NOTES FOR THE SEMINAR

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Abstract. Notes on our seminar

1. Definitions and notation

For notational purposes we assume that 0 is *not* a natural number.

2. Towards the exponential drift for $\mathrm{SL}_2(\mathbb{R})$

2.1. **The case of a finite IFS with uniform contraction rate.** In what follows we let $G = \mathrm{SL}_2(\mathbb{R})$ and denote by $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ its Lie algebra. We fix ourselves a constant $\varrho \in (0,1)$ and real numbers $x_1, \dots, x_r \in \mathbb{R}$ not all zero. We denote

$$g_i = \begin{pmatrix} \varrho^{-1} & -x_i \\ 0 & \varrho \end{pmatrix} \qquad (i = 1, \dots, r).$$

The example to think of is $\varrho = \frac{1}{3}$, $r = 2$, $x_1 = 0$, and $x_2 = \frac{2}{3}$. In what follows, we let $E = \{g_i : i = 1, \dots, r\}$. We assume that μ is a fully supported probability measure on E. Let $B = E^{\mathbb{N}}$ and $\beta = \mu^{\otimes \mathbb{N}}$.

We write

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and note that $\mathfrak{g} = \mathbb{R} - \mathrm{span}\{X, Y, H\}$. We will write

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+,$$

where $\mathfrak{g}^- = \mathbb{R}X$, $\mathfrak{g}^0 = \mathbb{R}H$, $\mathfrak{g}^+ = \mathbb{R}Y$. Let $a \in G$ be a diagonal matrix and note that $a = \pm\exp(tH)$ for some $t \in \mathbb{R}$. An elementary calculation shows that $\{X, Y, H\}$ is an eigenbasis of $\mathfrak{g}$ for $\mathrm{Ad}_a \in \mathrm{SL}(\mathfrak{g})$ given by

$$\mathrm{Ad}_a(v) = ava^{-1} \quad (v \in \mathfrak{g}).$$

In particular, we have

$$\mathrm{Ad}_a(X) = e^{2t}X, \quad \mathrm{Ad}_a(Y) = e^{-2t}Y, \quad \mathrm{Ad}_a(H) = H.$$

Lemma 2.1. *The adjoint representation of G on $\mathfrak{g}$ and the induced representation on $\wedge^2\mathfrak{g}$ are isomorphic.*

Proof. Using the classification of irreducible representations of G it suffices to prove that both representations are irreducible. As every representation of G is semisimple and as the trivial representation is the unique one-dimensional representation of G, it suffices to prove that both representations do not admit any non-trivial invariant vectors.

Using the correspondence between irreducible representations of G and representations of $\mathfrak{g}$, it suffices to show that the representations of $\mathfrak{g}$ on $\mathfrak{g}$ and on $\wedge^2\mathfrak{g}$ induced by the adjoint action do not admit any fixed vectors. To this end we recall that

$$\mathrm{ad}_Y(X) = -H, \quad \mathrm{ad}_Y(H) = 2Y, \quad \mathrm{ad}_Y(Y) = 0,$$

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$$\mathrm{ad}_X(X) = 0, \quad \mathrm{ad}_X(H) = -2X, \quad \mathrm{ad}_X(Y) = H.$$

Let $v \in \mathfrak{g}$, $w \in \wedge^2\mathfrak{g}$ be $\mathfrak{g}$-fixed vectors. Then

$$v = \mathrm{ad}_X^3(v) = 0 \quad \text{and} \quad w = \mathrm{ad}_X^3(w) = 0.$$

This proves the claim. □

Remark 2.2. (1) More explicitly, one calculates that $\wedge^2\mathrm{Ad}_a$ is diagonalizable with eigenbasis $\{X \wedge H, H \wedge Y, X \wedge Y\}$ satisfying

$$(\wedge^2\mathrm{Ad}_a)(X \wedge H) = e^{2t} X \wedge H,$$
$$(\wedge^2\mathrm{Ad}_a)(X \wedge Y) = X \wedge Y,$$
$$(\wedge^2\mathrm{Ad}_a)(H \wedge Y) = e^{-2t} H \wedge Y,$$

and it is not very difficult to see that the map $\Phi : \mathfrak{g} \to \wedge^2\mathfrak{g}$ defined by

$$X \mapsto -\frac{1}{2} X \wedge H,$$
$$H \mapsto X \wedge Y,$$
$$Y \mapsto -\frac{1}{2} H \wedge Y$$

defines an isomorphism of representations of G. *Note:* It is easier to check that the isomorphism is $\mathfrak{g}$-equivariant. To this end, we recall that for any $V, W \in \mathfrak{g}$ we have

$$(\wedge^2\mathrm{ad}_U)(V \wedge W) = (\mathrm{ad}_U V) \wedge W + V \wedge (\mathrm{ad}_U W).$$

