Definition: The hyperplane absolute game on $\mathbb{R}^d$ is played by two players, say Alice and Bob, who take turns making their moves.

Bob starts by choosing a parameter $0 < \beta < \frac{1}{3}$, which is fixed throughout the game, and a ball $B_0 \subseteq \mathbb{R}^d$ of radius $r_0 > 0$.

For $n = 0, 1, 2, \ldots$, first, Alice chooses a neighborhood A_{n+1} of any hyperplane in $\mathbb{R}^d$ of radius εr_n for some $0 < \varepsilon \leq \beta$; and second, Bob chooses a Ball $B_{n+1} \subset B_n \setminus A_{n+1}$ of radius $r_{n+1} \geq \beta r_n$ where r_n is the radius of B_n.

$$B_0 \supset B_0 \setminus A_1 \supset B_1 \supset B_1 \setminus A_2 \supset B_2 \supset \cdots$$

$r_0 > 0$ $\quad$ ϵr_0, $0 < \epsilon \leq \beta$ $\quad$ $r_1 > \beta r_0$ $\quad$ ϵr_1 $\quad$ $r_2 \geq \beta r_1$

* A set $S \subseteq \mathbb{R}^d$ is called hyperplane absolute winning (HAW) if Alice has a strategy which ensures

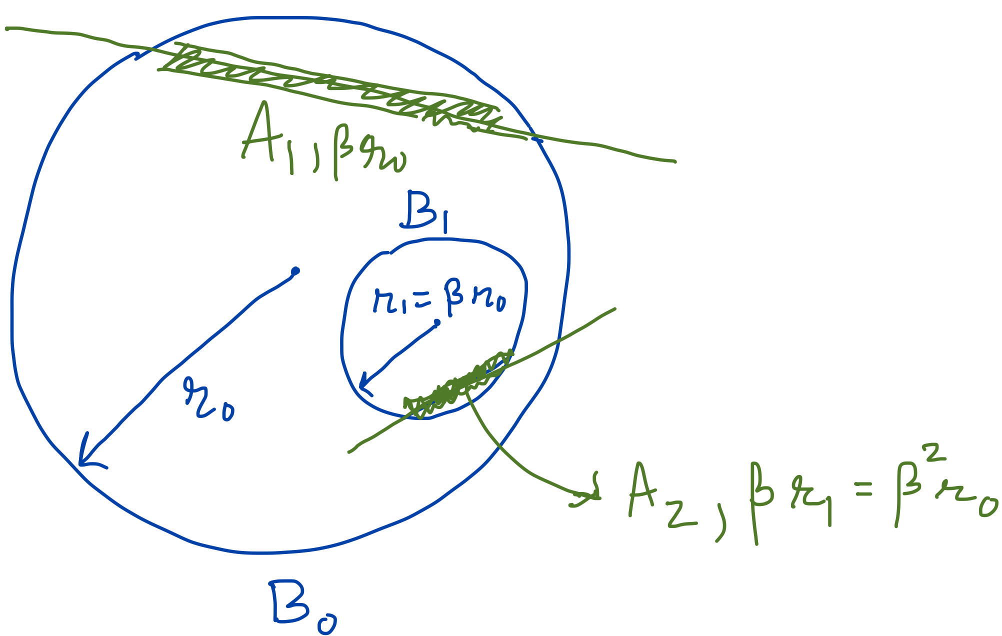
$$S \cap \left(\bigcap_{n \geq 0} B_n \right) \neq \phi .$$

Definition: The restricted hyperplane absolute game on $\mathbb{R}^d$ is played by two players, say Alice and Bob, who take turns making their moves.

Bob starts by choosing a parameter $0 < \beta < 1$, which is fixed throughout the game, and a ball $B_0 \subset \mathbb{R}^d$ of radius $r_0 > 0$.

