\overset{\alpha \circ \Phi_\omega = \Phi_{\varrho^{?}\omega} \circ \alpha}{=} \frac{1}{\tilde{\sigma}_x(\eta_{\varrho^{-2}} V_{\alpha(x)})} \int_{\eta_{\varrho^{-2}} V_{\alpha(x)}} (f \circ \Phi_{\varrho^{?}\omega})(\alpha(x))\, d\tilde{\sigma}_x(\omega)$$

$$= \frac{1}{(\eta_{\varrho^{?}})_* \tilde{\sigma}_x(V_{\alpha(x)})} \int_{V_{\alpha(x)}} (f \circ \Phi_\omega)(\alpha(x))\, d(\eta_{\varrho^{?}})_* \tilde{\sigma}_x(\omega)$$

$$\Rightarrow \tilde{\sigma}_{\alpha(x)} \propto (\eta_{\varrho^{?}})_* \tilde{\sigma}_x$$

∎