<ins>Recall:</ins> Given $\mathcal{F}$ a (contractive) IFS by similitudes and a probability vector $\lambda$ on $\mathcal{F}$, we obtained a uniquely defined pair $\mathcal{C} \in \mathbb{R}^d$, $\mu \in \mathcal{M}_c^1(\mathbb{R}^d)$ characterized by

- $\mathcal{C} = \bigcup \{ f(\mathcal{C}) : f \in \mathcal{F} \}$
- $\mu$ is the unique fixed point of $P_{(\mathcal{F},\lambda)} : \mathcal{M}_c^1(\mathbb{R}^d) \to \mathcal{M}_c^1(\mathbb{R}^d)$ given by

$$P(\nu) = P_{(\mathcal{F},\lambda)}(\nu) = \sum_{f \in \mathcal{F}} \lambda_f f_* \nu .$$

Moreover, we have $\operatorname{supp} \mu = \mathcal{C}$.

Last time we showed that for $\mathcal{F}$ a $\mathbb{Q}$-IFS such that $\mathcal{C}$ has sufficiently large Hausdorff codimension, the measure $\mu$ satisfies a full Khintchine theorem, assuming that $\exists \varkappa > 0\ \forall t \in \mathbb{R}\ \forall \varphi \in C_c^\infty(X_{d+1})$

$$\underbrace{\int_{\mathbb{R}^d} \varphi(g_t \Lambda_x)\, d\mu(x)}_{=: \mu_t(\varphi)} = \int_{X_{d+1}} \varphi(\Lambda)\, d\Lambda + O_\varphi(e^{-\varkappa t})$$

We start with a very rough sketch of the proof, which motivates us to introduce a homogeneous space $G/\Gamma$ $(\neq X_{d+1})$ and an operator

$$\mathcal{P} : C_c(G/\Gamma) \to C_c(G/\Gamma),$$

whose properties will be examined throughout the rest of this talk.

# Proposition

Let $\mathcal{F}$ a (rational) IFS and let $\lambda$ be a probability vector on $\mathcal{F}$. Let $r = \sum_{f \in \mathcal{F}} \lambda_f \rho_f$. Then there is $C \geq 0$ s.t. $\forall \varphi \in \mathrm{Lip}_b(\mathbb{R}^d), \forall \nu \in \mathcal{M}_c^1(\mathbb{R}^d), \forall n \in \mathbb{N}$

$$|\mu(\varphi) - P^n(\nu)(\varphi)| \leq C r^n \|\varphi\|_{\mathrm{Lip}}.$$

Lipschitz-constant of $\varphi$

Now assume that we have an operator $\mathcal{P}: C_c^\infty(X_{d+1}) \longrightarrow C_c^\infty(X_{d+1})$ with the property that

$$\int_{\mathbb{R}^d} \varphi(g_t \Lambda_x) dP(\nu)(x) = \int_{\mathbb{R}^d} \mathcal{P}\varphi(g_t \Lambda_x) d\nu(x) \qquad \text{(FALSE ASSUMPTION)}$$

and assume that $\nu$ is absolutely continuous. Let $m, n \in \mathbb{N}$. Then for $\varphi \in C_c^\infty(X_{d+1}) \cap L_0^2(G/\Gamma)$

$$\int_{\mathbb{R}^d} \varphi(g_t \Lambda_x) \mu(x) = \int_{\mathbb{R}^d} \mathcal{P}^n \varphi(g_t \Lambda_x) \underbrace{\frac{dP^m(\nu)}{dx}(x)}_{=:\psi_m(x)} dx + O_\varphi\left(r^{m+n} e^{\frac{d+1}{d}t}\right)$$

in $G(\mathbb{R})$: $a_t = \begin{pmatrix} e^{\frac{d+1}{d}t} \mathbb{1}_d & 0 \\ 0 & 1 \end{pmatrix}$

$$\ll \left( \int_{\operatorname{supp} \psi_m} \mathcal{P}^n \varphi(g_t \Lambda_x)^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^d} \psi_m(x)^2 dx \right)^{1/2} + O_\varphi\left(r^{m+n} e^{\frac{d+1}{d}t}\right)$$

Now $\operatorname{supp} \psi_m \subseteq M \Subset \mathbb{R}^d$ with $M$ independent of $m$ (note: $\|b_{f_m \circ \cdots f_1}\| = \|f_m \circ \cdots \circ f_1(0)\| \leq \max_{f \in \mathcal{F}} \|b_f\| \frac{1}{1 - \max_{f \in \mathcal{F}} \rho_f}$.