For $n = 0, 1, 2, \ldots$, first, Alice chooses a neighborhood A_{n+1} of some hyperplane of $\mathbb{R}^d$ of radius $\beta r_n = \beta^{n+1} r_0$; and second, Bob chooses a ball $B_{n+1} \subseteq B_n \setminus A_{n+1}$ of radius $r_{n+1} = \beta r_n = \beta^{n+1} r_0$, where r_n is the radius of B_n. If there is no such ball the game stops and Alice wins by default. Otherwise, the outcome is the unique point $\bigcap_{n \geq 0} B_n$.

$$\underset{r_0}{B_0} \supseteq B_0 \setminus \underset{=\beta r_0}{A_1} \supset \underset{r_1 = \beta r_0}{B_1} \supset B_1 \setminus \underset{\beta r_1 = \beta^2 r_0}{A_2} \supset \overset{r_2 = \beta r_1 = \beta^2 r_0}{B_2} \supset$$



* A set $S \subseteq \mathbb{R}^d$ is called restricted hyperplane absolute winning if Alice has a strategy which ensures that either she wins by default or the outcome lies in S.

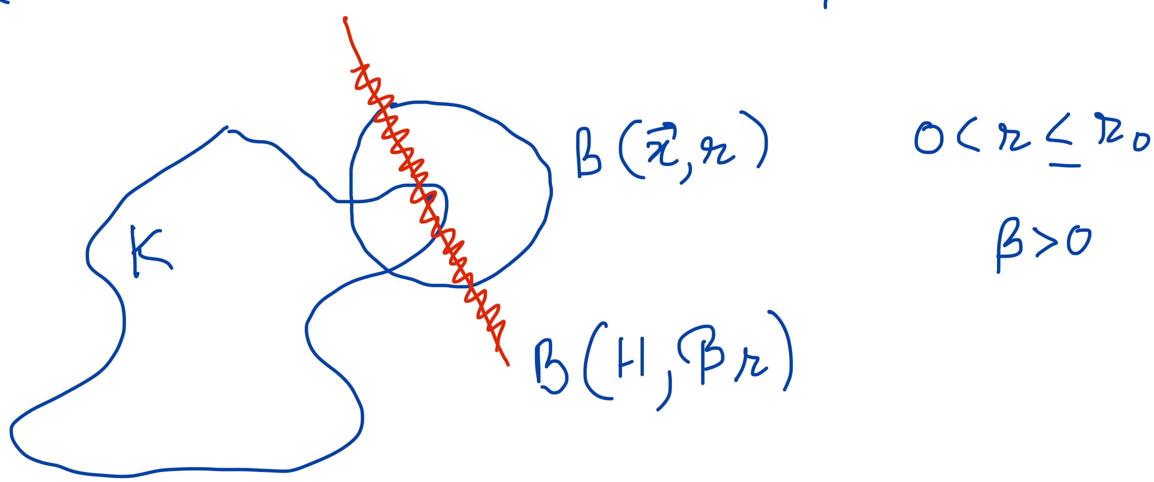
Proposition 8 [BNY] Let $S \subseteq \mathbb{R}^d$. Then S is restricted hyperplane abs. winning if and only if S is hyperplane abs. winning.

Proof: Later if time permits.

* Notation: For any $S \subset \mathbb{R}^d$, $r > 0$ denote $B(S, r) = \{x \in \mathbb{R}^d : d(x, S) \leq r\}$

Definition: A nonempty closed subset $K \subseteq \mathbb{R}^d$ is called hyperplane diffuse if there exists $\beta > 0$, $r_0 > 0$ s.t. for every $\vec{x} \in K$, $0 < r \leq r_0$ and every hyperplane $H \subseteq \mathbb{R}^d$ we have

$$K \cap \left(B(\vec{x}, r) \setminus B(H, \beta r)\right) \neq \phi .$$



Main Result

Proposition 13 [BNY]: If $S \subseteq \mathbb{R}^d$ is HAW then $S \cap K \neq \phi$ for any hyperplane diffuse set $K \subseteq \mathbb{R}^d$. Moreover, if S is Borel then converse is true.

