# Khintchine's theorem

## Dirichlet's theorem:

$\forall x \in \mathbb{R} \, \forall Q \in \mathbb{N} \, \exists 1 \leq q \leq Q$

$$|qx+p| < \frac{1}{Q} \qquad (1)$$

**Proof:** Pigeon-hole principle:

$$\underbrace{0 \cdot x + \mathbb{Z}, 1 \cdot x + \mathbb{Z}, \ldots, Q \cdot x + \mathbb{Z}}_{Q+1 \text{ pts.}} \in \mathbb{R}/\mathbb{Z} \cong [0,1)$$

$\blacksquare$

**Corollary:** $\forall x \in \mathbb{R} \, \exists \infty$-many $q \in \mathbb{N}$ s.t.

$$|qx+p| < \frac{1}{q}$$

**Proof:** If $x \in \mathbb{Q}$, this is clear. So assume $x \in \mathbb{R} - \mathbb{Q}$. Note that $q$ as in $(1)$ satisfies $q \xrightarrow{Q \to \infty} \infty$. Hence we find a sequence $(q_Q)_{Q \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ s.t. $q_Q \uparrow \infty$ and

$$|q_Q x + p| < \frac{1}{Q} \leq \frac{1}{q_Q}$$

$\blacksquare$

There are analogues of these in higher dimension. We generalize: Let

- $\|v\| := \max\{|v_i| : 1 \leq i \leq d\} \qquad (v \in \mathbb{R}^d)$,
- $\psi : \mathbb{N} \longrightarrow [0, \infty)$
- $W(\psi) = \left\{ v \in \mathbb{R}^d : \begin{array}{c} \exists \infty\text{-many } (p,q) \in \mathbb{Z}^d \times \mathbb{N} \text{ s.t.} \\ \|qv + p\| < \psi(q) \end{array} \right\}$

For $d=1$, $\psi_1(q) := \frac{1}{q^1}$, we have shown that

$$W(\psi_1) = \mathbb{R}$$

What happens for $\psi_\tau$ with $\tau > 1$?

**Theorem (Khintchine '26):**

Assume that $\psi$ is monotonic. Then

$$\mathrm{Leb}(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q\in\mathbb{N}} \psi(q)^d < \infty, \\ \text{FULL} & \text{if } \sum_{q\in\mathbb{N}} \psi(q)^d = \infty. \end{cases}$$

**Example:** $W(\psi_\tau) = 0$ if $\tau > 1$

**Remark:** Unsatisfactory, because "clearly" for $\tau_2 > \tau_1$ the set $W(\psi_{\tau_2})$ is smaller $W(\psi_{\tau_1})$. This is covered by Jarník's and Besicovitch's theorems:

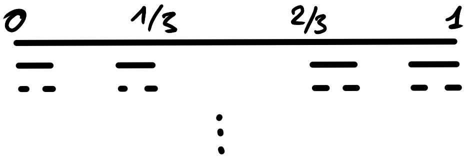
$$\dim_{\mathcal{H}} W(\psi_\tau) = \frac{2}{1+\tau}$$

This is the story of Diophantine approximation in Euclidean space. What can we say about subsets, e.g.,

- curves/manifolds (like $t \mapsto (t, t^2, t^3, \ldots, t^d)$),
- "fractals"?

# The middle third Cantor's set:

Geometric construction




limit is a compact set $C_3$ described as the numbers between 0 and 1 whose base 3 expansion does not contain the digit 1. It has Lebesgue measure 0.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{3}x,$$
$$g(x) = \frac{1}{3}x + \frac{2}{3},$$

then $C_3 \subseteq \mathbb{R}$ is the unique compact subset satisfying

$$C_3 = f(C_3) \cup g(C_3).$$ (Attractor of IFS)

Moreover, $\dim_H(C_3) = \frac{\log 2}{\log 3}$ (2 = # digits used, 3 = base) and the Hausdorff measure $\mu$ on $C_3$ is (up to scaling) the unique measure satisfying

$$\mu = \frac{1}{2} f_* \mu + \frac{1}{2} g_* \mu.$$

**Remark:** Hutchinson ('81) has shown (in greater generality) that for any (fully supported) probability vector $\lambda = (\lambda_f, \lambda_g)$ there is a unique probability measure $\mu_\lambda$ on $C_3$ s.t.

$$\mu_\lambda = \lambda_f f_* \mu_\lambda + \lambda_g g_* \mu_\lambda.$$

**Mahler's question ('84):** How close can irrational elements of Cantor's set be approximated by rational numbers
(i) in Cantor's set, and
(ii) by rational numbers not in Cantor's set.

