Rudolph’s x2 x3 Theorem

Plan of the talk

- On the theorem

Plan of the talk

- On the theorem
- The invertible extension

Plan of the talk

- On the theorem
- The invertible extension
- Certain conditional measures as translates of a measure on a group.

Theorem

Let $\mathbb{T} \stackrel{\textit{def}}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \stackrel{\textit{def}}{=} 2x$ *and* $S_3(x) \stackrel{\textit{def}}{=} 3x$*.*

Theorem

Let $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \stackrel{\text{def}}{=} 2x$ *and* $S_3(x) \stackrel{\text{def}}{=} 3x$*.*

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,

Theorem

Let $\mathbb{T} \overset{def}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \overset{def}{=} 2x$ *and* $S_3(x) \overset{def}{=} 3x$.

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,
satisfies $S_2^{-1}A = S_3^{-1}A = A$ if and only if $\mu(A) \in \{0, 1\}$ (namely μ is ergodic for the joint action of S_2 and S_3)

Theorem

Let $\mathbb{T} \overset{def}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \overset{def}{=} 2x$ *and* $S_3(x) \overset{def}{=} 3x$*.*

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,
satisfies $S_2^{-1}A = S_3^{-1}A = A$ if and only if $\mu(A) \in \{0, 1\}$ (namely μ is ergodic for the joint action of S_2 and S_3)
and $h_\mu(S_2) > 0$.

Theorem

Let $\mathbb{T} \overset{def}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \overset{def}{=} 2x$ *and* $S_3(x) \overset{def}{=} 3x$*.*

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,
satisfies $S_2^{-1}A = S_3^{-1}A = A$ if and only if $\mu(A) \in \{0,1\}$ (namely μ is ergodic for the joint action of S_2 and S_3)
and $h_\mu(S_2) > 0$.
Then $\mu = m_\mathbb{T}$ where $m_\mathbb{T}$ is the Haar measure on $\mathbb{T}$.

Theorem

Let $\mathbb{T} \overset{def}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \overset{def}{=} 2x$ *and* $S_3(x) \overset{def}{=} 3x$*.*

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,
satisfies $S_2^{-1}A = S_3^{-1}A = A$ if and only if $\mu(A) \in \{0, 1\}$ (namely μ is ergodic for the joint action of S_2 and S_3)
and $h_\mu(S_2) > 0$.
Then $\mu = m_{\mathbb{T}}$ where $m_{\mathbb{T}}$ is the Haar measure on $\mathbb{T}$.

Theorem

Let $\mathbb{T} \stackrel{def}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \stackrel{def}{=} 2x$ *and* $S_3(x) \stackrel{def}{=} 3x$*.*

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,
satisfies $S_2^{-1}A = S_3^{-1}A = A$ if and only if $\mu(A) \in \{0, 1\}$ (namely μ is ergodic for the joint action of S_2 and S_3)
and $h_\mu(S_2) > 0$.
Then $\mu = m_\mathbb{T}$ where $m_\mathbb{T}$ is the Haar measure on $\mathbb{T}$.

Remark.

Theorem

Let $\mathbb{T} \stackrel{def}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \stackrel{def}{=} 2x$ *and* $S_3(x) \stackrel{def}{=} 3x$*.*

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,
satisfies $S_2^{-1}A = S_3^{-1}A = A$ if and only if $\mu(A) \in \{0, 1\}$ (namely μ is ergodic for the joint action of S_2 and S_3)
and $h_\mu(S_2) > 0$.
Then $\mu = m_{\mathbb{T}}$ where $m_{\mathbb{T}}$ is the Haar measure on $\mathbb{T}$.

Remark.

1. It actually holds that
$h_\mu(S_2) > 0 \iff h_\mu(S_3) > 0 \iff h_\mu(S_2^m S_3^n) > 0$ for some $m, n \in \mathbb{N}$.
We will briefly explain how to show this later on

Theorem

Let $\mathbb{T} \overset{def}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \overset{def}{=} 2x$ *and* $S_3(x) \overset{def}{=} 3x$.

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,
satisfies $S_2^{-1}A = S_3^{-1}A = A$ if and only if $\mu(A) \in \{0, 1\}$ (namely μ is ergodic for the joint action of S_2 and S_3)
and $h_\mu(S_2) > 0$.
Then $\mu = m_\mathbb{T}$ where $m_\mathbb{T}$ is the Haar measure on $\mathbb{T}$.

Remark.

1. It actually holds that
$h_\mu(S_2) > 0 \iff h_\mu(S_3) > 0 \iff h_\mu(S_2^m S_3^n) > 0$ for some $m, n \in \mathbb{N}$.
We will briefly explain how to show this later on
2. The proof simplifies considerably if one assumes that μ is T_3 ergodic.

