NOTES FOR THE SEMINAR

MANUEL W. LUETHI

Abstract. Notes on our seminar

1. Definitions and notation

For notational purposes we assume that 0 is *not* a natural number.

2. Towards the exponential drift for $\mathrm{SL}_2(\mathbb{R})$

2.1. **The case of a finite IFS with uniform contraction rate.** In what follows we let $G = \mathrm{SL}_2(\mathbb{R})$ and denote by $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ its Lie algebra. We fix ourselves a constant $\varrho \in (0,1)$ and real numbers $x_1, \ldots, x_r \in \mathbb{R}$ not all zero. We denote

$$g_i = \begin{pmatrix} \varrho^{-1} & -x_i \\ 0 & \varrho \end{pmatrix} \qquad (i = 1, \ldots, r).$$

The example to think of is $\varrho = \frac{1}{3}$, $r = 2$, $x_1 = 0$, and $x_2 = \frac{2}{3}$. In what follows, we let $E = \{g_i : i = 1, \ldots, r\}$. We assume that μ is a fully supported probability measure on E. Let $B = E^{\mathbb{N}}$ and $\beta = \mu^{\otimes \mathbb{N}}$.

We write

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and note that $\mathfrak{g} = \mathbb{R} - \mathrm{span}\{X, Y, H\}$. We will write

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+,$$

where $\mathfrak{g}^- = \mathbb{R}X$, $\mathfrak{g}^0 = \mathbb{R}H$, $\mathfrak{g}^+ = \mathbb{R}Y$. Let $a \in G$ be a diagonal matrix and note that $a = \pm\exp(tH)$ for some $t \in \mathbb{R}$. An elementary calculation shows that $\{X, Y, H\}$ is an eigenbasis of $\mathfrak{g}$ for $\mathrm{Ad}_a \in \mathrm{SL}(\mathfrak{g})$ given by

$$\mathrm{Ad}_a(v) = ava^{-1} \quad (v \in \mathfrak{g}).$$

In particular, we have

$$\mathrm{Ad}_a(X) = e^{2t}X, \quad \mathrm{Ad}_a(Y) = e^{-2t}Y, \quad \mathrm{Ad}_a(H) = H.$$

Lemma 2.1. *The adjoint representation of G on $\mathfrak{g}$ and the induced representation on $\wedge^2\mathfrak{g}$ are isomorphic.*

Proof. Using the classification of irreducible representations of G it suffices to prove that both representations are irreducible. As every representation of G is semisimple and as the trivial representation is the unique one-dimensional representation of G, it suffices to prove that both representations do not admit any non-trivial invariant vectors.

Using the correspondence between irreducible representations of G and representations of $\mathfrak{g}$, it suffices to show that the representations of $\mathfrak{g}$ on $\mathfrak{g}$ and on $\wedge^2\mathfrak{g}$ induced by the adjoint action do not admit any fixed vectors. To this end we recall that

$$\mathrm{ad}_Y(X) = -H, \quad \mathrm{ad}_Y(H) = 2Y, \quad \mathrm{ad}_Y(Y) = 0,$$

Date: April 28, 2020.

$$\mathrm{ad}_X(X) = 0, \quad \mathrm{ad}_X(H) = -2X, \quad \mathrm{ad}_X(Y) = H.$$

Let $v \in \mathfrak{g}$, $w \in \wedge^2\mathfrak{g}$ be $\mathfrak{g}$-fixed vectors. Then

$$v = \mathrm{ad}_X^3(v) = 0 \quad \text{and} \quad w = \mathrm{ad}_X^3(w) = 0.$$

This proves the claim. □

Remark 2.2. (1) More explicitly, one calculates that $\wedge^2\mathrm{Ad}_a$ is diagonalizable with eigenbasis $\{X \wedge H, H \wedge Y, X \wedge Y\}$ satisfying

$$(\wedge^2\mathrm{Ad}_a)(X \wedge H) = e^{2t} X \wedge H,$$
$$(\wedge^2\mathrm{Ad}_a)(X \wedge Y) = X \wedge Y,$$
$$(\wedge^2\mathrm{Ad}_a)(H \wedge Y) = e^{-2t} H \wedge Y,$$

and it is not very difficult to see that the map $\Phi : \mathfrak{g} \to \wedge^2\mathfrak{g}$ defined by

$$X \mapsto -\frac{1}{2} X \wedge H,$$
$$H \mapsto X \wedge Y,$$
$$Y \mapsto -\frac{1}{2} H \wedge Y$$

defines an isomorphism of representations of G. *Note:* It is easier to check that the isomorphism is $\mathfrak{g}$-equivariant. To this end, we recall that for any $V, W \in \mathfrak{g}$ we have

$$(\wedge^2\mathrm{ad}_U)(V \wedge W) = (\mathrm{ad}_U V) \wedge W + V \wedge (\mathrm{ad}_U W).$$

(2) In what follows let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denote an invariant inner product on $\mathfrak{g}$. Then $B^2 = \Phi_* B : \wedge^2\mathfrak{g} \times \wedge^2\mathfrak{g} \to \mathbb{R}$ given by $B^2(v, w) = B(\Phi^{-1}v, \Phi^{-1}w)$ defines an inner product on $\wedge^2\mathfrak{g}$ and $\Phi : \mathfrak{g} \to \wedge^2\mathfrak{g}$ is a G equivariant isometry. It will be convenient to assume either that B is $\mathrm{SO}_2(\mathbb{R})$-invariant or that $\{X, Y, H\}$ is an orthonormal basis.

In what follows, we denote by $\|\cdot\|$ the norm on $\mathfrak{g}$ induced by B. We will abuse notation and use the same notation for the norm on $\wedge^2\mathfrak{g}$ induced by the isometry Φ.

Corollary 2.3. *Let E, μ, B, β as above. In the notation of* [SW19, Thm. 2.1] *we can choose $W^{\wedge 1} = \mathbb{R}X$ and $W^{\wedge 2} = \mathbb{R}(X \wedge H)$.*

Proof. The argument for this was provided earlier already in greater generality. We reproduce it for concreteness. Note that following the preceding discussion on isometry of the representations and using that all norms on $\mathfrak{g}$ and on $\wedge^2\mathfrak{g}$ are equivalent, the conclusion of Corollary 2.3 is independent of the choice of norms both on $\mathfrak{g}$ and (independently) on $\wedge^2\mathfrak{g}$. We can therefore assume that B is chosen so that $\{X, Y, H\}$ is an orthonormal basis with respect to the inner product B. As of Lemma 2.1 and by definition of the norm on $\wedge^2\mathfrak{g}$, it suffices to prove the statement for $d = 1$.