# Progress towards Mahler's questions

Call $x \in \mathbb{R}$ very-well approximable if

$$x \in \bigcup_{\tau > 1} W(\psi_\tau) =: VWA$$

By Khintchine $\mathrm{Leb}(VWA) = 0$

Jarník, Besicovitch $\dim_{\mathcal{H}}(VWA) = 1$

## Weiss ('01):

If $\sum q^{\frac{\log 2}{\log 3} - 1} \psi(q)^{\frac{\log 2}{\log 3}} < \infty$, then $C^3 \cap W(\psi)$ is null.

Almost no points on a Cantor set are very well approximable

Generalized by Kleinbock, Lindenstrauss, Weiss ('05), who defined a class of measures (including volume on non-degenerate manifolds and certain "self-similar" fractal measures), called friendly measures and proved

## Kleinbock, Lindenstrauss, Weiss ('05):

Almost no points w.r.t. a friendly measure are VWA

## Yu ('20)

$\dim_{\mathcal{H}}(C_{15} \cap VWA) = \dim_{\mathcal{H}}(C_{15})$

$\sum \psi(q) < \infty \Rightarrow \mu_{15}(W(\psi)) = 0$

At the opposite end are badly approximable numbers, i.e., $\exists c>0$ s.t.

$$\forall q\in\mathbb{N} \quad |qx+p|\geq \frac{c}{q}$$

($\Leftrightarrow$ cfe is bounded).

Folklore (more general statement by Kleinbock-Weiss ('05), Kristensen-Thorn-Velani ('06))

$$\dim_{\mathcal{H}}(\mathrm{Bad}\cap C_3)=\dim_{\mathcal{H}}(C_3)$$

Einsiedler-Fishman-Shapira ('11)

Almost no points in $C_3$ are badly approximable

Recent progress in terms of Schmidt games by Badziahin, Beresnevich, Farkas, Fraser, Harrap, Nesharim, Simmons, Yang.

## Rational IFS

A (finite) rational IFS (by contractive similarities) on $\mathbb{R}^d$ is a finite family $\mathcal{F}$ of maps of the form

$$f(x)=\rho_f O_f x+b_f,$$

where $\rho_f\in(0,1)\cap\mathbb{Q}$, $O_f\in SO_d(\mathbb{Q})$, $b_f\in\mathbb{Q}^d$

Theorem (Hutchinson '81)

Let $\mathcal{F}$ a rational IFS and let $\lambda$ a (fully supp) probability $\omega$ on $\mathcal{F}$.

1) $\exists!\,\mathcal{C}=\mathcal{C}_{\mathcal{F}}\Subset\mathbb{R}^d$ non-empty s.t. $\mathcal{C}=\bigcup\{f(\mathcal{C}):f\in\mathcal{F}\}$

2) $\exists!\,\mu=\mu_{\mathcal{F},\lambda}$ compactly supported probability on $\mathbb{R}^d$ s.t.

$$\mu=\sum_{f\in\mathcal{F}}\lambda_f f_*\mu.$$

Moreover $\operatorname{supp}\mu=\mathcal{C}$.

So, given a rational IFS and a probability measure $\lambda$, we obtain a natural tuple $(\mathcal{C}, \lambda)$ consisting of a fractal and a self-similar measure.