Hence $\exists D \geq 1$ s.t. $\mathbb{1}_{\operatorname{supp} \psi_k}(x) \leq D \psi_0\left(\frac{x}{D}\right)$ $\forall k \in \mathbb{N}$.

$$\Rightarrow \int_{\operatorname{supp} \psi_m} \mathcal{P}^n \varphi(g_t \Lambda_x)^2 dx \leq D \int_{\mathbb{R}^d} \mathcal{P}^n \varphi(g_t \Lambda_x)^2 \psi_0\left(\frac{x}{D}\right) dx$$

$$\ll_{\psi_0} \int_{X_{d+1}} \mathcal{P}^n \varphi(\Lambda)^2 d\Lambda + O_{\mathcal{P}^n_\varphi}\left(e^{-\kappa t}\right)$$

$\therefore$ We want to prove a spectral gap for $\mathcal{P}^n$ acting on $L^2(X_{d+1})$.

Defining $\mathcal{P}$: We only consider the case base-$p$ missing digit set.

In what follows, $\mathbb{G} := PGL_2/\mathbb{Q}$, i.e.,

$$\mathbb{G} = \left\{ T \in GL(\mathrm{Mat}_2) : T(vw) = T(v)T(w) \right\}$$

By the Skolem-Noether theorem, we know that for every $k|\mathbb{Q}$

$$\begin{aligned} \Phi : GL_2(k) &\longrightarrow \mathbb{G}(k) \\ g &\longmapsto (v \mapsto gvg^{-1}) \end{aligned}$$

is onto and $\mathbb{G}(k) \cong GL_2(k)/k^\times$.

We fix the standard integral structure on $\mathrm{Mat}_2$, i.e.,

$$\mathrm{Mat}_2(\mathbb{Z}) = \mathrm{span}_{\mathbb{Z}} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and define

$$\mathbb{G}(\mathbb{Z}) = \mathbb{G}(\mathbb{Q}) \cap GL(\mathrm{Mat}_2(\mathbb{Z})).$$

Similarly for $\mathbb{Z}_p$, where $p$ is a rational prime.

One can show that $\mathbb{G}(\mathbb{Z}_p) = \Phi(GL_2(\mathbb{Z}_p))$ for all $p$

and hence $\mathbb{G}(\mathbb{Z}) = \Phi(GL_2(\mathbb{Z}))$.

For later use, given $k \in \mathbb{N}$, let

$$\Gamma(p^k) := \begin{cases} \mathbb{G}(\mathbb{Z}) & \text{if } k = 0, \\ \ker\left(T \in GL(\mathrm{Mat}_2(\mathbb{Z})) \mapsto T \bmod N\right) \cap \mathbb{G}(\mathbb{Z}) & \text{else.} \end{cases}$$

"principal congruence subgroup of level $p^k$"

**Lemma:** $\mathbb{G}(\mathbb{R})/\mathbb{G}(\mathbb{Z}) \cong X_2$

(exercise)

Let $G_\infty = \mathbb{G}(\mathbb{R})$, $G_p = \mathbb{G}(\mathbb{Q}_p)$, $K_p = \mathbb{G}(\mathbb{Z}_p)$, $G = G_\infty \times G_p$, $K = \{e_\infty\} \times K_p$,
$\Gamma = \mathbb{G}(\mathbb{Z}[\frac{1}{p}]) \hookrightarrow G$.