Let $t \in \mathbb{R}$ arbitrary and

$$u(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

then

$$\mathrm{Ad}_{u(t)} X = X, \qquad \mathrm{Ad}_{u(t)} H = H - 2tX, \qquad \mathrm{Ad}_{u(t)} Y = Y + tH - t^2 X.$$

Furthermore, each $g_i \in E$ decomposes as $g_i = u_i a$, where

$$a = \begin{pmatrix} \varrho^{-1} & 0 \\ 0 & \varrho \end{pmatrix} \quad \text{and} \quad u_i = u(\varrho^{-1} x_i).$$

Using the relation
$$au(t)a^{-1} = u(\varrho^{-2}t),$$
one obtains that for any $\jmath : \mathbb{N} \to \{1, \dots, r\}$ we have
$$g_{\jmath(n)} \cdots g_{\jmath(1)} = u\left(\sum_{\ell=1}^{n} \varrho^{-2(n-\ell)-1} x_{\jmath(\ell)}\right) a^n. \tag{2.1}$$
This is proven by induction. If $n = 1$, the statement is clearly true. The induction step is then given by
$$\begin{aligned} g_{\jmath(n+1)} \cdots g_{\jmath(1)} &= u(x_{\jmath(n+1)}) a u\left(\sum_{\ell=1}^{n} \varrho^{-2(n-\ell)-1} x_{\jmath(\ell)}\right) a^n \\ &= u(x_{\jmath(n+1)}) u\left(\varrho^{-2}\sum_{\ell=1}^{n} \varrho^{-2(n-\ell)-1} x_{\jmath(\ell)}\right) a^{n+1} \\ &= u\left(\sum_{\ell=1}^{n+1} \varrho^{-2(n+1-\ell)-1} x_{\jmath(\ell)}\right) a^{n+1} \end{aligned}$$
Given $\jmath$ as above, we set
$$t_n(\jmath) = \sum_{\ell=1}^{n} \varrho^{-2(n-\ell)-1} x_{\jmath(\ell)}.$$
Therefore
$$\begin{aligned} &\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} Y = \varrho^{2n}\mathrm{Ad}_{u(t_n(\jmath))} Y = \varrho^{2n} Y + \varrho^{2n} t_n(\jmath) H - \varrho^{2n} t_n(\jmath)^2 X, \\ &\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} H = \mathrm{Ad}_{u(t_n(\jmath))} H = H - 2t_n(\jmath) X, \\ &\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} X = \varrho^{-2n}\mathrm{Ad}_{u(t_n(\jmath))} X = \varrho^{-2n} X. \end{aligned}$$
We have
$$|t_n(\jmath)| \ll \sum_{\ell=0}^{n-1} \varrho^{-2\ell} = \varrho^{-2(n-1)} \frac{1-\varrho^{2n}}{1-\varrho^2} \asymp \varrho^{-2n}.$$
Denote by π_X the canonical projection $\mathfrak{g} \to \mathfrak{g} / \mathbb{R}X$. As $gX \in \mathbb{R}X$ for all $g \in \mathrm{supp}\mu$, the adjoint action by elements in E descends to an action on $\mathfrak{g} / \mathbb{R}X$ and using the preceding calculations, we get
$$\begin{aligned} &\|\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} \pi_X(Y)\|^2 = \varrho^{4n} + \varrho^{4n} t_n(\jmath)^2, \\ &\|\mathrm{Ad}_{g_{\jmath(n)}\cdots g_{\jmath(1)}} \pi_X(H)\|^2 = 1. \end{aligned}$$
Therefore we can apply Lemma [SW19, Lem. 6.1] to deduce that for β-a.e. $b \in B$, i.e. β-a.e. $\jmath : \mathbb{N} \to \{1, \dots, r\}$, the subalgebra $\mathbb{R}X$ is complementary to
$$V_{<-2\log\varrho} = \left\{ v \in \mathfrak{g} : \lim_{n\to\infty} \frac{1}{n} \|\mathrm{Ad}_{b_n\cdots b_1} v\| < -2\log\varrho \right\}.$$
□

In what follows, we denote by ν_b, $b \in B$, the limit (which is defined β-a.s.) given by
$$\nu_b = \lim_{n\to\infty} (b_1 \cdots b_n)_* \nu.$$
Here, the measure ν is a μ-stationary measure on G / Λ. We let $X = G / \Lambda$, $B^X = B \times X$, $\mathcal{B}_{B^X}$ the Borel σ-algebra on B^X, and define
$$\beta^X = \int_B \delta_b \otimes \nu_b \mathrm{d}\beta(b).$$

Lemma 2.4. *Let $\pi_B : B^X \to B$ denote the canonical projection. Let $\mathcal{B}_B$ be the Borel σ-algebra on B and let $\mathcal{A}_B = \pi_B^{-1}(\mathcal{B}_B)$. Then $\mathcal{A}_B \subseteq \mathcal{B}^{\mathcal{X}}$ is countably generated, for β^X-a.e. $(b,x) \in B^X$ we have $[(b,x)]_{\mathcal{A}_B} = \{b\} \times X$, and $(\beta^X)^{\mathcal{A}_B}_{(b,x)} = \delta_b \otimes \nu_b$.*

Proof. One checks that the family $\{\delta_b \otimes \nu_b : b \in B\}$ satisfies the defining properties of a family of conditional measures. This is purely formal and left to the reader. □

Lemma 2.5. *Let $T_X : B^X \to B^X$, $T_X(b,x) = (Tb, b_1^{-1}x)$. Then β^X is T_X-invariant.*

Proof. This is essentially purely formal. Uses only definition of β^X and μ-stationarity of ν. □

The subspace $W^{\wedge 1}$ acts on B^X by

$$w.(b,x) = (b, e^w x) \qquad (w \in W^{\wedge 1}, b \in B, x \in X).$$

We denote by $\Phi_\cdot : B^X \times W^{\wedge 1} \to B^X$ the action map $\Phi_w(z) = w.z$. Let $\mathcal{A}_W \subseteq \mathcal{B}_{B^X}$ denote the σ-algebra of Φ-invariant sets. Then $\mathcal{A}_W$ is a refinement of $\mathcal{A}_B$, and therefore [EW11, Prop. 5.20] the leafwise measures for β^X and the action of $W^{\wedge 1}$ essentially agree with the leafwise measures for ν_b. In what follows, we denote by $\{\sigma_z : z \in B^X\}$ a family of leafwise measures for the action of the Lie subalgebra $W^{\wedge 1} \leq \mathfrak{g}$ on B^X and note that for β^X-a.e. $(b,x) \in B^X$ the measure $\sigma_{(b,x)}$ agrees with a leafwise measure for ν_b with respect to the action of $W^{\wedge 1}$ on X.

Given $w \in W^{\wedge 1}$, we denote by $\tau_w : W^{\wedge 1} \to W^{\wedge 1}$ the translation map $v \mapsto v + w$. Recall the following fact.