**Theorem** (Khalil-L. ('21))

$\forall d \in \mathbb{N}\ \exists \varepsilon_0 > 0$ s.t. the following is true. Suppose that $\mathcal{F}$ is a rational IFS on $\mathbb{R}^d$ and $\lambda$ a probability vector. Assume that $\mathcal{F}$ satisfies the open set condition. Let $s = \dim_{\mathcal{H}}(\mathcal{C})$ and assume that

$$\left(\frac{d \log \rho_{min}}{\log \lambda_{max}} - 1\right) \frac{\log \lambda_{min}}{s \log \rho_{max}} < \varepsilon_0$$

missing digit: $d = 1$, $\lambda_{min} = \lambda_{max} = \frac{1}{|\Lambda|}$, $\rho_{min} = \rho_{max} = \frac{1}{p}$

$$\left(\frac{\log p}{\log \Lambda} - 1\right) \frac{\log \Lambda}{s \log p} = \left(\frac{1}{s} - 1\right) < \varepsilon$$

Let $\psi : \mathbb{N} \to (0, \infty)$ non-increasing. Then

$$\mu\left(W(\psi)\right) = \begin{cases} 0 & \text{if } \sum \psi(q)^d < \infty, \\ 1 & \text{if } \sum \psi(q)^d = \infty. \end{cases}$$

$\ulcorner$ $\mathcal{F}$ satisfies the open set condition if $\exists\, U \subseteq \mathbb{R}^d$ non-empty open s.t.

(i) $\forall f \in \mathcal{F}$ $f(U) \subseteq U$,

(ii) $\forall f \neq g \in \mathcal{F}$ $f(U) \cap g(U) = \emptyset$. $\lrcorner$

## Theorem (Khalil–L. ('21))

Let $p$ be a prime, $\mathcal{C}$ a missing digit Cantor's set in base $p$, and let $\mu$ be the normalized Hausdorff measure on $\mathcal{C}$. If

$$\dim_H(\mathcal{C}) > 0.839,$$

and if $\psi : \mathbb{N} \to (0,\infty)$ non-increasing, then

$$\mu(W(\psi)) = \begin{cases} 0 & \text{if } \sum \psi(q) < \infty, \\ 1 & \text{if } \sum \psi(q) = \infty. \end{cases}$$

<u>Remark:</u> (i) A missing digit set for base is given as follows. Let

$$\Lambda \subseteq \{0, \dots, p-1\}$$

such that $|\Lambda| \geq 2$. Let

$$\mathcal{F} = \left\{ x \mapsto \frac{1}{p}x + \frac{k}{p} : k \in \Lambda \right\}.$$

Then $\mathcal{C}_{p,\Lambda}$ is the missing-digit set, $\dim_H(\mathcal{C}_{p,\Lambda}) = \frac{\log|\Lambda|}{\log|p|}$, and Hausdorff measure is the unique self-similar measure w.r.t. uniform distribution.

(ii) $\frac{\log 4}{\log 5} \sim 0.861$, $\frac{\log 2}{\log 3} \sim 0.631$

The proof is modelled after the proof of the Khintchine–Groshev theorem due to Kleinbock–Margulis ('98).

## The convergence case

From now on assume $\psi: [1,\infty) \to [0,\infty)$, $\psi(1)=1$, $\psi(q) \xrightarrow{q\to\infty} 0$ (for simplification)

Remark/Exercise: Note that on $[0,1]^d$, we have

$$\omega(\psi) = \bigcap_{q} \bigcup \left\{ B\left(t, \tfrac{\psi(q)}{q}\right) \cap [0,1]^d : t \in \tfrac{1}{q}\mathbb{Z}^d \cap [0,1]^d \right\} =: \overline{\lim_q} \tilde{A}_q$$

and

$$\mathrm{vol}(\tilde{A}_q) \ll \psi(q)^d.$$

Hence

$$\sum_q \psi(q)^d < \infty \Rightarrow \sum_q \mathrm{vol}(\tilde{A}_q) < \infty \overset{\text{Borel-Cantelli}}{\Rightarrow} \mathrm{vol}(\overline{\lim_q} \tilde{A}_q) = 0 \Rightarrow \mathrm{vol}(\omega(\psi)) = 0.$$