Before we prove Prop 13 we need

following Lemmas

Lemma 15 [BNY] : For any $\beta > 0$ there exists $0 < \beta' < \beta$ and N such that for every ball $B = B(\vec{x}, \rho) \subseteq \mathbb{R}^d$, there is a collection of at most N hyperplanes $\mathcal{H}_B$ such that for any hyperplane H' there exists $H \in \mathcal{H}_B$ for which

$$B(\vec{x}, \rho) \cap B(H', \beta' \rho) \subseteq B(H, \beta \rho) \quad (1)$$

Proof: W.L.O.G we will prove (1) for $B(\vec{0}, 1)$ instead of $B(\vec{x}, \vec{\rho})$.

We will show that

$$B(\vec{0}, 1) \cap B(H', \beta') \subseteq B(H, \beta) \quad (2)$$

* Denote by $\mathcal{H}$ the set of all Hyperplanes in $\mathbb{R}^d$. Recall that every $\ell \in \mathcal{H}$ can be expressed as

$$v_1 \vec{x}_1 + v_2 \vec{x}_2 + \cdots + v_d \vec{x}_d = t$$

where at least one of v_i is non-zero and t is constant.

W.L.O.G by scaling 't' we can assume

$$(v_1, \ldots, v_d) \in S^{d-1} \quad \text{i.e } \sqrt{v_1^2 + \cdots + v_d^2} = 1$$

* Thus $\exists$ a surjective map

$$\pi : S^{d-1} \times \mathbb{R} \longrightarrow \mathcal{H} \quad \text{s.t}$$

$$\pi(\vec{v}, t) = \{ \vec{x} \in \mathbb{R}^d : \vec{v} \cdot \vec{x} = t \}$$

* Let F be a maximal β'-seperated subset of $S^{d-1} \times [-(1+\beta'), (1+\beta')]$

i.e. if $(\vec{v}_1, t_1)$; $(\vec{v}_2, t_2) \in F$ then

$||\vec{v}_1 - \vec{v}_2|| \geq \beta'$ and $|t_1 - t_2| \geq \beta'$

* Denote by $\mathcal{F} = \pi(F)$. We claim $\mathcal{F}$ satisfies (2) i.e $\mathcal{F} = \mathcal{H}_B$.

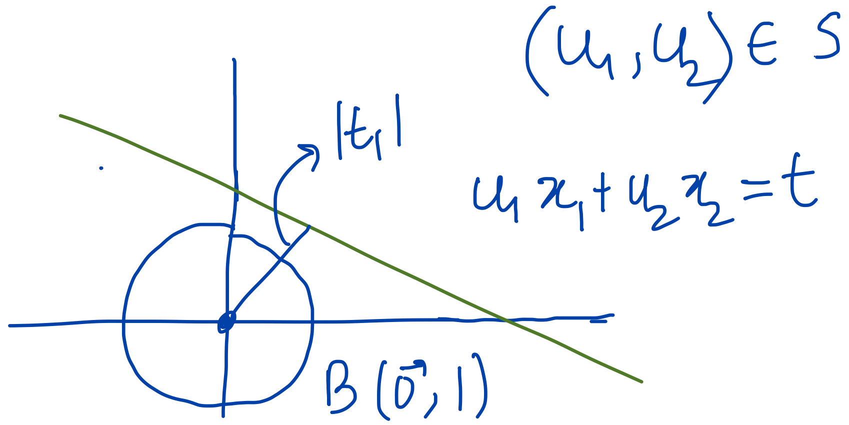
Since $|F| < \infty$ its image under π is also finite i.e $|\mathcal{F}| < \infty$.

Fix $H' = \pi(\vec{v}_1, t_1) \in \mathcal{H}$

If $|t_1| > 1 + \beta'$ then

$$B(\vec{0}, 1) \cap B(H', \beta') = \phi$$

To see this in $\mathbb{R}^2$



$$d(\vec{0}, H') = \frac{|u_1 0 + u_2 0 - t_1|}{\sqrt{u_1^2 + u_2^2}} = |t_1|$$

Hence (2) holds trivially.