Lemma 2.6 (cf. Weikun's talk)**.** *Let $z \in E$ and $w \in W^{\wedge 1}$ such that $\tau_w(z) \in E$. Then*

$$\sigma_z \propto (\tau_w)_* \sigma_{\Phi_w(z)}.$$

Proof. Cf. Weikun's talk and [EL10, Thm. 6.3(iii)]. Added later. □

It will be convenient to introduce the map $\eta_a : W^{\wedge 1} \to W^{\wedge 1}$ given by $\eta_a(w) = aw$ for $a > 0$.

Lemma 2.7. *For β-a.e. $z \in B^X$ and for all $n \in \mathbb{N}$ we have*

$$\sigma_z \propto (\eta_{\varrho^{-2n}})_* \sigma_{T_X^n(z)}.$$

Proof. It suffices to prove

$$\sigma_z \propto (\eta_{\varrho^{-2}})_* \sigma_{T_X(z)}.$$

on a set of full measure.

We note that for all $w \in W^{\wedge 1}$

$$T_X \circ \Phi_w = \Phi_{\varrho^2 w} \circ T_X.$$

Next we note that for β-a.e. $b \in B$ the map $\alpha_b : x \mapsto b_1^{-1}x$ defines an isomorphism of measure spaces $(X, \nu_b) \cong (X, \nu_{Tb})$. Indeed, for all $f \in \mathrm{C_c}(X)$ we have

$$\begin{aligned}\int_X f \mathrm{d}\nu_{Tb} &= \lim_{n\to\infty} \int_X f \mathrm{d}(b_2 \cdots b_n)_* \nu \\ &= \lim_{n\to\infty} \int_X f \circ b_1^{-1} \mathrm{d}(b_1 b_2 \cdots b_n)_* \nu \\ &= \int_X f \circ b_1^{-1} \mathrm{d}\nu_b,\end{aligned}$$

that is, $\nu_{Tb} = (b_1^{-1})_* \nu_b$. In particular, we get that the leafwise measures for ν_{Tb} and the leafwise measures for $(b_1^{-1})_* \nu_b$ agree.

We quickly recall the defining property of leafwise measures $\tilde{\sigma}_x$ for ν_b with respect to $W^{\wedge 1}$ acting on X. Let $F \subseteq X$ have finite, positive measure. Let $\mathcal{A}$ be a $W^{\wedge 1}$-subordinate σ-algebra on F. Let λ_F denote the normalized restriction of ν_b to F. Then for λ_F-a.e. $x \in F$ and for all $f \in \mathrm{C_c}(F)$ we have

$$\int_F f \mathrm{d}(\lambda_F)_x^{\mathcal{A}} = \frac{1}{\tilde{\sigma}_x(V_x)} \int_{V_x} (f \circ \Phi_w)(x) \mathrm{d}\tilde{\sigma}_x(w).$$

In particular, this completely characterizes the leafwise measures.

Let $(F, \mathcal{A}, \lambda_F)$ as above, i.e. $\mathcal{A}$ is $W^{\wedge 1}$-subordinate. Then $\alpha = \alpha(b)$ defines an isomorphism

$$(F, \mathcal{A}, \lambda_F) \cong (\alpha^{-1}F, \alpha^{-1}\mathcal{A}, \lambda_{\alpha^{-1}F}).$$

The $\tilde{\sigma}$-algebra $\alpha^{-1}\mathcal{A}$ is again W-subordinate, as for any $x \in \alpha^{-1}F$ we have

$$[x]_{\alpha^{-1}\mathcal{A}} = \alpha^{-1}[\alpha x]_{\mathcal{A}} = \alpha^{-1}(V_{b_1^{-1}x}.b_1^{-1}x) = (b_1 V_{b_1^{-1}x} b_1^{-1}).x.$$

In what follows, we denote by U_x the open subset of $W^{\wedge 1}$ such that $w \in W^{\wedge 1} \mapsto w.x$ is injective and $[x]_{\alpha^{-1}\mathcal{A}} = U_x.x$. Using the above calculation and the fact that the contraction ratios are all equal, we get $U_x = \varrho^{-2} V_{\alpha x}$. Using the characterizing property, we get

$$\begin{aligned}
\frac{1}{\tilde{\sigma}_{\alpha x}(V_{\alpha x})} \int_{V_{\alpha x}} (f \circ \Phi_w)(\alpha x) \mathrm{d}\tilde{\sigma}_{\alpha x}(w) &= \int_F f \mathrm{d}(\lambda_F)_x^{\mathcal{A}} \\
&= \int_{\alpha^{-1}F} (f \circ \alpha) \mathrm{d}(\lambda_{\alpha^{-1}F})_x^{\alpha^{-1}\mathcal{A}} \\
&= \frac{1}{\tilde{\sigma}_x(U_x)} \int_{U_x} (f \circ \alpha \circ \Phi_w)(x) \mathrm{d}\tilde{\sigma}_x(w) \\
&= \frac{1}{\tilde{\sigma}_x(U_x)} \int_{\varrho^{-2}V_{\alpha x}} (f \circ \Phi_{\varrho^2 w} \circ \alpha)(x) \mathrm{d}\tilde{\sigma}_x(w) \\
&= \frac{1}{(\eta_{\varrho^2})_* \tilde{\sigma}_x(V_{\alpha x})} \int_{V_{\alpha x}} (f \circ \Phi_w)(\alpha x) \mathrm{d}(\eta_{\varrho^2})_* \tilde{\sigma}_x(w).
\end{aligned}$$

This shows that $(\eta_{\varrho^{-2}})_* \tilde{\sigma}_{\alpha x} \propto \tilde{\sigma}_x$. Using the previous remarks, we know that for almost every $(b, x) \in B^X$ we have $\sigma_{(b,x)} = \tilde{\sigma}_x$, and therefore

$$\sigma_{(b,x)} \propto \tilde{\sigma}_x \propto (\eta_{\varrho^{-2}})_* \tilde{\sigma}_{b_1^{-1}x} \propto (\eta_{\varrho^{-2}})_* \sigma_{T_X(b,x)}.$$

□

We want to prove the following proposition.

Proposition 2.8. *Let $F \subseteq B^X$ of full measure. Then for β^X-a.e. $z_0 \in F$, for all $A > 0$, there exist $w \in W^{\wedge 1}$ and $z' \in F$ such that*

$$\|w\| < A \quad \textit{and} \quad \Phi_w(z') \in F \quad \textit{and} \quad \sigma_{\Phi_w(z')} = \sigma_{z'} = \sigma_{z_0}.$$

Remark 2.9. When interpreted correctly, the map σ. taking values among proportionality classes of Radon measures on $W^{\wedge 1}$ is a mesurable map between the topological space B^X and the metrizable space of proportionality classes of Radon measures satisfying a certain growth condition; cf. [EL10, Thm. 6.30]

For the extended proof it will be useful to fix some notation motivated later. Any upper triangular matrix $p \in G$ can be written uniquely in the form $g = a_g u_g$ for a diagonal matrix a_g and an upper triangular unipotent u_g. Explicitly, we have

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \begin{pmatrix} 1 & \beta/\alpha \\ 0 & 1 \end{pmatrix}.$$

We write P for the group of upper triangular matrices in G and note that P preserves $W^{\wedge 1} = \mathbb{R}X$ and acts by similarities an $W^{\wedge 1}$. Given a sequence $b =$

$(b_n)_{n\in\mathbb{N}} \in P^{\mathbb{N}}$ we will write $b_n = a_n u_n$ where $a_n = \mathrm{diag}(\alpha_n, 1/\alpha_n)$ and u_n is upper triangular unipotent.