Given $n \in \mathbb{N}$, let

$$A_n = \left\{ x \in \mathbb{R}^d : \begin{array}{l} \exists (p,q) \in \mathbb{Z}^{d+1}_{\text{prim}} \text{ such that} \\ 0 < q < 2^{n+1} \text{ and } \|qx+p\| < \psi(2^n) \end{array} \right\}$$

Then $\omega(\psi) \subseteq \overline{\lim} A_n$:

⌈Let $x \in \omega(\psi)$ and let $p,q$ s.t. $\|qx+p\| < \psi(q)$. Let $m = \gcd(p,q)$, then

$$\|qx+p\| = m\|q'x+p'\| < \psi(q)$$

$$\Rightarrow \|q'x+p'\| < \frac{\psi(q)}{m} \leq \psi(q').$$

Let $q_\ell \nearrow \infty$ and assume $\|q_\ell x + p_\ell\| < \psi(q_\ell)$

Case 1: $q'_\ell$ is bounded $\Rightarrow m_\ell \nearrow \infty$

$$\Rightarrow \|q'_\ell x + p'_\ell\| < \frac{\psi(q_\ell)}{m_\ell} \longrightarrow 0$$

$$\|p'_\ell\| \leq \|q'_\ell x + p'_\ell\| + \|q'_\ell x\| \ll 1 \Rightarrow q'_\ell \longrightarrow q_*,\ p'_\ell \longrightarrow p_*$$

$$\therefore\ q'_\ell x + p'_\ell \longrightarrow q_* x + p_* = 0 \Rightarrow x = -\frac{p_*}{q_*} \in \mathbb{Q}^d.$$

But then clearly $x \in A_n$ for all sufficiently large $n$: Let $x = -\frac{p}{q}$ a reduced form representation and let $n_0 \in \mathbb{N}$ s.t. $q < 2^{n_0+1}$, then for all $n \geq n_0$ $x \in A_n$.

Case 2: $q'_\ell$ is unbounded. Let $(n_\ell)_{\ell\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ s.t. $\forall \ell$ $2^{n_\ell} \leq q'_\ell < 2^{n_\ell+1}$. Then

$$\|q'_\ell x + p'_\ell\| < \frac{\psi(q_\ell)}{m_\ell} \leq \psi(q'_\ell) \leq \psi(2^{n_\ell})$$

$\therefore\ x \in A_{n_\ell}$ for all $n \in \mathbb{N}$

As $q'_\ell$ is unbounded, $x \in \overline{\lim_n} A_n$

⌋

Analogously, we will show that

$$\sum_{q\in\mathbb{N}} \psi(q)^d < \infty \Rightarrow \sum_{q\in\mathbb{N}} \mu(A_n) < \infty.$$

In this generality, the argument is not as straight-forward and we will rely on a homogeneous dynamics argument for the proof.

## Setup

- $G = GL_{d+1}(\mathbb{R})/\mathbb{R}^\times$
- $\Gamma$ = image of $GL_{d+1}(\mathbb{Z})$ in $G$
- $X_{d+1} = G/\Gamma \underset{\Uparrow}{=}$ space of unimodular lattices in $\mathbb{R}^{d+1}$

  (i) every coset $[g]\in G/\Gamma$ has a representative in $SL_{d+1}(\mathbb{R})$

  (ii) $[g] \mapsto g\mathbb{Z}^{d+1}$
- Given $x\in\mathbb{R}^d$, let

$$\Lambda_x = \underbrace{\begin{pmatrix} \mathbb{1}_d & x \\ 0 & 1 \end{pmatrix}}_{=:u(x)} \mathbb{Z}^{d+1} \in X_{d+1}$$

- Given $t\in\mathbb{R}$, $\Lambda\in X_{d+1}$, let

$$g_t\Lambda = \underbrace{\begin{pmatrix} e^{t/d}\mathbb{1}_d & 0 \\ 0 & e^{-t} \end{pmatrix}}_{=:a_t} \Lambda.$$

<u>Goal</u>: Relate the property $x\in A_n$ to cusp excursions of $\Lambda_x$ under $\{g_t : t\geq 0\}$.