**Proposition:** $\Gamma \leq G$ is a lattice. Moreover $G_\infty \curvearrowright K \backslash G / \Gamma$ is transitive and
$\mathrm{Stab}_{G_\infty}(K\Gamma) = \mathbb{G}(\mathbb{Z}) =: \Gamma(1)$.
In particular, $G/\Gamma \longrightarrow G_\infty/\Gamma(1) \cong X_2$ is a covering with fibres $\cong K$.

**Some notation:** Let $\Lambda \subseteq \{0, \dots, p-1\}$ be the set of admissible digits.

We let $\Lambda^* := \bigcup_{k \geq 0} \Lambda^k$. Given $\omega = (\omega_1, \dots, \omega_n) \in \Lambda^*$, we let

$$f_\omega := f_{\omega_1} \circ \cdots \circ f_{\omega_n},$$

where, for $j \in \Lambda$, $f_j(x) = \frac{1}{p}x + \frac{j}{p}$. Given $\omega \in \Lambda^*$, let

$$\gamma_\omega = \left( \begin{pmatrix} p^{-|\omega|} & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p^{-|\omega|} & -b_\omega \\ 0 & 1 \end{pmatrix} \right) \in G. \qquad (*)$$

and note

$$\gamma_\omega (u(x), e_p)\Gamma = (\gamma_{\omega,\infty} u(x), \gamma_{\omega,p})\Gamma = (\gamma_{\omega,\infty} u(x) \gamma_{\omega,p}^{-1}, e_p)\Gamma.$$

One calculates $\gamma_{\omega,p}^{-1} = \begin{pmatrix} p^{|\omega|} & p^{|\omega|} b_\omega \\ 0 & 1 \end{pmatrix}$ and $\gamma_{\omega,\infty} u(x) \gamma_{\omega,p}^{-1} = \begin{pmatrix} 1 & f_\omega(x) \\ 0 & 1 \end{pmatrix}$.

$$\therefore \boxed{\gamma_\omega (u_x, e_{fin})\Gamma = (u_{f_\omega(x)}, e_{fin})\Gamma.}$$

In view of the sketch and the fixed point characterization of $\mu$, we are naturally led to the following definition.

## Definition:

Define $\mathcal{P}_\lambda : C(G/\Gamma) \longrightarrow C(G/\Gamma)$ by (given $\alpha \in \Lambda^*$)

$$(\alpha\cdot\mathcal{P}_\lambda)(\varphi)(x) = \frac{1}{|\Lambda|}\sum_{\omega\in\Lambda}\varphi(\gamma_\omega^\alpha\cdot x) \qquad (x\in G/\Gamma),$$

where $\gamma_\omega^\alpha = \gamma_\alpha\gamma_\omega\gamma_\alpha^{-1}$.

Note that

$$(\alpha\cdot\mathcal{P}_\lambda)^n(\varphi)(x) = \frac{1}{|\Lambda|^n}\sum_{\omega\in\Lambda^n}\varphi(\gamma_\omega^\alpha\cdot x)$$

and

$$\alpha=\emptyset \implies \alpha\cdot\mathcal{P}_\lambda = \mathcal{P}_\lambda .$$

**Proposition (Spectral gap – Khalil-L. ('21))**

Let $W_p \leq G_p$ a compact-open subgroup and let $\varphi \in L^2_{00}(G/\Gamma)\cap C^\infty(G/\Gamma)$ be $W_p$-invariant. Let $r>4$, then

$$\|(\alpha\cdot\mathcal{P}_\lambda)^n(\varphi)\|^2_{L^2(G/\Gamma)} \ll_{r,W_p,p} S_{2,1}(\varphi)^2|\Lambda|^{-\frac{1}{r}n}.$$

This needs a bit of preparation. In what follows, we fix norms $|||\cdot|||_\sigma$ on $\mathrm{Mat}_2(\mathbb{Q}_\sigma)$, namely

$$|||v|||_\infty := \Big(\sum_{1\le i,j\le 2} v_{ij}^2\Big)^{1/2} \qquad (v\in\mathrm{Mat}_2(\mathbb{R}))$$

$$|||v|||_p := \max\{|v_{ij}|_p : 1\le i,j\le 2\} \qquad (v\in\mathrm{Mat}_2(\mathbb{Q}_p))$$