For what follows, it will be useful to note that given $\tilde{F} \subseteq B^X$ and a $W^{\wedge 1}$-subordinate σ-algebra $\mathcal{A}$ on F, the σ-algebra $T_X^{-1}\mathcal{A}$ on $T_X^{-1}F$ is *not* $W^{\wedge 1}$-subordinate if $r = |E| > 1$. Indeed, let $z = (b, x) \in T_X^{-1}F$, $z' = T_X z$ and let $V \subseteq W^{\wedge 1}$ be open and bounded such that the map $w \in V \mapsto w.T_X z$ is injective and $[T_X z]_{\mathcal{A}} = V.z'$, then

$$[z]_{T_X^{-1}\mathcal{A}} = \bigsqcup_{i=1}^{r} \big(\{g_i T b\} \times \eta_{\varrho^{-1}} V g_i b_1^{-1} x\big).$$

Note that each atom is therefore a finite union of open $W^{\wedge 1}$-plaques.

Remark 2.10. Recall [EL10, Def. 6.2]: A subset A of the orbit $W^{\wedge 1}.z$ is an open $W^{\wedge 1}$-plaque if for every z' the preimage of the orbit map, i.e. the set

$$\{w \in W^{\wedge 1} : w.z' \in A\},$$

is open and bounded.

So let us show for completeness that for β^X-a.e. $z \in B^X$ the stabilizer of z in $W^{\wedge 1}$ is trivial.

We first show that for $\beta \otimes \nu$-a.e. $z \in B \times X$ the stabilizer of z in $W^{\wedge 1}$ is trivial. To this end we consider the forward random walk $T_{\mathrm{f}} : B \times X \to B \times X$ given by

$$T_{\mathrm{f}}(b, x) = (Tb, b_1 x) \qquad ((b, x) \in B \times X).$$

The measure $\beta \otimes \nu$ is T_{f}-invariant.

The set $P \subseteq X$ of $W^{\wedge 1}$-periodic points is measurable, which in this setup e.g. follows from [EW11, Lem. 11.29]. The map T_{f} preserves $B \times P$ as E is contained in the normalizer of $W^{\wedge 1}$. Given a point $z \in B \times P$, let $\lambda(z)$ be the volume of $W^{\wedge 1}.z$, which depends measurably on z. Let $K \subseteq B \times P$ compact such that $\lambda|_K$ is continuous and

$$\beta\big((B \times P) \setminus K\big) < \varepsilon.$$

Using Poincaé recurrence, for $\beta \otimes \nu$-a.e. $z \in K$ there exists a sequence $(n_k)_{k\in\mathbb{N}}$ such that $n_k \to \infty$ as $k \to \infty$, $T_{\mathrm{f}}^{n_k}(z) \in K$ for all $k \in \mathbb{N}$ and $T_{\mathrm{f}}^{n_k}(z) \to z$ as $k \to \infty$. Therefore $\lambda(T_{\mathrm{f}}^{n_k}(z)) \to z$ as $k \to \infty$.

On the other hand we have

$$\mathrm{Stab}_{W^{\wedge 1}}\big(T_{\mathrm{f}}^n(z)\big) = \eta_{\varrho^{-2n}}\big(\mathrm{Stab}_{W^{\wedge 1}}(z)\big)$$

and therefore $\lambda(T_{\mathrm{f}}^n(z)) = \varrho^{-2n}\lambda(z)$ for all $n \to \infty$. It follows that $(\beta \otimes \nu)(K) = 0$. Hence inner regularity of $\beta \otimes \nu$ and Lusin's theorem imply that $B \times P$ is a nullset and therefore P is a nullset.

We now deduce that for β^X-a.e. $z \in B^X$ the stabilizer of z in $W^{\wedge 1}$ is trivial. First of all we recall that

$$\nu = \int_B \nu_b \mathrm{d}\beta(b)$$

and therefore the above argument implies that $\nu_b(P) = 0$ for β-a.e. $b \in B$. By definition of β^X we hence also find $\beta^X(B \times P) = 0$ or equivalently

$$\beta^X(B \times (X \setminus P)) = 1.$$

This proves the claim.

Proposition 2.11. *For β-a.e. $z = (b, x) \in B^X$ the measure ν_b is non-atomic and $\nu_b(\Phi_{W^{\wedge 1}}(x)) = 0$.*

Proof. In what follows we denote by $R_n : B \to \mathrm{Homeo}(X)$ the map given by

$$R_n(b)x = b_n^{-1} \cdots b_1^{-1} x \quad (x \in X, b \in B). \tag{2.2}$$

In the light of Florent's talk we only have to show that for β^X-a.e. $z \in B^X$

$$\Phi_{W^{\wedge 1}}(x) \subseteq \{y \in X : d(R_n(b)x, R_n(b)y) \overset{n\to\infty}{\longrightarrow} 0\}, \tag{2.3}$$

where d is the metric on $X = G\,/\,\Lambda$ given by

$$d(g\Lambda, h\Lambda) = \inf_{\lambda\in\Lambda} d_G(g\lambda, h) \qquad (g, h \in G)$$

for some right-invariant metric d_G on G inducing the topology on G.

In order to prove (2.3) $x \in X$ arbitrary, $w \in W^{\wedge 1}$ and $y = \exp(w)x$. Then

$$R_n(b)y = \exp(\mathrm{Ad}_{b_n^{-1}\cdots b_1^{-1}} w)R_n(b)x = \exp(\varrho^{2n}w)R_n(b)x.$$

Using right-invariance of the metric d_G, we get

$$d(R_n(b)y, R_n(b)x) = d(\exp(\varrho^{2n}w)R_n(b)x, R_n(b)x) \leq d_G(\exp(\varrho^{2n}w), 1) \overset{n\to\infty}{\longrightarrow} 0.$$

□

Corollary 2.12 (cf. [BQ11, Cor 6.15])**.** *Let $L \subseteq B^X$ measurable. Then there is a measurable subset $L' \subseteq L$ such that $\beta^X(L \setminus L') = 0$ and for all $z = (b, x) \in L'$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathfrak{g} \setminus W^{\wedge 1}$ such that $u_n \to 0$ as $n \to \infty$ and $z_n = (b, \exp(u_n)x) \in L$.*

The proof makes use of the following technical lemma, which we include for completeness.