## Preparations

<u>Proposition</u> (Kleinbock-Margulis ('99))

$\exists r = r_\psi : [0,\infty) \longrightarrow \mathbb{R}$ continuous s.t.

(i) $\lambda(t) := t - r(t)$ sat's $\lambda(t)\uparrow\infty$ strictly,

(ii) $L(t) := t + dr(t)$ is non-decreasing,

(iii) $\psi(e^{\lambda(t)})^d = e^{-L(t)}$

(iv) $\forall t_2\geq t_1\geq 0$

$r(t_2) - r(t_1) \geq -\frac{1}{d}(t_2-t_1)$ "weak monotonicity"

$\lambda(t_2)-\lambda(t_1) \leq \frac{d+1}{d}(t_2-t_1)$ "moderate growth"

In what follows $(t_n)_{n\in\mathbb{N}}$ is chosen so that

$$e^{\lambda(t_n)}=2^n$$

The weak monotonicity gives

$$t_{n+1}-t_n\geq\frac{d}{d+1}\log 2$$

Let $V_n=\left\{(\omega',\omega_{d+1})\in\mathbb{R}^d\times\mathbb{R} : \begin{array}{l}\|\omega'\|_\infty<e^{-r(t_n)}\\ |\omega_{d+1}|<2e^{-r(t_n)}\end{array}\right\}$

**Lemma ("Dani correspondence")**

For sufficiently large $n\in\mathbb{N}$ the following are equivalent:

(i) $x\in A_n$

(ii) $\exists v\in\Lambda_{x,\mathrm{prim}}$ s.t. $a_{t_n}v\in V_n$

(iii) $\exists v\in\mathbb{Z}^{d+1}_{\mathrm{prim}}$ s.t. $a_{t_n}u(x)v\in V_n$

Proof: (ii)$\Leftrightarrow$(iii): clear/immediate ← immediate

(i)$\Rightarrow$(iii): $x\in A_n\Rightarrow\exists(p,q)\in\mathbb{Z}^{d+1}_{\mathrm{prim}}$ s.t. ← formal application of Kleinbock-Margulis Prop. (use that $\psi(2^n)<1$ for $n$ suff. large)

$0<q<2^{n+1}$ and $\|qx+p\|<\psi(2^n)$

$$a_{t_n}u(x)\begin{pmatrix}p\\q\end{pmatrix}=a_{t_n}\begin{pmatrix}qx+p\\q\end{pmatrix}=\begin{pmatrix}e^{t_n/d}(qx+p)\\e^{-t_n}q\end{pmatrix}$$

$$\|e^{t_n/d}(qx+p)\|_\infty<\psi(e^{\lambda(t_n)})e^{t_n/d}=e^{-(L(t_n)-t_n)/d}=e^{-r(t_n)}$$

$$|e^{-t_n}q|<e^{-t_n}2e^{\lambda(t_n)}=2e^{-r(t_n)}$$

$$\therefore\ a_{t_n}u(x)\begin{pmatrix}p\\q\end{pmatrix}\in V_n$$

(iii)$\Rightarrow$(i): $a_{t_n}u(x)\begin{pmatrix}p\\q\end{pmatrix}\in V_n\Leftrightarrow a_{t_n}u(x)\begin{pmatrix}-p\\-q\end{pmatrix}\in V_n$

$\therefore$ w.l.o.g. $q\geq0$.

If $q=0$, then $\|p\|_\infty=\|qx+p\|_\infty\leq e^{-(t_n+dr(t_n))/d}=e^{-L(t_n)/d}=\psi(e^{\lambda(t_n)})=\psi(2^n)<1$ (for $n$ suff. large),

hence $p=0$ ↯

So $q>0$ and $q=|q|<2e^{t_n-r(t_n)}=2e^{\lambda(t_n)}=2^{n+1}$.

$\therefore\ x\in A_n$.