Theorem

Let $\mathbb{T} \stackrel{def}{=} \mathbb{R}/\mathbb{Z}$ *and consider the maps* $S_2(x) \stackrel{def}{=} 2x$ *and* $S_3(x) \stackrel{def}{=} 3x$.

Assume that $\mu \in \mathcal{P}(\mathbb{T})$ invariant under S_2 and S_3,
satisfies $S_2^{-1}A = S_3^{-1}A = A$ if and only if $\mu(A) \in \{0, 1\}$ (namely μ is ergodic for the joint action of S_2 and S_3)
and $h_\mu(S_2) > 0$.
Then $\mu = m_{\mathbb{T}}$ where $m_{\mathbb{T}}$ is the Haar measure on $\mathbb{T}$.

Remark.

1. It actually holds that
$h_\mu(S_2) > 0 \iff h_\mu(S_3) > 0 \iff h_\mu(S_2^m S_3^n) > 0$ for some $m, n \in \mathbb{N}$.
We will briefly explain how to show this later on
2. The proof simplifies considerably if one assumes that μ is T_3 ergodic.

Open question (Furstenberg). Is it true that the Haar measure of $\mathbb{T}$ is the unique non-atomic measure invariant under S_2 and S_3?

An implication of Rudolph's theorem

Furstenberg proved the following topological version of the mentioned open question.

An implication of Rudolph's theorem

Furstenberg proved the following topological version of the mentioned open question.

Theorem. Assume that $A \subseteq \mathbb{T}$ is a forward invariant under S_2 and S_3 (namely $\forall x \in A$, $S_i x \in A$, for $i \in \{2, 3\}$). Then either A is finite or A is dense.

An implication of Rudolph's theorem

Furstenberg proved the following topological version of the mentioned open question.

Theorem. Assume that $A \subseteq \mathbb{T}$ is a forward invariant under S_2 and S_3 (namely $\forall x \in A$, $S_i x \in A$, for $i \in \{2,3\}$). Then either A is finite or A is dense.

By Rudolph's theorem we obtain the following result which can give some insight (in some cases) that Furstenberg's result can't.

Corollary from Rudolph's theorem (Exercise 9.3.2. ELW book). Let μ be an S_3 invariant and ergodic probability measure with positive entropy. Then μ almost every $x \in \mathbb{R}/\mathbb{Z}$ has a dense orbit under S_2.

An implication of Rudolph's theorem

Furstenberg proved the following topological version of the mentioned open question.

Theorem. Assume that $A \subseteq \mathbb{T}$ is a forward invariant under S_2 and S_3 (namely $\forall x \in A$, $S_i x \in A$, for $i \in \{2,3\}$). Then either A is finite or A is dense.

By Rudolph's theorem we obtain the following result which can give some insight (in some cases) that Furstenberg's result can't.

Corollary from Rudolph's theorem (Exercise 9.3.2. ELW book). Let μ be an S_3 invariant and ergodic probability measure with positive entropy. Then μ almost every $x \in \mathbb{R}/\mathbb{Z}$ has a dense orbit under S_2.

Example. Consider the middle third cantor set

$$C = \left\{ \sum_{i=1}^{\infty} \frac{a_i}{3^i} \mid a_i \in \{0,2\} \right\},$$

which is clearly S_3 invariant. The bernouli shift on two symbols gives C an S_3 invariant ergodic meaure μ_C with positive entropy.

The invertible extension

We will change the setting to the space

$$X \stackrel{\text{def}}{=} \left\{ x \in \mathbb{T}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} = 2x_n,\ x_{\mathbf{n}+\mathbf{e}_2} = 3x_n,\ \forall \mathbf{n} \in \mathbb{Z}^2 \right\},$$

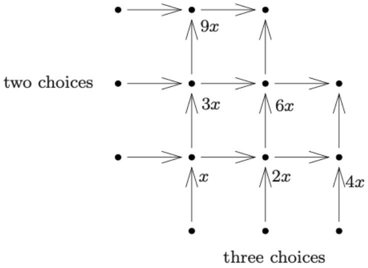
which will allow us to understand the dynamics more clearly.

The invertible extension

We will change the setting to the space

$$X \overset{\text{def}}{=} \left\{ x \in \mathbb{T}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} = 2x_n,\ x_{\mathbf{n}+\mathbf{e}_2} = 3x_n,\ \forall \mathbf{n} \in \mathbb{Z}^2 \right\},$$

which will allow us to understand the dynamics more clearly.



$$X \overset{\text{def}}{=} \left\{ x \in \mathbb{T}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} = 2x_n,\ x_{\mathbf{n}+\mathbf{e}_2} = 3x_n,\ \forall \mathbf{n} \in \mathbb{Z}^2 \right\}$$

X is a closed subgroup of the compact group $\mathbb{T}^{\mathbb{Z}^2}$ hence X is a compact abelian group with respect to the induced topology τ_X.