Lemma 2.13. *Let Y be a second countable metric space and let λ be a Borel measure on Y. Let $\mathcal{T}$ be a topology of open sets on Y and*

$$\mathrm{supp}\lambda = \{y \in Y : \forall V \in \mathcal{T}\, y \in V \implies \lambda(V) > 0\}.$$

Then suppλ *is measureable and $Y \setminus \mathrm{supp}\lambda$ is a nullset.*

In the current discussion X, B, and B^X are of course second countable metric spaces.

Proof. One easily sees that $Y \setminus \mathrm{supp}\lambda$ is open, therefore suppλ is closed and in particular measurable. Let now $A \subseteq Y \setminus \mathrm{supp}\lambda$. For every $y \in A$ there exists $V_y \in \mathcal{T}$ an open neighborhood of y such that $\lambda(V_y) = 0$. As suppλ is closed, we can assume that $V_y \subseteq Y \setminus \mathrm{supp}\lambda$. The family $\{V_y : y \in A\}$ is an open cover of A. As A is a subspace of a product of a second countable metric space, it is Lindelöf. Therefore there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in A such that $\{V_{y_n} : n \in \mathbb{N}\}$ is an open cover of A. Hence

$$\lambda(A) \leq \sum_{n\in\mathbb{N}} \lambda(V_{y_n}) = 0.$$

□

Proof of Corollary 2.12. If $\beta^X(L) = 0$, the statement is trivially true. So assume now that $\beta^X(L) > 0$. As of inner regularity of β^X we can assume without loss of generality that L is compact. Let $\pi_B : B^X \to B$ denote the canonical projection and set $F' = \pi_B(L)$. The set F' is compact and therefore measureable. Given $b \in F'$ set

$$L_b = \{x \in X : (b, x) \in L\}.$$

Then

$$\begin{aligned}\beta^X(L) &= \int_B \int_X \chi_L(b, x)\mathrm{d}\nu_b(x)\mathrm{d}\beta(b)\\ &= \int_{F'} \nu_b(L_b)\mathrm{d}\beta(b).\end{aligned}$$

Let $F \subseteq F'$ be given by

$$F = \{b \in F' : \nu_b(L_b) \neq 0\}.$$

Then

$$\begin{aligned}\beta^X(\pi_B^{-1}(F) \cap L) &= \int_B \int_X \chi_F(b)\chi_L(b,x)\mathrm{d}\nu_b(x)\mathrm{d}\beta(b) \\ &= \int_F \nu_b(L_b)\mathrm{d}\beta(b) \\ &= \int_{F'} \nu_b(L_b)\mathrm{d}\beta(b) = \beta^X(L).\end{aligned}$$

Therefore the set $L_1 = \pi_B^{-1}(F) \cap L$ is a conull subset of L such that for all $(b,x) \in L_1$ we have $\nu_b(L_b) > 0$. The support of ν_b restricted to L_b is a conull subset of L_b and therefore for ν_b-a.e. $x \in L_b$ we have $\nu_b(L_b \cap \exp(U)x) > 0$ for every neighborhood $U \subseteq \mathfrak{g}$ of 0. Let now $(U_n)_{n\in\mathbb{N}}$ be neighborhood basis of $0 \in \mathfrak{g}$. As of Proposition 2.11 we have

$$\nu_b(L_b \cap \exp(U_n \setminus W^{\wedge 1})x) > 0$$

for ν_b-a.e. $x \in L_b$ and for every $b \in F'$. Let $u_n \in U_n \setminus W^{\wedge 1}$ such that $\exp(u_n)x \in L_b$, then $(b, \exp(u_n)x) \in L$ by definition and therefore the claim follows.

□

Given $n \in \mathbb{N}$ and $z = (b,x) \in B^X$, we define

$$h_{n,z} : E^n \to B^X, \quad h_{n,z}(a) \mapsto (aT^n b, a_1 \cdots a_n b_n^{-1} \cdots b_1^{-1} x),$$

where

$$(aT^n b)_i = \begin{cases} g_i & \text{if } 1 \leq i \leq n, \\ b_i & \text{otherwise.} \end{cases}$$

Note that in this case the image of a under $h_{n,z}$ agrees with $T_X^{-n}\{T_X^n z\}$

Lemma 2.14 (cf. [BQ12, Lem. 4.8]). *Let $K \subseteq B^X$ measurable. Then for β^X-a.e. $z \in B^X$ the limit*

$$\psi_z = \lim_{n\to\infty} \sum_{a \in E^n} \chi_K \circ h_{n,z}(a)\mu^{\otimes n}(a) \tag{2.4}$$

exists and satisfies

$$\int_{B^X} \psi_z \mathrm{d}\beta^X(z) = \beta^X(K).$$

In what follows, we denote

$$\mathcal{Q}_n^X = T_X^{-n}(\mathcal{B}_{B^X}), \qquad \mathcal{Q}_\infty^X = \bigcap_{n\in\mathbb{N}} \mathcal{Q}_n^X.$$

Note that $\mathcal{Q}_n^X \subseteq \mathcal{Q}_{n+1}^X$ for all $n \in \mathbb{N}$. Let $z = (b,x) \in B^X$, then

$$[z]_{\mathcal{Q}_n^X} = T_X^{-n}\{T_X^n z\} = h_{n,z}(E^n). \tag{2.5}$$

Proof. We first claim that for β^X-a.e. $z \in B^X$

$$(\beta^X)_z^{\mathcal{Q}_n^X} = \sum_{a\in E^n} h_{n,z}(a)\mu^{\otimes n}(a).$$

We denote the right hand side by $\lambda_{n,z}$. As of (2.5), the map $z \mapsto \lambda_{n,z}$ is $\mathcal{Q}_n^X$-measurable. Let $f \in \mathscr{L}(B^X)$ arbitrary. We will show that

$$\int_{B^X} \int_{B^X} f \mathrm{d}\lambda_{n,z}\mathrm{d}\beta^X(z) = \int_{B^X} f(z)\mathrm{d}\beta^X(z). \tag{2.6}$$

Assuming (2.6), the claim follows from [EW11, Prop. 5.19]. Therefore it only remains to prove (2.6). Recall the definition of $R_n : B \to \mathrm{Homeo}(X)$ in (2.2). Whenever convenient, we will abuse notation and consider R_n as a map from E^n to $\mathrm{Homeo}(X)$. The map R_n on B is then obtain by precomposing the map on E^n with the projection $\pi_n : B \to E^n$ onto the first n components.