**Theorem (Siegel ('45))**

Given $f\in L^1(\mathbb{R}^{d+1})$, let $E_f:X_{d+1}\to\mathbb{C}$ be

$$E_f(\Lambda)=\sum_{v\in\Lambda_{\mathrm{prim}}}f(v).$$

Then $E_f\in L^1(X_{d+1})$ and

$$\int_{X_{d+1}}E_f(\Lambda)\,d\Lambda\propto_d\int_{\mathbb{R}^{d+1}}f(v)\,dv$$

## Theorem (Effective equidistribution – Khalil-L. ('21))

$\exists \kappa>0$ s.t. $\forall f\in C_c^\infty(X_{d+1})$
$$\int_{\mathbb{R}^d} f(g_t\Lambda_x)\,d\mu(x) = \int_{X_{d+1}} f(\Lambda)\,d\Lambda + O_f(e^{-\kappa t})$$

This is one of the main results of the paper and will be the content of the third meeting.
Assuming the theorem, we morally have for $\chi_n$ the indicator function of $V_n$

$$\mu(A_n) \leq \int_{\mathbb{R}^d} E_{\chi_n}(a_{t_n}\Lambda_x)\,d\mu(x)$$
$$\leq \int_{X_{d+1}} E_{\chi_n}(\Lambda)\,d\Lambda + O_{E_{\chi_n}}(e^{-\kappa t_n})$$
$$\ll_d \mathrm{vol}(V_n) + e^{-\kappa t_n}.$$

Note that $L(t_n) = \lambda(t_n) + (d+1)r(t_n)$
$$\therefore\ 2^n\psi(2^n)^d = e^{\lambda(t_n)-L(t_n)} = e^{-(d+1)r(t_n)}$$
Also $t_n \geq -\frac{d}{d+1}\lambda(0) + \frac{d\log 2}{d+1}n$.

Hence
$$\sum_n \mu(A_n) \ll_d \sum_n 2e^{-(d+1)r(t_n)} + \sum_n e^{-\kappa t_n}$$
$$\ll \underbrace{\sum_n 2^n\psi(2^n)^d}_{\substack{<\infty \\ \text{by Cauchy} \\ \text{condensation}}} + \underbrace{\sum_n e^{-\epsilon n}}_{<\infty} \quad \text{for some } \epsilon>0$$

$\blacksquare$

## Divergence case:

The divergence case is more complicated. It is well-known that
$$\sum_n P(B_n) = \infty \not\Rightarrow P(\overline{\lim}\, B_n) > 0.$$

Example: Let $P = \mathrm{leb}$, $B_n = [0,\frac{1}{n}]$, then $\sum_n P(B_n) = \infty$ and
$$\overline{\lim}\, B_n = \bigcap_n \bigcup_{m\geq n} B_n = \bigcap_n \bigcup_{m\geq n} [0,\tfrac{1}{n}] = \bigcap_n [0,\tfrac{1}{n}] = \{0\} \therefore P(\overline{\lim}\, B_n) = 0.$$

However, a standard result is that if $\{B_n : n \in \mathbb{N}\}$ is independent, then

$$\sum_n P(B_n) = \infty \Rightarrow P(\overline{\lim}\, B_n) = 1 .$$

This is too strong but $\underbrace{\text{"asymptotic" independence}}$ can sometimes be achieved.

$\Rightarrow$ decay of correlations

In the proof of Khintchine - Groshev (i.e., for Lebesgue measure) one can deduce decay of correlations from decay of matrix coefficients, whose error term depends on the $L^2$-Sobolev norm of smooth approximations of indicators of cusps.

This is not true for us: Our effective equidistribution statement depends on the test function via $L^\infty$-Sobolev norm, which is $O(1)$.