(2) In what follows let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denote an invariant inner product on $\mathfrak{g}$. Then $B^2 = \Phi_* B : \wedge^2\mathfrak{g} \times \wedge^2\mathfrak{g} \to \mathbb{R}$ given by $B^2(v, w) = B(\Phi^{-1}v, \Phi^{-1}w)$ defines an inner product on $\wedge^2\mathfrak{g}$ and $\Phi : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is a G equivariant isometry. It will be convenient to assume either that B is $\mathrm{SO}_2(\mathbb{R})$-invariant or that $\{X, Y, H\}$ is an orthonormal basis.

In what follows, we denote by $\|\cdot\|$ the norm on $\mathfrak{g}$ induced by B. We will abuse notation and use the same notation for the norm on $\wedge^2\mathfrak{g}$ induced by the isometry Φ.

Corollary 2.3. *Let E, μ, B, β as above. In the notation of* [SW19, Thm. 2.1] *we can choose $W^{\wedge 1} = \mathbb{R}X$ and $W^{\wedge 2} = \mathbb{R}(X \wedge H)$.*

Proof. The argument for this was provided earlier already in greater generality. We reproduce it for concreteness. Note that following the preceding discussion on isometry of the representations and using that all norms on $\mathfrak{g}$ and on $\wedge^2\mathfrak{g}$ are equivalent, the conclusion of Corollary 2.3 is independent of the choice of norms both on $\mathfrak{g}$ and (independently) on $\wedge^2\mathfrak{g}$. We can therefore assume that B is chosen so that $\{X, Y, H\}$ is an orthonormal basis with respect to the inner product B. As of Lemma 2.1 and by definition of the norm on $\wedge^2\mathfrak{g}$, it suffices to prove the statement for $d = 1$.

Let $t \in \mathbb{R}$ arbitrary and

$$u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

then

$$\mathrm{Ad}_{u(t)}X = X, \qquad \mathrm{Ad}_{u(t)}H = H - 2tX, \qquad \mathrm{Ad}_{u(t)}Y = Y + tH - t^2X.$$

Furthermore, each $g_i \in E$ decomposes as $g_i = u_i a$, where

$$a = \begin{pmatrix} \varrho^{-1} & 0 \\ 0 & \varrho \end{pmatrix} \quad \text{and} \quad u_i = u(\varrho^{-1}x_i).$$

Using the relation

$$au(t)a^{-1} = u(\varrho^{-2}t),$$

one obtains that for any $\jmath : \mathbb{N} \to \{1, \dots, r\}$ we have

$$g_{\jmath(n)} \cdots g_{\jmath(1)} = u\left(\sum_{\ell=1}^{n} \varrho^{-2(n-\ell)-1} x_{\jmath(\ell)}\right) a^n. \tag{2.1}$$

This is proven by induction. If $n = 1$, the statement is clearly true. The induction step is then given by

$$\begin{aligned} g_{\jmath(n+1)} \cdots g_{\jmath(1)} &= u(x_{\jmath(n+1)})au\left(\sum_{\ell=1}^{n} \varrho^{-2(n-\ell)-1} x_{\jmath(\ell)}\right) a^n \\ &= u(x_{\jmath(n+1)})u\left(\varrho^{-2}\sum_{\ell=1}^{n} \varrho^{-2(n-\ell)-1} x_{\jmath(\ell)}\right) a^{n+1} \\ &= u\left(\sum_{\ell=1}^{n+1} \varrho^{-2(n+1-\ell)-1} x_{\jmath(\ell)}\right) a^{n+1} \end{aligned}$$

Given $\jmath$ as above, we set

$$t_n(\jmath) = \sum_{\ell=1}^{n} \varrho^{-2(n-\ell)-1} x_{\jmath(\ell)}.$$

Therefore

$$\begin{aligned} &\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} Y = \varrho^{2n}\mathrm{Ad}_{u(t_n(\jmath))} Y = \varrho^{2n} Y + \varrho^{2n} t_n(\jmath) H - \varrho^{2n} t_n(\jmath)^2 X, \\ &\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} H = \mathrm{Ad}_{u(t_n(\jmath))} H = H - 2t_n(\jmath) X, \\ &\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} X = \varrho^{-2n}\mathrm{Ad}_{u(t_n(\jmath))} X = \varrho^{-2n} X. \end{aligned}$$

We have

$$|t_n(\jmath)| \ll \sum_{\ell=0}^{n-1} \varrho^{-2\ell} = \varrho^{-2(n-1)} \frac{1-\varrho^{2n}}{1-\varrho^2} \asymp \varrho^{-2n}.$$