In what follows, given $g\in G_\sigma$, we let $\|g\|_\sigma$ denote the operator norm of $\Phi(g)$ induced by $|||\cdot|||_\sigma$, i.e.,

$$\|g\|_\sigma = \sup\{|||gvg^{-1}|||_\sigma : v\in\mathrm{Mat}_2(\mathbb{Q}_\sigma),\ |||v|||_\sigma=1\}$$

Given $(g_\infty,g_p)\in G$, denote $\|(g_\infty,g_p)\| = \|g_\infty\|_\infty\|g_p\|_p$

**Lemma** (KAK-decomposition for $G(\mathbb{Q}_p)$)

Let $g \in G_p$. Then there exists a unique $n \in \mathbb{N}_0$ and there exist $m_1, m_2 \in K_p$ such that

$$g = m_1 \begin{pmatrix} p^{-n} & 0 \\ 0 & 1 \end{pmatrix} m_2 .$$

Moreover, $\|g\|_p = p^n$.

**Corollary:**

$$G_p = \bigsqcup_{n \in \mathbb{N}_0} K_p \begin{pmatrix} p^{-n} & 0 \\ 0 & 1 \end{pmatrix} K_p$$

One can show that $\mathrm{vol}\left(K_p \begin{pmatrix} p^{-n} & 0 \\ 0 & 1 \end{pmatrix} K_p\right) \asymp p^n$.

**Corollary**

Let $\eta : G \longrightarrow (0,1]$, $\eta(g) = \|g\|^{-1}$. Then $\eta \in L^{1+\varepsilon}(G)$.

Pf: It suffices to show that $g \mapsto \|g\|_\sigma^{-1}$ is $L^{1+\varepsilon}(G_\sigma)$. We first check the case $\sigma = p$. Then

$$\int_{G_p} \|g\|_p^{-1-\varepsilon} dg = \sum_{n \in \mathbb{N}_0} \int_{K_p \begin{pmatrix} p^{-n} & 0 \\ 0 & 1 \end{pmatrix} K_p} \|g\|_p^{-1-\varepsilon} dg \asymp \sum_{n \in \mathbb{N}_0} p^{-n\varepsilon} = \frac{1}{1 - p^{-\varepsilon}} < \infty .$$

For $\sigma = \infty$, there is an analogue of the KAK-decomposition, i.e.,

$$g = m_1 \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} m_2 \qquad (m_1, m_2 \in SO_2(\mathbb{R}),\ t \in \mathbb{R}_{\geq 0})$$

and one has the following formula for the Haar measure

$$\int_{G_\infty} f(g) dg = \int_{SO_2(\mathbb{R})} \int_{SO_2(\mathbb{R})} \int_0^\infty f\left(m_1 \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} m_2\right) \sinh(t)\, dt\, dm_1\, dm_2 \quad (f \in C_c(G_\infty))$$

One similarly has $\|g\|_\infty = e^t$, hence the result. $\square$

## Corollary

$$\boxed{\eta|_\Gamma \in \ell^{1+\varepsilon}(\Gamma)}$$

Proof: Note that $\|g\| = \|g^{-1}\|$. Hence submultiplicativity of the operator norm gives

$$\|g_1 g_2\| \geq \|g_1\|^{-1}\|g_2\| \qquad (g, \gamma \in G). \qquad (*)$$

Therefore

$$\infty > \int_G \eta(g)^{1+\varepsilon} = \int_{G/\Gamma} \sum_{\gamma \in \Gamma} \eta(g\gamma)^{1+\varepsilon}\, dg\Gamma$$

$$\Rightarrow \sum_{\gamma\in\Gamma} \eta(g\gamma)^{1+\varepsilon} < \infty \quad \forall g \in G$$