$$X \overset{\text{def}}{=} \left\{ x \in \mathbb{T}^{\mathbb{Z}^2} \mid x_{\mathbf{n}+\mathbf{e}_1} = 2x_n,\ x_{\mathbf{n}+\mathbf{e}_2} = 3x_n,\ \forall \mathbf{n} \in \mathbb{Z}^2 \right\}$$

X is a closed subgroup of the compact group $\mathbb{T}^{\mathbb{Z}^2}$ hence X is a compact abelian group with respect to the induced topology τ_X.
Let $I \subseteq \mathbb{Z}^2$ be a finite set and for each $\mathbf{n} \in I$ let $E_\mathbf{n} \subseteq \mathbb{T}$ be an open set, and define

$$[E_\mathbf{n}]_{\mathbf{n} \subseteq I} \overset{\text{def}}{=} \{x \in X \mid x_\mathbf{n} \in E_\mathbf{n},\ \mathbf{n} \in I\}.$$

Then the sets $[E_\mathbf{n}]_{\mathbf{n} \subseteq I}$ form a basis for τ_X.

Cylindrical sets in view of coordinate projections

Let

$$\pi_{m,n} : X \to \mathbb{T},$$

be the projection to the (m, n) coordinate, namely $\pi_{m,n}(x) = x_{m,n}$.

Cylindrical sets in view of coordinate projections

Let

$$\pi_{m,n} : X \to \mathbb{T},$$

be the projection to the (m,n) coordinate, namely $\pi_{m,n}(x) = x_{m,n}$. Observe that for $I \subseteq \mathbb{Z}^2$ finite, if $m_0 = \min_m\{(m,n) \in I\}$ and $n_0 = \min_m\{(m,n) \in I\}$, then

$$[E_{\mathbf{n}}]_{\mathbf{n}\subseteq I} = \left\{ x \in X \mid x_{m_0,n_0} \in \bigcap_{(m,n)\in I} S_2^{-(m-m_0)} S_3^{-(n-n_0)} E_{m,n} \right\} =$$

$$\pi_{m_0,n_0}^{-1}\left(\bigcap_{(m,n)\in I} S_2^{-(m-m_0)} S_3^{-(n-n_0)} E_{m,n} \right).$$

Cylindrical sets in view of coordinate projections

Let

$$\pi_{m,n} : X \to \mathbb{T},$$

be the projection to the (m,n) coordinate, namely $\pi_{m,n}(x) = x_{m,n}$. Observe that for $I \subseteq \mathbb{Z}^2$ finite, if $m_0 = \min_m\{(m,n) \in I\}$ and $n_0 = \min_m\{(m,n) \in I\}$, then

$$[E_{\mathbf{n}}]_{\mathbf{n} \subseteq I} = \left\{ x \in X \mid x_{m_0,n_0} \in \bigcap_{(m,n)\in I} S_2^{-(m-m_0)} S_3^{-(n-n_0)} E_{m,n} \right\} =$$

$$\pi^{-1}_{m_0,n_0}\left(\bigcap_{(m,n)\in I} S_2^{-(m-m_0)} S_3^{-(n-n_0)} E_{m,n} \right).$$

Hence $\tau_{m,n} \stackrel{\text{def}}{=} \pi^{-1}_{m,n}\tau_{\mathbb{T}}$ generate the topology, and moreover $\tau_{m-1,n} \supseteq \tau_{m,n}$, $\tau_{m,n-1} \supseteq \tau_{m,n}$. We conclude

$$\tau_{m,n} \nearrow \tau_X.$$

The borel σ-algebra

Definition. Let $\mathcal{B}_{\mathbb{T}}$ be the borel σ-algebra on $\mathbb{T}$.

The borel σ-algebra

Definition. Let $\mathcal{B}_{\mathbb{T}}$ be the borel σ-algebra on $\mathbb{T}$.
The σ-algebra generated by the $(m, n) \in \mathbb{Z}^2$ coordinate $\mathcal{B}_{m,n}$ is defined to be

$$\mathcal{B}_{m,n} \stackrel{\text{def}}{=} \pi_{m,n}^{-1}(\mathcal{B}_{\mathbb{T}})$$

The borel σ-algebra

Definition. Let $\mathcal{B}_{\mathbb{T}}$ be the borel σ-algebra on $\mathbb{T}$.
The σ-algebra generated by the $(m,n) \in \mathbb{Z}^2$ coordinate $\mathcal{B}_{m,n}$ is defined to be

$$\mathcal{B}_{m,n} \stackrel{\text{def}}{=} \pi_{m,n}^{-1}(\mathcal{B}_{\mathbb{T}})$$

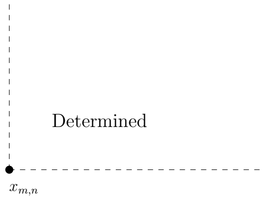
Then we conclude that $\mathcal{B}_{m-1,n} \supseteq \mathcal{B}_{m,n}$, $\mathcal{B}_{m,n-1} \supseteq \mathcal{B}_{m,n}$, and $\bigvee_{n=0}^{\infty} \bigvee_{m=0}^{\infty} \mathcal{B}_{-m,-n} = \mathcal{B}_X$, where $\mathcal{B}_X$ is the Borel σ-algebra on X.