* Now assume $|t_1| \leq 1 + \beta'$, we need to find $H \in \mathcal{F}$ s.t

$$B(\vec{0}, 1) \cap B(H', \beta') \subseteq B(H, \beta) \quad \text{— (3)}$$

Pick $H = \pi(\vec{v}_2, t_2) \in \mathcal{F}$ such that

$\|v_1 - v_2\| < \beta'$ and $|t_1 - t_2| < \beta'$

For $\vec{x} \in B(\vec{0}, 1) \cap B(H', \beta')$, we have

$$|\vec{v}_2 \cdot \vec{x} - t_2| \leq |(\vec{v}_2 - \vec{v}_1 + \vec{v}_1) \cdot \vec{x} - t_2|$$

$$= |(\vec{v}_2 - \vec{v}_1) \cdot \vec{x} + \vec{v}_1 \cdot \vec{x} - t_2 - t_1 + t_1|$$

$$\leq |(\vec{v}_2 - \vec{v}_1) \cdot \vec{x}| + |\vec{v}_1 \cdot \vec{x} - t_1| + |t_1 - t_2|$$

$$\leq \|(\vec{v}_2 - \vec{v}_1)\| \, \|\vec{x}\| + |\vec{v}_1 \cdot \vec{x} - t_1| + |t_1 - t_2|$$

$$< \underbrace{\beta' \cdot 1}_{\vec{x} \in B(0,1)} + \underbrace{\beta'}_{\text{since } \vec{x} \in B(H', \beta')} + \beta'$$

put $\beta' = \beta/3$

$= \beta$. Hence ③ holds $\boxed{///}$

<u>Lemma 16</u>: Let $K \subset \mathbb{R}^d$ be hyperplane diffuse. Then there exist $0 < \beta_0 < \frac{1}{3}$ and $r_0 > 0$ such that for any $0 < r \leq r_0$ any $\vec{x} \in K$ and any hyperplane H there exists $\vec{x}' \in K$ such that

$$B(\vec{x}', \beta_0 r) \subset B(\vec{x}, r) \setminus B(H, \beta_0 r)$$

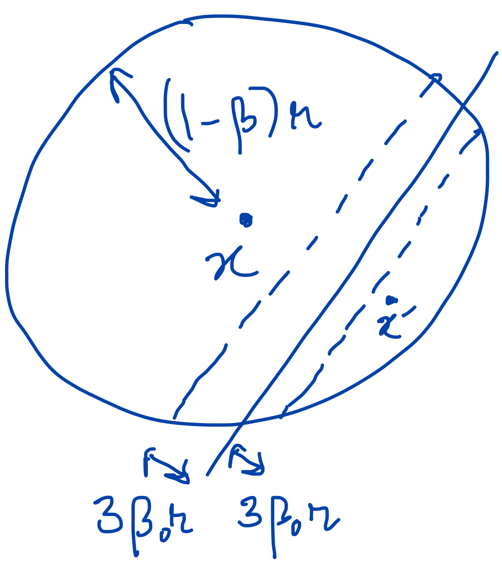
Proof: Since K is hyperplane diffuse, there exist $\vec{x}' \in K$ s.t

$$x' \in B(\vec{x}, (1-\beta) r) \setminus B(H, \beta(1-\beta) r)$$

(we assume w.l.o.g $0<\beta<1$, thus $(1-\beta) r \leq r_0$)

Put $\beta(1-\beta) = 3\beta_0$ then $\beta_0 = \frac{\beta}{3}(1-\beta) < \frac{\beta}{3}$

and $x' \in B(\vec{x}, (1-\beta) r) \setminus B(H, 3\beta_0 r)$



$$\Rightarrow B(\vec{x}', \beta_0 r) \subset B(\vec{x}, r) \setminus B(H, \beta_0 r)$$

$\square$

Now we prove the main result (Prop 13). Recall

<u>Proposition 13</u> [BNY]: If $S \subseteq \mathbb{R}^d$ is HAW then $S \cap K \neq \emptyset$ for any hyperplane diffuse set $K \subseteq \mathbb{R}^d$. Moreover, if S is Borel then converse is true.