Now $(*)$ yields

$$\sum_{\gamma\in\Gamma} \eta(\gamma)^{1+\varepsilon} \leq \|g^{-1}\| \sum_{\gamma\in\Gamma} \eta(g\gamma)^{1+\varepsilon} < \infty$$

for typical $g$, hence the claim. $\blacksquare$

We let

$$Z_r(\Gamma) := \left( \sum_{\gamma\in\Gamma} \|\gamma\|^{-1-\frac{r-4}{4r+4}} \right)^{\frac{1}{r}}.$$

Selberg, Gelbart-Jacquet, CHH, Clozel-Ullmo

Theorem (Decay of matrix coefficients - Clozel-Oh-Ullmo, Gorodnik-Maucourant-Oh, EMV)

Let $\Omega_p \leq G_p$ compact-open. For all $g \in G$ and for all $\Omega_p$-invariant $\varphi_1, \varphi_2 \in L^2_{00}(G/\Gamma) \cap C^\infty(G/\Gamma)$

$$|\langle g.\varphi_1, \varphi_2\rangle_{L^2(G/\Gamma)}| \ll_{\Omega_p, \varepsilon} S_{2,1}(\varphi_1) S_{2,1}(\varphi_2) \|g\|^{-\frac{1}{4}+\varepsilon}. \qquad ((g.\varphi_1)(x) = \varphi_1(gx))$$

Proof of proposition "Spectral gap": We assume w.l.o.g. that $\varphi$ is $\mathbb{R}$-valued.

For the sake of exposition, we assume $\alpha = \phi$, i.e., $(\alpha \cdot \mathbb{P}_\lambda)^n = \mathbb{P}_\lambda^n$. Let $w \in \Lambda^n$ and set

$$\tilde{\gamma}_w = \underbrace{\left( \begin{pmatrix} 1 & -b_w \\ 0 & 1 \end{pmatrix}, e_p \right)}_{\tau_w^{-1}} \gamma_w.$$

Note (i) $\tilde{\gamma}_w \in \Gamma$

(ii) $\tilde{\gamma}_w \in \mathcal{O}\gamma_w$, where $\mathcal{O} \subseteq G$ is a cpt., symmetric neighbourhood of $e \in G$.

It follows that

$$\int_{G/\Gamma}\Big(\frac{1}{|\Lambda|^n}\sum_{\omega\in\Lambda^n}\varphi(\gamma_\omega\cdot x)\Big)^2dx=\frac{1}{|\Lambda|^{2n}}\sum_{\omega,\beta\in\Lambda^n}\underbrace{\langle\gamma_\omega\cdot\varphi,\gamma_\beta\cdot\varphi\rangle_{L^2(G/\Gamma)}}_{=\langle\tau_\omega\tilde{\gamma}_\omega\cdot\varphi,\tau_\beta\tilde{\gamma}_\beta\cdot\varphi\rangle_{L^2(G/\Gamma)}}$$

$(-\frac{1}{4}+\varepsilon)r=-1-\varepsilon \Leftrightarrow (4\varepsilon-1)r=-4-4\varepsilon \Leftrightarrow r=\frac{4+4\varepsilon}{1-4\varepsilon}$

$\varepsilon(4+4r)=r-4$

$\varepsilon:=\frac{r-4}{4r+4}$

$$\ll_{r,W_p}\frac{S_{2,1}(\varphi)^2}{|\Lambda|^{2n}}\sum_{\omega,\beta\in\Lambda^n}\underbrace{\|\tau_\omega\tilde{\gamma}_\omega\tilde{\gamma}_\beta^{-1}\tau_\beta^{-1}\|^{-\frac{1}{4}+\varepsilon}}_{\ll_{J,\varepsilon}\|\tilde{\gamma}_\omega\tilde{\gamma}_\beta^{-1}\|^{-\frac{1}{4}+\varepsilon}}$$