The borel σ-algebra

Definition. Let $\mathcal{B}_{\mathbb{T}}$ be the borel σ-algebra on $\mathbb{T}$.
The σ-algebra generated by the $(m, n) \in \mathbb{Z}^2$ coordinate $\mathcal{B}_{m,n}$ is defined to be

$$\mathcal{B}_{m,n} \stackrel{\text{def}}{=} \pi_{m,n}^{-1}(\mathcal{B}_{\mathbb{T}})$$

Then we conclude that $\mathcal{B}_{m-1,n} \supseteq \mathcal{B}_{m,n}$, $\mathcal{B}_{m,n-1} \supseteq \mathcal{B}_{m,n}$, and $\bigvee_{n=0}^{\infty} \bigvee_{m=0}^{\infty} \mathcal{B}_{-m,-n} = \mathcal{B}_X$, where $\mathcal{B}_X$ is the Borel σ-algebra on X.



$$[x]_{\mathcal{B}_{m,n}} = \{y \in X \mid x_{a,b} = y_{a,b}, \ \forall a \geq m, \ b \geq n\}$$

.

Shift maps

We consider the left shift map $T_2(x)_{(m,n)} \overset{\text{def}}{=} x_{(m+1,n)}$ and the down shift map $T_3(x)_{(m,n)} \overset{\text{def}}{=} x_{(m,n+1)}$ which are invertible and keep X invariant.

$$\begin{CD} X @>{T_2^m T_3^n}>> X \\ @V{\pi_{m,n}}VV @VV{\pi_{m,n}}V \\ \mathbb{T} @>{S_2^m S_3^n}>> \mathbb{T} \end{CD}$$

Shift maps

We consider the left shift map $T_2(x)_{(m,n)} \stackrel{\text{def}}{=} x_{(m+1,n)}$ and the down shift map $T_3(x)_{(m,n)} \stackrel{\text{def}}{=} x_{(m,n+1)}$ which are invertible and keep X invariant.

$$\begin{CD} X @>{T_2^m T_3^n}>> X \\ @V{\pi_{m,n}}VV @VV{\pi_{m,n}}V \\ \mathbb{T} @>{S_2^m S_3^n}>> \mathbb{T} \end{CD}$$

Lemma (without proof). Assume that μ is S_2 and S_3 invariant borel probability measure on $\mathbb{T}$.

Shift maps

We consider the left shift map $T_2(x)_{(m,n)} \stackrel{\text{def}}{=} x_{(m+1,n)}$ and the down shift map $T_3(x)_{(m,n)} \stackrel{\text{def}}{=} x_{(m,n+1)}$ which are invertible and keep X invariant.

$$\begin{array}{ccc} X & \xrightarrow{T_2^m T_3^n} & X \\ {\scriptstyle \pi_{m,n}} \downarrow & & \downarrow {\scriptstyle \pi_{m,n}} \\ \mathbb{T} & \xrightarrow{S_2^m S_3^n} & \mathbb{T} \end{array}$$

Lemma (without proof). Assume that μ is S_2 and S_3 invariant borel probability measure on $\mathbb{T}$.
Then there exists a borel probability measure μ_X on X which is T_2,T_3 invariant and $\left(\pi_{m,n}\right)_* \mu_X = \mu$ for all $(m,n) \in \mathbb{Z}^2$.

Shift maps

We consider the left shift map $T_2(x)_{(m,n)} \overset{\text{def}}{=} x_{(m+1,n)}$ and the down shift map $T_3(x)_{(m,n)} \overset{\text{def}}{=} x_{(m,n+1)}$ which are invertible and keep X invariant.

$$\begin{array}{ccc} X & \xrightarrow{T_2^m T_3^n} & X \\ {\scriptstyle \pi_{m,n}}\downarrow & & \downarrow{\scriptstyle \pi_{m,n}} \\ \mathbb{T} & \xrightarrow{S_2^m S_3^n} & \mathbb{T} \end{array}$$

Lemma (without proof). Assume that μ is S_2 and S_3 invariant borel probability measure on $\mathbb{T}$.
Then there exists a borel probability measure μ_X on X which is T_2,T_3 invariant and $\left(\pi_{m,n}\right)_* \mu_X = \mu$ for all $(m,n) \in \mathbb{Z}^2$.
Moreover, if μ is ergodic for the joint S_2,S_3 action, then μ_X is ergodic for the joint T_2, T_3 action.

