Quantitative Recurrence

Poincaré recurrence theorem is a fundamental result which states that any measure-preserving system exhibits a non-trivial recurrence to any measurable set with positive measure. In the case of metric spaces, the theorem can be stated as follows: Let (X, d) be a separable metric space, $T : X \to X$ a transformation and μ a T -invariant Borel measure, then for almost all x

$$\liminf_{n \geq 1} d\left(x, T^n\left(x\right)\right) = 0.$$

We will see that under some additional assumptions the rate of recurrence can be quantified and it depends on the Hausdorff dimension. We will also see some applications of the theorem.

Exercises

1. Prove the metric spaces case: Let (X, d) be a separable metric space, $T : X \to X$ a transformation and μ a T -invariant Borel measure. Then for almost all x
$$\liminf_{n \geq 1} d\left(x, T^n\left(x\right)\right) = 0.$$

2. Prove Kac Lemma: Let T be an ergodic measure-preserving transformation of the probability space $(X, \mathscr{B}, \mu)$. Let $E \in \mathscr{B}$ be such that $\mu\left(E\right) > 0$. Denote
$$n_1\left(x\right) = \inf\left\{n \geq 1 \ : \ T^n\left(x\right) \in E\right\}$$
Then
$$\int_E n_1\left(x\right) d\mu = 1.$$