<u>Proposition</u> (Double equidistribution)

open set condition $\Rightarrow$ small overlaps

If $\mu$ is as in the main theorem and satisfies a strengthening of our effective equidistribution theorem, then $\exists\, \delta, \varepsilon_* > 0$, $C_* > 1$ s.t. $\forall\, \varphi, \psi \in B^\infty_{\infty,\ell}(G/\Gamma)$ invariant under non-negative

$$\left\{ k_\rho := \begin{pmatrix} O_\rho & 0 \\ 0 & 1 \end{pmatrix} : \rho \in \mathcal{F} \right\}$$

and for all $t \geq s > 0$ satisfying

$$\underbrace{t \geq C_* s}_{\text{long range}} \quad \text{or} \quad \underbrace{s \leq t \leq (1+\varepsilon_*) s}_{\text{intermediate range}}$$

we have

$$\int \varphi(a_t u(x)\Gamma)\, \psi(a_s u(x)\Gamma)\, d\mu(x) \leq \int \varphi \int \psi(a_s u(x)\Gamma)\, d\mu(x) + O\left(S_{\infty,\ell}(\varphi) S_{\infty,\ell}(\psi) e^{-\delta|t-s|}\right)$$

<u>Bottom line:</u> We get sort of a double equidistribution for long-range and intermediate range correlations.

**Proposition** (Chaika-Fairchild/Khalil-L.)

Taylor-made converse to the Borel-Cantelli lemma aiming at decay in long and intermediate range (giving positive mass: in our case $\frac{1}{144}$)

Let $(X,\mu)$ a probability space and assume that there are constants $D\geq 1$, $0<\sigma<1$, $0<a\leq\frac{1}{6}$ such that

(1) $\mu(E_n)>0$ for all $n\gg 1$ and $\sum_{n\in\mathbb{N}}\mu(E_n)=\infty$

(2) $\exists C_*, C_\#\geq 1, \varepsilon_*>0$ s.t. $\forall m,n\in\mathbb{N}$ with $m\gg 1$ and

$$n\geq C_* m \quad \text{or} \quad m\leq n\leq(1+\varepsilon_*)m,$$

we have "quasi-independence"

$$\mu(E_m\cap E_n)\leq C_\#\mu(E_m)\mu(E_n)+D\left(e^{-\sigma m}\mu(E_n)+e^{-\sigma(n-m)}\right).$$

(3) $\forall m,n\in\mathbb{N}$ with $1\ll m\leq n$ "weak quasi-independence"

$$\mu(E_m\cap E_n)\leq D\mu(E_m)\max\{\mu(E_n)^\sigma, 2^{-\sigma(n-m)}\}$$

(4) $\forall m,n\in\mathbb{N}$ with $1\ll m\leq n\leq m+\lceil -a\log\mu(E_m)\rceil$, "monotonicity"

$$\mu(E_n)\leq D\mu(E_m)^\sigma.$$

Then $\mu(\overline{\lim}\, E_n)\geq\frac{1}{C_\#}$.

Define

$$A_n^*(\psi)=\left\{x\in\mathbb{R}^d : \begin{array}{l}\exists (p,q)\in\mathbb{Z}^{d+1}_{prim} \text{ s.t. } 2^{n-1}\leq|q|<2^n \text{ and}\\ \|qx+p\|<\psi(2^n)\end{array}\right\}$$

One "checks" that $\overline{\lim}\, A_n^*(\psi)\subseteq W(\psi)$ and that the collection satisfies the assumptions of the divergence Borel-Cantelli lemma (using double equidistribution, simplex lemma).

Davenport: small balls intersect rational vectors of bounded denominators in hyperplanes.

As the lower bound obtained in the divergence BC, we can iterate and apply to pieces of the fractal.

Then we apply an analogue to Cassels' zero-full law to deduce that $W^*(\psi)$ has full measure.

Proposition (Cassels)

If $A \subseteq [0,1]$ is Lebesgue (m)-able, $c > 0$, and for all intervals $I \subseteq [0,1]$ we have that $\lambda(A \cap I) \geq c\lambda(I)$. Then $\lambda(A) = 1$.

Proposition (Khalil-L. ('21))

Let $\mu$ as in the theorem. Let $A \subseteq \mathbb{R}^d$ Borel and assume $c > 0$ s.t. for all $n \in \mathbb{N}$ and for all $(f_1, \dots, f_n) \in \mathcal{F}^n$

$$\mu\left(A \cap (f_n \circ \dots \circ f_1)(\mathcal{C})\right) \geq c\,\mu\left((f_n \circ \dots \circ f_1)(\mathcal{C})\right).$$

Then $\mu(A) = 1$.