In the proof of Lemma 2.7 we have argued that $R_1(b)_*\nu_b = \nu_{Tb}$ for almost every $b \in B^X$ and therefore induction implies that for all $a \in E^n$ and for almost all $b \in B$

$$(R_n(b))_*\nu_b = \nu_{T^n b} \quad \text{and} \quad (R_n(a)^{-1})_*\nu_b = \nu_{ab}.$$

Using T-invariance of β this implies

$$\begin{aligned}
\int_{B^X}\int_{B^X} f \mathrm{d}\lambda_{n,z} \mathrm{d}\beta^X(z) &= \int_B\int_X\int_{B^X} f \mathrm{d}\lambda_{n,(b,x)} \mathrm{d}\nu_b(x)\mathrm{d}\beta(b) \\
&= \int_B\int_X \sum_{a\in E^n} f(aT^n b, R_n(a)^{-1}R_n(b)x)\mu^{\otimes n}(a)\mathrm{d}\nu_b(x)\mathrm{d}\beta(b) \\
&= \sum_{a\in E^n}\int_B\int_X f(aT^n b, R_n(a)^{-1}x)\mathrm{d}\nu_{T^n b}(x)\mathrm{d}\beta(b)\mu^{\otimes n}(a) \\
&= \sum_{a\in E^n}\int_B\int_X f(ab, R_n(a)^{-1}x)\mathrm{d}\nu_b(x)\mathrm{d}\beta(b)\mu^{\otimes n}(a) \\
&= \sum_{a\in E^n}\int_B\int_X f(ab, x)\mathrm{d}\nu_{ab}(x)\mathrm{d}\beta(b)\mu^{\otimes n}(a) \\
&= \int_B\int_X f(b,x)\mathrm{d}\nu_b(x)\mathrm{d}\beta(b) = \int_{B^X} f\mathrm{d}\beta^X.
\end{aligned}$$

This shows in particular that $\mathbb{E}(\chi_K|\mathcal{Q}_n^X)(z) = \lambda_{m,z}(f)$ for β^X-a.e. $z \in B^X$.

Now the decreasing martingale theorem, cf. [EW11, Thm. 5.8], implies that

$$\lim_{n\to\infty} \mathbb{E}(f|\mathcal{Q}_n^X) \to \mathbb{E}(f|\mathcal{Q}_\infty^X)$$

β-almost surely and in $L^1(B^X, \beta^X)$. In particular, the limit

$$\psi_z = \lim_{n\to\infty} \lambda_{n,z}(\chi_K)$$

exists β^X-almost surely and

$$\int_{B^X} \psi_z \mathrm{d}\beta^X(z) = \int_{B^X} \mathbb{E}(\chi_K|\mathcal{Q}_\infty^X)\mathrm{d}\beta^X = \beta^X(K).$$

□

The following is a simple but crucial observation which is due to the uniform contraction ratio. It is important to keep in mind that the situation is more complicated for more general sets E.

Proposition 2.15. *Let $F \subseteq B^X$ be a conull subset so that for all $z \in B^X$ the conclusion of Lemma 2.7 applies. Then for all $z \in F$ and for all $n \in \mathbb{N}$ and $a \in E^n$ such that $h_{n,z}(a) \in F$, we have*

$$\sigma_z = \sigma_{h_{n,z}(a)} \qquad (a \in E^n).$$

Proof. We note that $T_X^n(h_{n,z}(a)) = T_X^n(z)$. Therefore $z, h_{n,z}(a) \in F$ implies

$$\sigma_{h_{n,z}(a)} = (\eta_{\varrho^{-2n}})_*\sigma_{T_X^n(h_{n,z}(a))} = (\eta_{\varrho^{-2n}})_*\sigma_{T_X^n(z)} = \sigma_z.$$

□

For what follows, we make the following observation. Let $x, y \in X$ and assume $y = \exp(v)x$, where $v \in \mathfrak{g}$. Let $b \in B$ arbitrary and $a \in E^n$. Then

$$\begin{aligned} h_{n,(b,y)}(a) &= \big(aT^n b, R_n(a)^{-1} R_n(b)\exp(v)x\big) \\ &= \big(aT^n b, \exp(\mathrm{Ad}_{a_1\cdots a_n b_n^{-1}\cdots b_1^{-1}} v) R_n(a)^{-1} R_n(b)x\big) \\ &= \exp(\mathrm{Ad}_{a_1\cdots a_n b_n^{-1}\cdots b_1^{-1}} v).h_{n,(b,x)} \end{aligned}$$

In what follows, we will abuse notation and use $R_n(b)$ to denote $\mathrm{Ad}_{b_n^{-1}\cdots b_1^{-1}} \in \mathrm{SL}(\mathfrak{g})$. We also define $F_{n,b}(a) = R_n(a)^{-1} \circ R_n(b) \in \mathrm{SL}(\mathfrak{g})$. In this notation, the above calculation can be written more concisely in the form

$$h_{n,(b,y)}(a) = \exp(F_{n,b}(a)v).h_{n,(b,x)}(a). \tag{2.7}$$

Lemma 2.16 (cf. [BQ11, Lem. 7.3])**.** *For all $\varepsilon, \eta > 0$ there are $r_0 \geq 1, n_0 \in \mathbb{N}$ such that for β-a.e. $b \in B$, for all $n \geq n_0$ and for all $u \in \mathfrak{g} \setminus \{0\}$*

$$\beta\left\{a \in B : \frac{1}{r_0} \leq \frac{\|F_{n,b}(a)u\|}{\varrho^{-2n}\|R_n(b)u\|} \leq r_0\right\} > 1 - \varepsilon \tag{2.8}$$

and

$$\beta\left\{a \in B : d(\mathbb{R}F_{n,b}(a)u, W^{\wedge 1}) \leq \eta\right\} > 1 - \varepsilon. \tag{2.9}$$

Proof. Recall from [SW19, Prop. 3.1] that for all $\varepsilon' > 0$ there are $c_0 > 0$, $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $v \in \mathfrak{g} \setminus \{0\}$

$$\beta\left\{a \in B : c_0 \leq \frac{\|R_n(a)^{-1}v\|}{\|R_n(a)^{-1}\|\|v\|}\right\} > 1 - \varepsilon \tag{2.10}$$

By construction $R_n(a)$ corresponds to conjugation with a matrix g_n of the form

$$g_n = \begin{pmatrix} \varrho^{-n} & y_n \\ 0 & \varrho^n \end{pmatrix},$$

where

$$y_n = \sum_{\ell=1}^{n} \varrho^{-n+2\ell-1} x_{\jmath(\ell)}$$

for some $\jmath : \mathbb{N} \to \{1, \ldots, r\}$; cf. the proof of Corollary 2.3. Moreover, we have shown that $|y_n| \ll \varrho^{-n}$. As all norms on $\mathrm{Mat}_3(\mathbb{R}) \cong \mathrm{End}(\mathfrak{g})$ are equivalent, one obtains that $\|R_n(a)^{-1}\| \asymp \varrho^{-2n}$. More explicitly, let