Define a probability measure $\vartheta_n$ on $\Gamma$ by $\vartheta_n(\tilde{\gamma}_\omega)=\frac{1}{|\Lambda|^n}$ for $\omega\in\Lambda^n$, $0$ otherwise.
Let $\check{\vartheta}_n(\gamma)=\vartheta_n(\gamma^{-1})$. Let

$$(\vartheta_n*\check{\vartheta}_n)(\gamma):=\sum_{\substack{\gamma_1,\gamma_2\in\Gamma\\ \gamma_1\gamma_2=\gamma}}\vartheta_n(\gamma_1)\check{\vartheta}_n(\gamma_2).$$

Note: $(\vartheta_n*\check{\vartheta}_n)(\gamma)=\int_\Gamma\vartheta_n(\gamma_1)\check{\vartheta}_n(\gamma_1^{-1}\gamma)d\gamma_1$, i.e., it is indeed a convolution of compactly supported functions.

$$\Big[(\vartheta_n*\check{\vartheta}_n)(\gamma)=\sum_{\substack{\gamma_1,\gamma_2\in\Gamma\\ \gamma_1\gamma_2=\gamma}}\vartheta_n(\gamma_1)\check{\vartheta}_n(\gamma_2)=\sum_{\substack{\gamma_1,\gamma_2\in\Gamma\\ \gamma_2=\gamma_1^{-1}\gamma}}\vartheta_n(\gamma_1)\check{\vartheta}_n(\gamma_2)$$

$$=\sum_{\gamma_1\in\Gamma}\vartheta_n(\gamma_1)\check{\vartheta}_n(\gamma_1^{-1}\gamma)=\int_\Gamma\vartheta_n(\gamma_1)\check{\vartheta}_n(\gamma_1^{-1}\gamma)d\gamma_1\Big]$$

Therefore Young's convolution inequality implies that for any $1\le q,\theta\le\infty$ satisfying $\frac{1}{q}+\frac{1}{q}=\frac{2}{q}=\frac{1}{\theta}+1$ we have

$$\Big(\sum_{\gamma\in\Gamma}(\vartheta_n*\check{\vartheta}_n)(\gamma)^\theta\Big)^{\frac{1}{\theta}}\le\Big(\sum_{\gamma\in\Gamma}\vartheta_n(\gamma)^q\Big)^{\frac{2}{q}}=|\Lambda|^{2\frac{1-q}{q}n}.$$

Then $\frac{1}{|\Lambda|^{2n}} \sum_{\omega,\beta\in\Lambda^n} \|\tilde{\gamma}_\omega \tilde{\gamma}_\beta^{-1}\|^{-\frac{1}{4}+\varepsilon} = \sum_{\gamma\in\Gamma} \|\gamma\|^{-\frac{1}{4}+\varepsilon} (\vartheta_n * \check{\vartheta}_n)(\gamma)$

$(-\frac{1}{4}+\varepsilon)r = -1-\varepsilon$

$$\leq \left( \sum_{\gamma\in\Gamma} \|\gamma\|^{-1-\varepsilon} \right)^{\frac{1}{r}} \left( \sum_{\gamma\in\Gamma} (\vartheta_n * \check{\vartheta}_n)(\gamma)^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}}$$

$$\leq Z_r(\Gamma) |\Lambda|^{-\frac{1}{r}n} \ll_{r,p} |\Lambda|^{-\frac{1}{r}n}$$

$\frac{2}{q} = \frac{r-1}{r} + 1 = \frac{2r-1}{r}$

$\iff q = \frac{2r}{2r-1}$

$\Rightarrow \frac{1-q}{q} = \frac{1-\frac{2r}{2r-1}}{\frac{2r}{2r-1}} = -\frac{1}{2r}$