Denote by π_X the canonical projection $\mathfrak{g} \to \mathfrak{g} / \mathbb{R}X$. As $gX \in \mathbb{R}X$ for all $g \in \mathrm{supp}\mu$, the adjoint action by elements in E descends to an action on $\mathfrak{g} / \mathbb{R}X$ and using the preceding calculations, we get

$$\begin{aligned} &\|\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} \pi_X(Y)\|^2 = \varrho^{4n} + \varrho^{4n} t_n(\jmath)^2, \\ &\|\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} \pi_X(H)\|^2 = 1. \end{aligned}$$

Therefore we can apply Lemma [SW19, Lem. 6.1] to deduce that for β-a.e. $b \in B$, i.e. β-a.e. $\jmath : \mathbb{N} \to \{1, \dots, r\}$, the subalgebra $\mathbb{R}X$ is complementary to

$$V_{<-2\log\varrho} = \left\{ v \in \mathfrak{g} : \lim_{n\to\infty} \frac{1}{n} \|\mathrm{Ad}_{b_n\cdots b_1} v\| < -2\log\varrho \right\}.$$

□

In what follows, we denote by ν_b, $b \in B$, the limit (which is defined β-a.s.) given by

$$\nu_b = \lim_{n\to\infty} (b_1 \cdots b_n)_* \nu.$$

Here, the measure ν is a μ-stationary measure on G / Λ. We let $X = G / \Lambda$, $B^X = B \times X$, $\mathcal{B}_{B^X}$ the Borel σ-algebra on B^X, and define

$$\beta^X = \int_B \delta_b \otimes \nu_b \mathrm{d}\beta(b).$$

Lemma 2.4. *Let $\pi_B : B^X \to B$ denote the canonical projection. Let $\mathcal{B}_B$ be the Borel σ-algebra on B and let $\mathcal{A}_B = \pi_B^{-1}(\mathcal{B}_B)$. Then $\mathcal{A}_B \subseteq \mathcal{B}^{\mathcal{X}}$ is countably generated, for β^X-a.e. $(b,x) \in B^X$ we have $[(b,x)]_{\mathcal{A}_B} = \{b\} \times X$, and $(\beta^X)^{\mathcal{A}_B}_{(b,x)} = \delta_b \otimes \nu_b$.*

Proof. One checks that the family $\{\delta_b \otimes \nu_b : b \in B\}$ satisfies the defining properties of a family of conditional measures. This is purely formal and left to the reader. Maybe we include this later. □

Lemma 2.5. *Let $T_X : B^X \to B^X$, $T_X(b,x) = (Tb, b_1^{-1}x)$. Then β^X is T_X-invariant.*

Proof. This is essentially purely formal. Uses only definition of β^X and μ-stationarity of ν. □

The subspace $W^{\wedge 1}$ acts on B^X by

$$w.(b,x) = (b, e^w x) \qquad (w \in W^{\wedge 1}, b \in B, x \in X).$$

We denote by $\Phi_. : B^X \times W^{\wedge 1} \to B^X$ the action map $\Phi_w(z) = w.z$. Let $\mathcal{A}_W \subseteq \mathcal{B}_{B^X}$ denote the σ-algebra of Φ-invariant sets. Then $\mathcal{A}_W$ is a refinement of $\mathcal{A}_B$, and therefore [EW11, Prop. 5.20] the leafwise measures for β^X and the action of $W^{\wedge 1}$ essentially agree with the leafwise measures for ν_b. In what follows, we denote by $\{\sigma_z : z \in B^X\}$ a family of leafwise measures for the action of the Lie subalgebra $W^{\wedge 1} \leq \mathfrak{g}$ on B^X and note that for β^X-a.e. $(b,x) \in B^X$ the measure $\sigma_{(b,x)}$ agrees with a leafwise measure for ν_b with respect to the action of $W^{\wedge 1}$ on X.

It will be convenient to introduce the map $\eta_a : W^{\wedge 1} \to W^{\wedge 1}$ given by $\eta_a(w) = aw$ for $a > 0$.

Lemma 2.6. *For β-a.e. $z \in B^X$ we have*

$$\sigma_z \propto (\eta_{\varrho^{-2}})_* \sigma_{T_X(z)}.$$

Proof. We note that for all $w \in W^{\wedge 1}$

$$T_X \circ \Phi_w = \Phi_{\varrho^2 w} \circ T_X.$$

Next we note that for β-a.e. $b \in B$ the map $\alpha_b : x \mapsto b_1^{-1}x$ defines an isomorphism of measure spaces $(X, \nu_b) \cong (X, \nu_{Tb})$. Indeed, for all $f \in \mathrm{C_c}(X)$ we have

$$\begin{aligned}\int_X f \mathrm{d}\nu_{Tb} &= \lim_{n\to\infty} \int_X f \mathrm{d}(b_2 \cdots b_n)_* \nu \\ &= \lim_{n\to\infty} \int_X f \circ b_1^{-1} \mathrm{d}(b_1 b_2 \cdots b_n)_* \nu \\ &= \int_X f \circ b_1^{-1} \mathrm{d}\nu_b,\end{aligned}$$

that is, $\nu_{Tb} = (b_1^{-1})_* \nu_b$. In particular, we get that the leafwise measures for ν_{Tb} and the leafwise measures for $(b_1^{-1})_* \nu_b$ agree.