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

then

$$\begin{aligned} \mathrm{Ad}_g X &= a^2 X, \\ \mathrm{Ad}_g H &= -2abX + adH, \\ \mathrm{Ad}_g Y &= -b^2 X + bdH + d^2 Y. \end{aligned}$$

Therefore, the map Ad_g is represented by the matrix

$$[\mathrm{Ad}_g]_{\{X,H,Y\}} = \begin{pmatrix} a^2 & -2ab & -b^2 \\ 0 & ad & bd \\ 0 & 0 & d^2 \end{pmatrix}.$$

We therefore get

$$\|\mathrm{Ad}_g\|_\infty \asymp \|g\|_\infty^2.$$

Applying (2.10) with vector $v = R_n(b)u$, we get that for all $\varepsilon > 0$ there are $c_0 > 0$, $n_0 \in \mathbb{N}$ such that for all $u \in \mathfrak{g} \setminus \{0\}$ and for all $n \geq n_0$ we have

$$\beta \left\{ a \in B : c_0 \leq \frac{\|F_{n,b}(a)u\|}{\varrho^{-2n}\|R_n(b)u\|} \right\} > 1 - \varepsilon.$$

Note that for all $u \in \mathfrak{g} \setminus \{0\}$ we have

$$\frac{\|F_{n,b}(a)u\|}{\varrho^{-2n}\|R_n(b)u\|} \leq \frac{\|R_n(a)^{-1}\|}{\varrho^{-2n}} \leq M_0$$

for some $M_0 > 0$ which only depends on E. Therefore (2.8) follows after setting $r_0 = \max\{M_0, c_0^{-1}\}$.

In order to prove (2.9), we apply [SW19, Prop. 3.1b] with vector $v = R_n(b)u$. $\square$

Proof of Prop. 2.8. Let $\varepsilon \in (0,1)$ arbitrary. As of remark 2.9 there is $K \subseteq F$ compact such that $\beta^X(K) > 1 - \varepsilon^2$ and the map $z \mapsto \sigma_z$ is continuous on K. Denote $F_\varepsilon = \{\mathbb{E}(\chi_K | \mathcal{Q}_\infty^X) > 1 - \varepsilon\}$. On the one hand we β-almost surely have $1 \geq \mathbb{E}(\chi_K | \mathcal{Q}_\infty^X)$. Therefore Lemma 2.14 implies

$$\begin{aligned} 1 - \varepsilon^2 < \beta^X(K) &= \int_{B^X} \mathbb{E}(\chi_K | \mathcal{Q}_\infty^X) \mathrm{d}\beta^X \\ &\leq (1-\varepsilon)\beta^X(B^X \setminus F_\varepsilon) + \beta^X(F_\varepsilon) \\ &= (1-\varepsilon) + \varepsilon\beta^X(F_\varepsilon). \end{aligned}$$

It follows that

$$1 - \varepsilon \leq \beta^X(F_\varepsilon).$$

Let $F_1 \subseteq B^X$ a set of full measure such that for all $z \in F_1$ we have

$$\mathbb{E}(\chi_K | \mathcal{Q}_n^X)(z) \overset{n \to \infty}{\longrightarrow} \mathbb{E}(\chi_K | \mathcal{Q}_\infty^X)(z)$$

and for all $n \in \mathbb{N}$

$$\mathbb{E}(\chi_K | \mathcal{Q}_n^X)(z) = \sum_{a \in E^n} \chi_K \circ h_{n,z}(a).$$

Using Lusin's theorem, we choose $L_1 \subseteq F \cap F_1$ such that $\beta^X(L_1) > 1 - \varepsilon$ and the restriction of $f = \mathbb{E}(\chi_K | \mathcal{Q}_\infty^X)$ to L_1 is continuous. Using Egorov's theorem, we can find $L_2 \subseteq L_1$ compact such that $\beta^X(L_2) > 1 - \varepsilon$ and the convergence $\mathbb{E}(\chi_K | \mathcal{Q}_n^X) \to \mathbb{E}(\chi_K | \mathcal{Q}_\infty^X)$ is uniform on L_2. In particular, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $z \in L$ we have

$$1 - \varepsilon < \mathbb{E}(\chi_K | \mathcal{Q}_n^X)(z).$$

Therefore we have for all $z \in L_2$ and for all $n \geq n_0$

$$\beta^X\big(b \in B : h_{n,z} \circ \pi_n(b) \in K\big) = \mu^{\otimes n}\big(\{a \in E^n : h_{n,z}(a) \in K\}\big) > 1 - \varepsilon.$$

We next intersect L_2 with the set

$$Z = \{(b,x) \in B^X : \nu_b \text{ is non-atomic and } \nu_b(\Phi_{W^{\wedge 1}} x) = 0\}$$

to obtain a conull subset in L_2 by Proposition 2.11. In what follows, $L \subseteq L_2$ is a compact subset satisfying $\beta^X(L) > 1 - \varepsilon$.

As of Corollary 2.12, for β^X-a.e. $(b,x) \in L$ there exists a sequence $(u_m)_{m \in \mathbb{N}}$ in $\mathfrak{g} \setminus W^{\wedge 1}$ such that $u_m \to 0$ as $n \to \infty$ and $(b, \exp(u_m)x) \in L$ for all $m \in \mathbb{N}$. In particular, letting $z_0 = (b,x)$ and $z_m = (b, \exp(u_m)x)$, we have

$$\beta(\{a \in B : h_{n,z_m}(a) \in K\}) > 1 - \varepsilon$$

for all $n \geq n_0$ and for all $m \in \mathbb{N} \cup \{0\}$.

Given $m \in \mathbb{N}$ consider the sequence

$$r_{n,m} = \varrho^{-2n}\|R_n(b)u_m\|;$$

cf. (2.8). As $u_m \notin W^{\wedge 1}$ for all $m \in \mathbb{N}$ and referring to the proof of Corollary 2.3 (with inverted g_i's), we see that the sequence $r_{n,m}$ is unbounded for every $m \in \mathbb{N}$. We also note that

$$\frac{r_{n+1,m}}{r_{n,m}} \leq \varrho^{-2} \| R_1(b_{n+1}) \| = \varrho^{-4}. \tag{2.11}$$

Let now $A > 0$ arbitrary, then (assuming that u_m is sufficiently small), we let $n_m \in \mathbb{N}$ minimal such that

$$\frac{A}{r_0 \varrho^{-4}} \leq r_{n_m,m}.$$

As $u_m \to 0$ in m, we have $n_m \to \infty$ as $m \to \infty$. We note that minimality of n_m implies

$$\frac{A}{r_0} \geq \frac{A}{r_0 \varrho^{-4}} \frac{r_{n_m,m}}{r_{n_m-1,m}} > r_{n_m,m}$$

as of (2.11). Recall (2.9) and note that we can construct a sequence $\eta_m > 0$ such that $\eta_m \to 0$ and for β-a.e. $b \in B$, for all $u \in \mathfrak{g} \setminus \{0\}$ and for all m we have

$$\beta(\{a \in B : d(\mathbb{R} F_{n_m,b}(a) u, W^{\wedge 1}) \leq \eta_m\}) > 1 - \varepsilon. \tag{2.12}$$

As $\varepsilon > 0$ was chosen very small, there now exists $a_m \in B$ contained in the intersections of the sets in (2.8) and (2.12) and such that both $h_{n_m,z_0}(a_m) \in K$ and $h_{n_m,z_m}(a_m) \in K$.