Consider the partition $\xi_{\mathbb{T}} \stackrel{\text{def}}{=} \{[0, \frac{1}{6}), [\frac{1}{6}, \frac{2}{6}), ..., [\frac{5}{6}, 1)\}$, and $\xi_X \stackrel{\text{def}}{=} \pi_{0,0}^{-1}(\xi_{\mathbb{T}})$.

Consider the partition $\xi_{\mathbb{T}} \stackrel{\text{def}}{=} \{[0, \frac{1}{6}), [\frac{1}{6}, \frac{2}{6}), ..., [\frac{5}{6}, 1)\}$, and $\xi_X \stackrel{\text{def}}{=} \pi_{0,0}^{-1}(\xi_{\mathbb{T}})$.
Then $h_{\mu_X}(T_2, \xi_X) = h_{\mu}(S_2, \xi_{\mathbb{T}})$.

Consider the partition $\xi_{\mathbb{T}} \stackrel{\text{def}}{=} \{[0, \frac{1}{6}), [\frac{1}{6}, \frac{2}{6}), ..., [\frac{5}{6}, 1)\}$, and $\xi_X \stackrel{\text{def}}{=} \pi_{0,0}^{-1}(\xi_{\mathbb{T}})$.
Then $h_{\mu_X}(T_2, \xi_X) = h_{\mu}(S_2, \xi_{\mathbb{T}})$.
In fact:

$$H_{\mu_X}(\bigvee_{i=0}^{n} T_2^{-i}(\pi_0^{-1}\xi_{\mathbb{T}})) \underbrace{=}_{T_2 \circ \pi_0 = \pi_0 \circ S_2} H_{\mu_X}(\bigvee_{i=0}^{n} \pi_0^{-1}(S_2^{-i}\xi_{\mathbb{T}}))$$

Consider the partition $\xi_{\mathbb{T}} \stackrel{\text{def}}{=} \{[0, \frac{1}{6}), [\frac{1}{6}, \frac{2}{6}), ..., [\frac{5}{6}, 1)\}$, and $\xi_X \stackrel{\text{def}}{=} \pi_{0,0}^{-1}(\xi_{\mathbb{T}})$.
Then $h_{\mu_X}(T_2, \xi_X) = h_\mu(S_2, \xi_{\mathbb{T}})$.
In fact:

$$H_{\mu_X}(\bigvee_{i=0}^{n} T_2^{-i}\left(\pi_0^{-1}\xi_{\mathbb{T}}\right)) \underbrace{=}_{T_2\circ\pi_0=\pi_0\circ S_2} H_{\mu_X}(\bigvee_{i=0}^{n} \pi_0^{-1}\left(S_2^{-i}\xi_{\mathbb{T}}\right))$$

$$\underbrace{=}_{(\pi_0)_*\mu_X=\mu} H_\mu(\bigvee_{i=0}^{n} S_2^{-i}\xi_{\mathbb{T}})$$

Consider the partition $\xi_{\mathbb{T}} \overset{\text{def}}{=} \{[0, \frac{1}{6}), [\frac{1}{6}, \frac{2}{6}), ..., [\frac{5}{6}, 1)\}$, and $\xi_X \overset{\text{def}}{=} \pi_{0,0}^{-1}(\xi_{\mathbb{T}})$.
Then $h_{\mu_X}(T_2, \xi_X) = h_\mu(S_2, \xi_{\mathbb{T}})$.
In fact:

$$H_{\mu_X}(\bigvee_{i=0}^{n} T_2^{-i}\,(\pi_0^{-1}\xi_{\mathbb{T}})) \underbrace{=}_{T_2\circ\pi_0=\pi_0\circ S_2} H_{\mu_X}(\bigvee_{i=0}^{n} \pi_0^{-1}\,(S_2^{-i}\xi_{\mathbb{T}}))$$

$$\underbrace{=}_{(\pi_0)_*\mu_X=\mu} H_\mu(\bigvee_{i=0}^{n} S_2^{-i}\xi_{\mathbb{T}})$$

By the same argument $h_{\mu_X}(T_3, \xi_X) = h_\mu(S_3, \xi_{\mathbb{T}})$.

Consider the partition $\xi_{\mathbb{T}} \overset{\text{def}}{=} \{[0, \frac{1}{6}), [\frac{1}{6}, \frac{2}{6}), ..., [\frac{5}{6}, 1)\}$, and $\xi_X \overset{\text{def}}{=} \pi_{0,0}^{-1}(\xi_{\mathbb{T}})$.
Then $h_{\mu_X}(T_2, \xi_X) = h_\mu(S_2, \xi_{\mathbb{T}})$.
In fact:

$$H_{\mu_X}(\bigvee_{i=0}^{n} T_2^{-i}(\pi_0^{-1}\xi_{\mathbb{T}})) \underbrace{=}_{T_2\circ\pi_0=\pi_0\circ S_2} H_{\mu_X}(\bigvee_{i=0}^{n} \pi_0^{-1}(S_2^{-i}\xi_{\mathbb{T}}))$$

$$\underbrace{=}_{(\pi_0)_*\mu_X=\mu} H_\mu(\bigvee_{i=0}^{n} S_2^{-i}\xi_{\mathbb{T}})$$

By the same argument $h_{\mu_X}(T_3, \xi_X) = h_\mu(S_3, \xi_{\mathbb{T}})$.
Now ξ_X and $\xi_{\mathbb{T}}$ are generators for both S_2 and S_3, thus we get
Corollary. $h_{\mu_X}(T_l, \xi_X) = h_\mu(S_l, \xi_{\mathbb{T}}) = h_\mu(S_l)$, for $l \in \{2, 3\}$.