We quickly recall the defining property of leafwise measures $\tilde{\sigma}_x$ for ν_b with respect to $W^{\wedge 1}$ acting on X. Let $F \subseteq X$ have finite, positive measure. Let $\mathcal{A}$ be a $W^{\wedge 1}$-subordinate σ-algebra on F. Let λ_F denote the normalized restriction of ν_b to F. Then for λ_F-a.e. $x \in F$ and for all $f \in \mathrm{C_c}(F)$ we have

$$\int_F f \mathrm{d}(\lambda_F)^{\mathcal{A}}_x = \frac{1}{\tilde{\sigma}_x(V_x)} \int_{V_x} (f \circ \Phi_w)(x) \mathrm{d}\tilde{\sigma}_x(w).$$

In particular, this completely characterizes the leafwise measures.

Let $(F, \mathcal{A}, \lambda_F)$ as above, i.e. $\mathcal{A}$ is $W^{\wedge 1}$-subordinate. Then $\alpha = \alpha(b)$ defines an isomorphism

$$(F, \mathcal{A}, \lambda_F) \cong (\alpha^{-1}F, \alpha^{-1}\mathcal{A}, \lambda_{\alpha^{-1}F}).$$

The $\tilde{\sigma}$-algebra $\alpha^{-1}\mathcal{A}$ is again W-subordinate, as for any $x \in \alpha^{-1}F$ we have

$$[x]_{\alpha^{-1}\mathcal{A}} = \alpha^{-1}[\alpha x]_{\mathcal{A}} = \alpha^{-1}(V_{b_1^{-1}x}.b_1^{-1}x) = (b_1 V_{b_1^{-1}x} b_1^{-1}).x.$$

In what follows, we denote by U_x the open subset of $W^{\wedge 1}$ such that $w \in W^{\wedge 1} \mapsto w.x$ is injective and $[x]_{\alpha^{-1}\mathcal{A}} = U_x.x$. Using the above calculation and the fact that the contraction ratios are all equal, we get $U_x = \varrho^{-2}V_{\alpha x}$. Using the characterizing property, we get

$$\begin{aligned}
\frac{1}{\tilde{\sigma}_{\alpha x}(V_{\alpha x})}\int_{V_{\alpha x}}(f\circ\Phi_w)(\alpha x)\mathrm{d}\tilde{\sigma}_{\alpha x}(w) &= \int_F f\mathrm{d}(\lambda_F)_x^{\mathcal{A}}\\
&= \int_{\alpha^{-1}F}(f\circ\alpha)\mathrm{d}(\lambda_{\alpha^{-1}F})_x^{\alpha^{-1}\mathcal{A}}\\
&= \frac{1}{\tilde{\sigma}_x(U_x)}\int_{U_x}(f\circ\alpha\circ\Phi_w)(x)\mathrm{d}\tilde{\sigma}_x(w)\\
&= \frac{1}{\tilde{\sigma}_x(U_x)}\int_{\varrho^{-2}V_{\alpha x}}(f\circ\Phi_{\varrho^2 w}\circ\alpha)(x)\mathrm{d}\tilde{\sigma}_x(w)\\
&= \frac{1}{(\eta_{\varrho^2})_*\tilde{\sigma}_x(V_{\alpha x})}\int_{V_{\alpha x}}(f\circ\Phi_w)(\alpha x)\mathrm{d}(\eta_{\varrho^2})_*\tilde{\sigma}_x(w).
\end{aligned}$$

This shows that $(\eta_{\varrho^{-2}})_*\tilde{\sigma}_{\alpha x} \propto \tilde{\sigma}_x$. Using the previous remarks, we know that for almost every $(b,x) \in B^X$ we have $\sigma_{(b,x)} = \tilde{\sigma}_x$, and therefore

$$\sigma_{(b,x)} \propto \tilde{\sigma}_x \propto (\eta_{\varrho^{-2}})_*\tilde{\sigma}_{b_1^{-1}x} \propto (\eta_{\varrho^{-2}})_*\sigma_{T_X(b,x)}.$$

□

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