After possibly passing to a subsequence we find that

(1) $\zeta'_m = h_{n_m,z_0}(a_m)$ converges to a limit $z' \in K$,
(2) $\zeta''_m = h_{n_m,z_m}(a_m)$ converges to a limit $z'' \in K$,
(3) $F_{n_m,b}(a_m) u_m$ converges to a vector $w \in W^{\wedge 1}$ satisfying

$$\frac{A}{r_0^2 \varrho^{-4}} \leq \|w\| \leq \frac{A}{r_0^2}.$$

Note that $\pi_B(\zeta'_m) = \pi_B(\zeta''_m)$ for all $m \in \mathbb{N}$. Moreover (2.7) implies that

$$\zeta''_m = \exp(F_{n_m,b}(a_m) u_m) \zeta'_m.$$

In particular

$$z'' = \Phi_w z'.$$

Since $[\sigma]$—the map sending a point $z \in B^X$ to the proportionality class of the leafwise measure σ_z—is continuous on K, we get

$$[\sigma_{z'}] = \lim_{m \to \infty} [\sigma_{\zeta'_m}] = [\sigma_{z_0}],$$
$$[\sigma_{\Phi_w z'}] = \lim_{m \to \infty} [\sigma_{\zeta''_m}] = \lim_{m \to \infty} [\sigma_{z_m}] = [\sigma_{z_0}].$$

As $r_0 \geq 1$, the claim follows. □

Corollary 2.17. *For β^X-a.e. $z \in B^X$ we have*

$$\mathrm{Stab}_{W^{\wedge 1}}([\sigma_z]) = W^{\wedge 1}.$$

Proof. As of Lemma 2.6, there is a β^X-conull subset $F \subseteq B^X$ such that for all $z \in F$ and for all $w \in W^{\wedge 1}$, if $\Phi_w z \in F$, then

$$\sigma_z = (\tau_w)_* \sigma_{\Phi_w z}. \tag{2.13}$$

As of Proposition 2.8 for β^X-a.e. $z \in F$ and for all $\varepsilon > 0$ there exits a non-zero element $w \in W^{\wedge 1}$ of norm less than ε and a point $z' \in F$ such that $\Phi_w z' \in F$ and

$$\sigma_{\Phi_w z'} \propto \sigma_{z'} \propto \sigma_z.$$

On the other hand (2.13) implies

$$\sigma_{z'} = (\tau_w)_* \sigma_{\Phi_w z'}$$

and therefore

$$(\tau_{-w})_*\sigma_z \propto \sigma_z.$$

As $\varepsilon > 0$ was arbitrary, we have shown that the stabilizer of $[\sigma_z]$ contains a non-discrete non-trivial subgroup of $W^{\wedge 1}$. As the stabilizer is closed and $W^{\wedge 1}$ is one-dimensional, the claim follows. $\square$

Corollary 2.18. *For β^X-a.e. $z \in B^X$ we have*

$$\mathrm{Stab}_{W^{\wedge 1}}(\sigma_z) = W^{\wedge 1}.$$

Proof. As of Corollary 2.17, we know that for β^X-a.e. $z \in B^X$ there is a linear map $\chi_z : W^{\wedge 1} \to \mathbb{R}$ such that for all $w \in W^{\wedge 1}$

$$(\tau_w)_*\sigma_z = e^{\chi_z(w)}\sigma_z.$$

Recall from Lemma 2.7 we can fix normalizations and a constant $c > 0$ (depending on z and n) such that $\sigma_{T_X^n z} = c(\eta_{\varrho^{2n}})_*\sigma_z$. Therefore for all $w \in W^{\wedge 1}$ we get

$$\begin{aligned}(\tau_w)_*\sigma_{T_X^n z} &= c(\eta_{\varrho^{2n}})_*\big((\tau_{\varrho^{-2n}w})_*\sigma_z\big)\\ &= ce^{\varrho^{-2n}\chi_z(w)}(\eta_{\varrho^{2n}})_*\sigma_z\\ &= e^{\varrho^{-2n}\chi_z(w)}\sigma_{T_X^n z}.\end{aligned}$$

In particular

$$\chi_{T_X^n z} = \varrho^{-2n}\chi_z.$$

The character χ_z depends measurably on z and therefore combining Lusin's theorem and Poincaré recurrence it follows that β^X-almost surely $\chi_z = 0$. $\square$

References

BQ11. Y. Benoist and J.-F. Quint. Mesures stationnaires et fermés invariants des espaces homogènes. *Ann. of Math.* (2) 174 (2011), no. 2, 1111–1162.

BQ12. Y. Benoist and J.-F. Quint. Introduction to random walks on homogeneous spaces. *Jpn. J. Math.* 7 (2012), no. 2, 135–166.

EL10. M. Einsiedler and E. Lindenstrauss. Diagonal actions on locally homogeneous spaces. *Homogeneous flows, moduli spaces and arithmetic*, 155–241, Clay Math. Proc., 10, *Amer. Math. Soc.*, *Providence, RI*, 2010.

EW11. M. Einsiedler and T. Ward. Ergodic theory with a view towards number theory. Graduate Texts in Mathematics, 259. *Springer-Verlag London, Ltd.*, *London*, 2011.

Ka13. O. Karpenkov. Geometry of Continued Fractions. Algorithms and Computation in Mathematics, 26. *Springer-Verlag Berlin Heidelberg*, 2013.

Kh64. A. Khintchine. Continued Fractions. *The University of Chicago Press*, 1964.

Mo56. G. Mostow. Fully Reducible Subgroups of Algebraic Groups. *Amer. J. Math.* (1) 78 (1956), 200–221.

Ne99. J. Neukirch. Algebraic number theory. Grundlehren der Mathematischen Wissenschaften, 322. *Springer-Verlag Berlin*, 1999.

Se85. C. Series. The modular surface and continued fractions. *J. London Math. Soc.* (2) 31 (1985), no. 1, 69–80.

SW19. D. Simmons and B. Weiss. Random walks on homogeneous spaces and Diophantine approximation on fractals. *Invent. Math.* 216 (2019), no. 2, 337–394.

Email address: manuel.luethi@math.ethz.ch