The reduction of the problem

Assuming that μ_X is T_2,T_3 invariant and ergodic, such that $h_{\mu_X}(T_2, \xi_X) > 0$, our goal will be to show

$$h_{\mu_X}(T_2, \xi_X) = \log(2).$$

The reduction of the problem

Assuming that μ_X is T_2,T_3 invariant and ergodic, such that $h_{\mu_X}(T_2, \xi_X) > 0$, our goal will be to show

$$h_{\mu_X}(T_2, \xi_X) = \log(2).$$

This will finish our proof by the following assertion.

The reduction of the problem

Assuming that μ_X is T_2,T_3 invariant and ergodic, such that $h_{\mu_X}(T_2, \xi_X) > 0$, our goal will be to show

$$h_{\mu_X}(T_2, \xi_X) = \log(2).$$

This will finish our proof by the following assertion.
Lemma. $h_\mu(S_2) = \log(2) \iff \mu$ is the Haar measure.

The reduction of the problem

Assuming that μ_X is T_2,T_3 invariant and ergodic, such that $h_{\mu_X}(T_2, \xi_X) > 0$, our goal will be to show

$$h_{\mu_X}(T_2, \xi_X) = \log(2).$$

This will finish our proof by the following assertion.

Lemma. $h_\mu(S_2) = \log(2) \iff \mu$ is the Haar measure.

Proof. Consider the generator $\xi_0 \stackrel{\text{def}}{=} \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ for S_2.

The reduction of the problem

Assuming that μ_X is T_2,T_3 invariant and ergodic, such that $h_{\mu_X}(T_2, \xi_X) > 0$, our goal will be to show

$$h_{\mu_X}(T_2, \xi_X) = \log(2).$$

This will finish our proof by the following assertion.

Lemma. $h_\mu(S_2) = \log(2) \iff \mu$ is the Haar measure.

Proof. Consider the generator $\xi_0 \overset{\text{def}}{=} \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ for S_2. The partition $\bigvee_{i=0}^{N-1} S_2^{-i}\xi_{\mathbb{T}}$ consists of 2^N dyadic intervals $I_{j,N} \overset{\text{def}}{=} [\frac{j}{2^N}, \frac{j+1}{2^N})$ of length $\frac{1}{2^N}$. Once we will show that $\mu(I_{j,N}) = \frac{1}{2^N}$ for all $j \leq N$ and $N \in \mathbb{N}$, it will follow that $\mu = m_{\mathbb{T}}$.

Assume for contradiction that there exists $I_{j,N}$ such that $|I_{j,N}| \neq \frac{1}{2^N}$.

Assume for contradiction that there exists $I_{j,N}$ such that $|I_{j,N}| \neq \frac{1}{2^N}$. Now recall that in general, if ξ is a partition of N elements then $H_\nu(\xi) \leq \log N$ and

$$H_\nu(\xi) = \log N \iff \nu(P) = \frac{1}{N}, \ \forall P \in \xi.$$

Assume for contradiction that there exists $I_{j,N}$ such that $|I_{j,N}| \neq \frac{1}{2^N}$. Now recall that in general, if ξ is a partition of N elements then $H_\nu(\xi) \leq \log N$ and

$$H_\nu(\xi) = \log N \iff \nu(P) = \frac{1}{N}, \ \forall P \in \xi.$$

Hence

$$\frac{1}{N} H_\mu \left(\bigvee_{i=0}^{N-1} S_2^{-i} \xi_{\mathbb{T}} \right) < \frac{1}{N} \log(2^N) = \log(2).$$

Assume for contradiction that there exists $I_{j,N}$ such that $|I_{j,N}| \neq \frac{1}{2^N}$. Now recall that in general, if ξ is a partition of N elements then $H_\nu(\xi) \leq \log N$ and

$$H_\nu(\xi) = \log N \iff \nu(P) = \frac{1}{N}, \ \forall P \in \xi.$$

Hence

$$\frac{1}{N} H_\mu \left(\bigvee_{i=0}^{N-1} S_2^{-i} \xi_{\mathbb{T}} \right) < \frac{1}{N} \log(2^N) = \log(2).$$

and since $h_\mu(S_2) = h_\mu(S_2, \xi_0) = \inf_{n \geq 1} \frac{1}{n} H_\mu \left(\bigvee_{i=0}^{n-1} S_2^{-i} \xi_{\mathbb{T}} \right)$, we have a contradiction.