## Remarks:

- decay of matrix coefficients is
$$|\langle g\varphi_1, \varphi_2\rangle| \ll_{\omega_p} \xi_G(g)^{\epsilon(d)} S_{2,\ell}(\varphi_1) S_{2,\ell}(\varphi_2),$$
where $\ell = \dim(SO_{d+1})$, and
$$\xi_G(g) \in \ell^{2v(d)+\varepsilon}(\Gamma)$$
for $v(d) = \left(\lfloor \frac{d}{2} \rfloor + 1\right)\lceil \frac{d}{2} \rceil \approx d^2$. Our guess is $\xi_G \in \ell^{2d+1}(\Gamma)$; cf. Amitay.
maybe even better?
- Decay is given by
$$\left( \sum_{\omega\in\Lambda^n} \lambda_\omega^{\frac{2r}{2r-1}} \right)^{2\frac{2r-1}{2r}} = \left( \sum_{\omega\in\Lambda} \lambda_\omega^q \right)^{\frac{2}{q}n},$$
$q = \frac{2r}{2r-1}$
where $r = \frac{2v(d)+\varepsilon}{\epsilon(d)}$, $\epsilon(d) = \begin{cases} \frac{25}{32} & d=1, \\ 1 & \text{else.} \end{cases}$
- This argument does not rely on base $p$ being prime and generalizes to the most general cases we can handle. To exploit prime base, instead consider
$$\gamma_\omega := \left(e_\infty, \begin{pmatrix} p^{-|\omega|} & -b_\omega \\ 0 & 1 \end{pmatrix}\right)$$
and notice that $\gamma_\omega \gamma_\beta^{-1} = \left(e_\infty, \begin{pmatrix} p^{-|\omega|} & -b_\omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{|\beta|} & p^{|\beta|}b_\beta \\ 0 & 1 \end{pmatrix}\right) = \left(e_\infty, \begin{pmatrix} 1 & b_\beta - b_\omega \\ 0 & 1 \end{pmatrix}\right)$
and then explicitely bound
$$\sum_{\beta,\omega\in\Lambda^n} \|b_\beta - b_\omega\|_p^{-\frac{25}{64}+\varepsilon}.$$

What remains to be done? In the sketch we started with a measure on $X_2$, namely $\mu_t(\varphi) = \int_{X_2} \varphi(g_t \Lambda_x) d\mu(x)$ and then applied $\mathcal{P}_\lambda^n$ to $\varphi$. However, this is not well-defined. Instead, we should use

$$\tilde{\mu}_t(\tilde{\varphi}) = \int_{G/\Gamma} \tilde{\varphi}\Big(\underbrace{\Big(\begin{pmatrix} e^{\frac{d+1}{d}t}\mathbb{1}_d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbb{1}_d & x \\ 0 & 1 \end{pmatrix}.e_p\Big)\Gamma}_{=:[\Lambda_{x,t}]}\Big) d\mu(x) \quad (\tilde{\varphi} \in C_c(G/\Gamma))$$

Then

$$\begin{array}{ccccc} G/\Gamma & \longrightarrow & K\backslash G/\Gamma & \cong & X_2 \\ [\Lambda_{x,t}] & \longmapsto & K[\Lambda_{x,t}] & \cong & g_t\Lambda_x \end{array}$$

and hence for $\varphi \in C_c(X_2)$ let $L\varphi \in C_c(G/\Gamma)$ be given by $L\varphi(x) := \varphi(Kx)$, so that

$$\tilde{\mu}_t(L\varphi) = \mu_t(\varphi).$$

Next time, we show that under a hypothesis of the form $\frac{\log|\Lambda|}{\log p} > \frac{1}{1+\varepsilon_0}$ we get (more than)

$$\tilde{\mu}_t(L\varphi) = \int_{X_2} \varphi + O_\varphi(e^{-\kappa t}).$$

What is difficult here? For the sketch we have argued that

$$\int_{X_{d+1}} \varphi(g_t\Lambda_x) d\mu(x) \ll \int_{\mathbb{R}^d} \mathcal{P}_\lambda^n(L\varphi)([\Lambda_{x,t}])^2 \psi_0\Big(\frac{x}{D}\Big) dx$$

and then applied equidistribution of long horocycles in the space of lattices. However, $\mathcal{P}_\lambda^n(L\varphi)$ is <u>not</u> $K$-invariant but rather invariant under the group

$$W = \bigcap_{\omega \in \Lambda^n} (K \cap \gamma_\omega^{-1} K \gamma_\omega) \leq K.$$

This introduces problems (because $n$ varies <u>with</u> $t$).