Pf: Lebesgue density + coding map.

# Sketch effective equidistribution

- Define
$$\mu_t(\varphi) = \int_{\mathbb{R}^d} \varphi(g_t \Lambda_x)\, d\mu(x) \qquad (\varphi \in C_c(G/\Gamma)).$$
- Construct an $a_t$-equiv. operator $P$ on $C_c(G/\Gamma)$ that leaves $\mu_t$ invariant.
- Show that $P$ has a spectral gap for the $L^2$-norm.
- Approximate $\mu$ by pol. eff. equidist. $\nu$ on $\mathbb{R}^d$, i.e. satisfying
$$\int_{\mathbb{R}^d} \varphi(g_t \Lambda_x)\, d\nu(x) = \int_{X_{d+1}} \varphi + O_\varphi(e^{-\varkappa t})$$
- Use spectral gap of $P$ to beat approx. error via Cauchy-Schwarz.

  ↑ suff. large spectral gap needed.

## Proposition:

Let $\mathcal{F}$ a rational IFS, $\lambda$ a probability vector on $\mathcal{F}$. Let $P: \mathcal{M}_c^1(\mathbb{R}^d) \to \mathcal{M}_c^1(\mathbb{R}^d)$
$$P(\nu) = \sum_{f \in \mathcal{F}} \lambda_f f_* \nu.$$

$\exists r \in (0,1)$, $C \geq 0$ s.t. $\forall \varphi \in \mathrm{Lip}_b(\mathbb{R}^d)$ and for all $n \in \mathbb{N}$
$$|\mu(\varphi) - P^n(\nu)(\varphi)| \leq C r^n \|\varphi\|_{\mathrm{Lip}}.$$

In fact $r = \sum_{f \in \mathcal{F}} \lambda_f \rho_f$

Assume that $P$ satisfies $\forall v \in \mathcal{M}_c^1(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(g_t \Lambda_x)\, dP(v)(x) = \int_{\mathbb{R}^d} P(\varphi)(g_t \Lambda_x)\, dv(x).$$

Assume $v$ absolutely continuous. Set $\psi_m(x) = \frac{dP^m(v)(x)}{dx}$. Then

$$\mu_t(\varphi) = \int_{\mathbb{R}^d} \underbrace{\mathcal{P}^n(\varphi)(g_t \Lambda_x)}_{\Theta(x)}\, dP^m(v)(x) + O_\varphi(r^{m+n} e^{(d+1)t/d})$$

$$= \underbrace{\int_{\mathbb{R}^d} \Theta(x) \psi_m(x)\, dx}_{\text{apply Cauchy-Schwarz, equidist of long horocycles, spectral gap}} + O_\varphi\left(r^{m+n} e^{\frac{(d+1)}{d} t}\right)$$

$$\therefore \int_{\operatorname{supp} \psi_m} \Theta(x)^2\, dx \overset{\text{eq}}{=} \|\mathcal{P}^n(\varphi)\|^2_{L^2(X_{d+1})} + O\left(e^{-\kappa \frac{d+1}{d} t} S((\mathcal{P}^n(\varphi))^2)\right)$$

$$\ll O_\varphi\left(r^{2nb} + r^{\kappa\tau - 2nc}\right) \quad \left(r^{-\tau} = e^{\frac{d+1}{d} t}\right)$$

Open set condition

$$\Rightarrow \|\psi_m\|_{L^2(\mathbb{R}^d)} \ll r^{-am}.$$

$$\therefore |\mu_t(\varphi)| \ll r^{m+n-\tau} + r^{-am} \sqrt{r^{2nb} + r^{\kappa\tau - 2nc}} \ll e^{-\delta \frac{d+1}{d} t}$$

with $\delta > 0$ if $2a(b+c) < \kappa(b+a)$.

Here $a, b, c$ depend on $(\mathcal{F}, \lambda)$, $d$, bounds on mat. coeff's, and degree of Sobolev norms.

<u>Next time:</u> Define $\mathcal{P}$ and prove spectral gap.