By the future formula for entropy $h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1)$, where

$$\mathcal{A}_1 \stackrel{\text{def}}{=} \bigvee_{i=1}^{\infty} T_2^{-i}\xi_X = T_2^{-1}\pi_0^{-1}\mathcal{B}_{\mathbb{T}}.$$

By the future formula for entropy $h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1)$, where

$$\mathcal{A}_1 \stackrel{\text{def}}{=} \bigvee_{i=1}^{\infty} T_2^{-i} \xi_X = T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}}.$$

Lemma. For each $n \in \mathbb{N}$ we have

$$(T_3^{-n} \xi_X) \vee \mathcal{A}_1 = \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}} = \xi_X \vee \mathcal{A}_1$$

By the future formula for entropy $h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1)$, where

$$\mathcal{A}_1 \overset{\text{def}}{=} \bigvee_{i=1}^{\infty} T_2^{-i} \xi_X = T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}}.$$

Lemma. For each $n \in \mathbb{N}$ we have

$$(T_3^{-n} \xi_X) \vee \mathcal{A}_1 = \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}} = \xi_X \vee \mathcal{A}_1$$

Proof. We are trying to show $T_3^{-n} \xi_X \vee T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}} = \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}}$.

By the future formula for entropy $h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1)$, where

$$\mathcal{A}_1 \stackrel{\text{def}}{=} \bigvee_{i=1}^{\infty} T_2^{-i}\xi_X = T_2^{-1}\pi_0^{-1}\mathcal{B}_{\mathbb{T}}.$$

Lemma. For each $n \in \mathbb{N}$ we have

$$(T_3^{-n}\xi_X) \vee \mathcal{A}_1 = \pi_{\mathbf{0}}^{-1}\mathcal{B}_{\mathbb{T}} = \xi_X \vee \mathcal{A}_1$$

Proof. We are trying to show $T_3^{-n}\xi_X \vee T_2^{-1}\pi_0^{-1}\mathcal{B}_{\mathbb{T}} = \pi_{\mathbf{0}}^{-1}\mathcal{B}_{\mathbb{T}}$.
Note that $T_3^{-n}\xi_X \vee T_2^{-1}\pi_0^{-1}\mathcal{B}_{\mathbb{T}} \subseteq \pi_{\mathbf{0}}^{-1}\mathcal{B}_{\mathbb{T}}$. So to prove equality it suffices to show that $\exists\epsilon > 0$ such that any interval of length smaller then ϵ is in $S_3^{-n}\xi_{\mathbb{T}} \vee S_2^{-1}\pi_0^{-1}\mathcal{B}_{\mathbb{T}}$, where $\xi_{\mathbb{T}} = \{[0, 1/6), .., [5/6, 1)\}$.

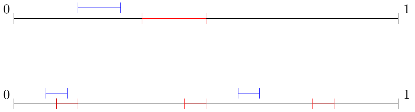
By the future formula for entropy $h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1)$, where

$$\mathcal{A}_1 \overset{\text{def}}{=} \bigvee_{i=1}^{\infty} T_2^{-i} \xi_X = T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}}.$$

Lemma. For each $n \in \mathbb{N}$ we have

$$(T_3^{-n} \xi_X) \vee \mathcal{A}_1 = \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}} = \xi_X \vee \mathcal{A}_1$$

Proof. We are trying to show $T_3^{-n} \xi_X \vee T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}} = \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}}$.
Note that $T_3^{-n} \xi_X \vee T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}} \subseteq \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}}$. So to prove equality it suffices to show that $\exists \epsilon > 0$ such that any interval of length smaller then ϵ is in $S_3^{-n} \xi_{\mathbb{T}} \vee S_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}}$, where $\xi_{\mathbb{T}} = \{[0, 1/6), .., [5/6, 1)\}$.
Proof by picture



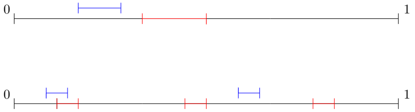
By the future formula for entropy $h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1)$, where

$$\mathcal{A}_1 \stackrel{\text{def}}{=} \bigvee_{i=1}^{\infty} T_2^{-i} \xi_X = T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}}.$$

Lemma. For each $n \in \mathbb{N}$ we have

$$(T_3^{-n} \xi_X) \vee \mathcal{A}_1 = \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}} = \xi_X \vee \mathcal{A}_1$$

Proof. We are trying to show $T_3^{-n} \xi_X \vee T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}} = \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}}$.
Note that $T_3^{-n} \xi_X \vee T_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}} \subseteq \pi_{\mathbf{0}}^{-1} \mathcal{B}_{\mathbb{T}}$. So to prove equality it suffices to show that $\exists \epsilon > 0$ such that any interval of length smaller then ϵ is in $S_3^{-n} \xi_{\mathbb{T}} \vee S_2^{-1} \pi_0^{-1} \mathcal{B}_{\mathbb{T}}$, where $\xi_{\mathbb{T}} = \{[0, 1/6), .., [5/6, 1)\}$.
Proof by picture



Proof of the picture: Note that if $|x - a| < \frac{1}{2 \cdot 3^{n+1}}$ and $x \in (a, b)$ such that $b - a = \frac{1}{2 \cdot 3^{n+1}}$ then its impossible that $x + \frac{1}{2} \in (a + \frac{j}{3^n}, b + \frac{j}{3^n})$. In fact, if we assume the contrary, then

$$\frac{1}{2 \cdot 3^n} - \frac{1}{2 \cdot 3^{n+1}} \leq \left| \frac{1}{2} - \frac{j}{3^n} \right| - |x - a| \leq \left| \left(x + \frac{1}{2} \right) - \left(a + \frac{j}{3^n} \right) \right| < b - a = \frac{1}{2 \cdot 3^{n+1}},$$

which is a contradiction.

$$(T_3^{-n}\xi_X) \vee \mathcal{A}_1 = \xi_X \vee \mathcal{A}_1 \implies \xi_X \vee T_3^n \mathcal{A}_1 = T_3^n(\xi_X \vee \mathcal{A}_1)$$

For $n \geq 0$ we get

$$h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1) =$$

$$(T_3^{-n}\xi_X) \vee \mathcal{A}_1 = \xi_X \vee \mathcal{A}_1 \implies \xi_X \vee T_3^n \mathcal{A}_1 = T_3^n (\xi_X \vee \mathcal{A}_1)$$

For $n \geq 0$ we get

$$h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \mid T_3^n \mathcal{A}_1) =$$

$(T_3^{-n}\xi_X) \vee \mathcal{A}_1 = \xi_X \vee \mathcal{A}_1 \implies \xi_X \vee T_3^n \mathcal{A}_1 = T_3^n (\xi_X \vee \mathcal{A}_1)$

For $n \geq 0$ we get

$$h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \vee T_3^n \mathcal{A}_1 \mid T_3^n \mathcal{A}_1) =$$

$$(T_3^{-n}\xi_X) \vee \mathcal{A}_1 = \xi_X \vee \mathcal{A}_1 \implies \xi_X \vee T_3^n \mathcal{A}_1 = T_3^n(\xi_X \vee \mathcal{A}_1)$$

For $n \geq 0$ we get

$$h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \vee T_3^n \mathcal{A}_1 \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(\xi_X \vee T_3^n \mathcal{A}_1 \mid T_3^n \mathcal{A}_1) =$$

$$(T_3^{-n}\xi_X) \vee \mathcal{A}_1 = \xi_X \vee \mathcal{A}_1 \implies \xi_X \vee T_3^n \mathcal{A}_1 = T_3^n(\xi_X \vee \mathcal{A}_1)$$

For $n \geq 0$ we get

$$h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \vee T_3^n \mathcal{A}_1 \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(\xi_X \vee T_3^n \mathcal{A}_1 \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(\xi_X \mid T_3^n \mathcal{A}_1).$$

Let $\mathcal{A} \stackrel{\text{def}}{=} \bigvee_{n=0}^{\infty} T_3^n \mathcal{A}_1$, which is the σ-algebra generated by the coordinates in the right-half plane $\{(m,n) \in \mathbb{Z}^2 \mid m > 0\}$.

$$(T_3^{-n}\xi_X) \vee \mathcal{A}_1 = \xi_X \vee \mathcal{A}_1 \implies \xi_X \vee T_3^n \mathcal{A}_1 = T_3^n (\xi_X \vee \mathcal{A}_1)$$

For $n \geq 0$ we get

$$h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(T_3^n \xi_X \vee T_3^n \mathcal{A}_1 \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(\xi_X \vee T_3^n \mathcal{A}_1 \mid T_3^n \mathcal{A}_1) =$$

$$H_{\mu_X}(\xi_X \mid T_3^n \mathcal{A}_1).$$

Let $\mathcal{A} \stackrel{\text{def}}{=} \bigvee_{n=0}^{\infty} T_3^n \mathcal{A}_1$, which is the σ-algebra generated by the coordinates in the right-half plane $\left\{(m,n) \in \mathbb{Z}^2 \mid m > 0\right\}$.
Then $T_3^n \mathcal{A}_1 \nearrow \mathcal{A}$ and

$$h_{\mu_X}(T_2, \xi_X) = H_{\mu_X}(\xi_X \mid \mathcal{A}_1) = \lim_{n\to\infty} H_{\mu_X}(\xi_X \mid T_3^n \mathcal{A}_1) = H_{\mu_X}(\xi_X \mid \mathcal{A}).$$