<u>Proof</u> "$\Rightarrow$" Assume that S is HAW and K is hyperplane diffuse. Let β_0 and r_0 be as in Lemma 16 and suppose Alice and Bob play <u>restricted HAW</u>.

Suppose that on the first move Bob chooses $\beta = \beta_0$ and a Ball B_0 of radius r_0 centered in K. Note that this is a valid assumption since $\beta_0 \in (0,1)$, K is not empty, and B_0 can be arbitary.

Let $n \geq 0$ and suppose that $B_0, \ldots, B_n$

are the balls arising from the restricted hyperplane absolute game that Alice and Bob play with Alice using winning strategy.

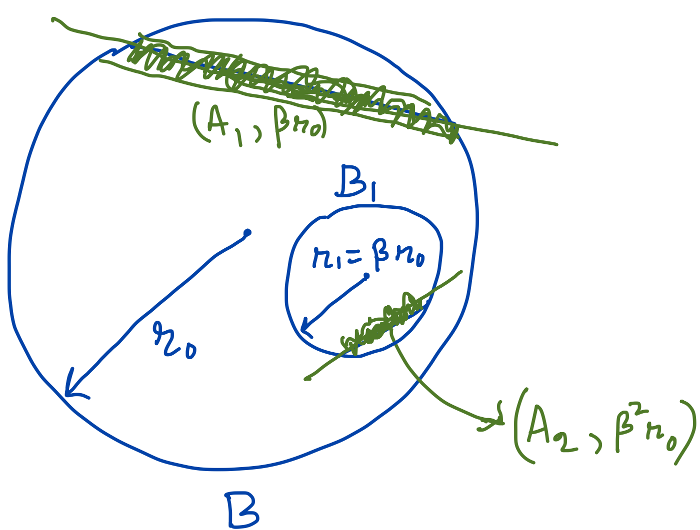
* We can assume that all B_i's have centers in K. To see this note that

for $n=0$, B_0 has center in K by choice.

* Let A_{n+1} be the nbhd of any hyperplane of radius $\beta^{n+1} r_0$ that Alice chose.

Recall

$$\underset{\downarrow \atop r_0}{B_0} \supseteq \underset{\downarrow \atop =\beta r_0}{B_0 \setminus A_1} \supset \underset{\downarrow \atop r_1 = \beta r_0}{B_1} \supset \underset{\downarrow \atop \beta r_1 = \beta^2 r_0}{B_1 \setminus A_2} \supset \underset{\downarrow \atop r_2 = \beta r_1 = \beta^2 r_0}{B_2} \supset$$



Then By Lemma 16, Bob can choose a ball B_{n+1} of radius $\beta^{n+1} r_0$ which is contained in $B_n \setminus A_{n+1}$ and centered at K.

* To see this, define B_{n+1} to be $B(x', \beta_0 r)$ from Lemma 16 with $B(x, r) = B_n$ and $B(H, \beta_0 r) = A_{n+1}$.

* Thus for every n Bob has a legal move, which means Alice cannot win by default, and the sequence $(B_n)_{n \geq 0}$ can be made infinite

* Also we have shown that Bob can force the centers of B_n to be in K. the unique point $\bigcap_n B_n$ lies in K.

* On the other hand Alice wins.

$\Rightarrow \bigcap_n B_n \in S \Rightarrow K \cap S \neq \phi$.

" $\Leftarrow$ " To prove the other direction we would need <u>Borel determinacy Theorem</u> from [FLC14].

<u>Theorem [FLC]</u>: Let (X, d) be a complete metric space. Fix $0 < \beta < 1/3$ and $S \subseteq X$. If S is Borel then (β, S)-absolute winning game are determined.

" $\Leftarrow$ " Assume S is Borel s.t $\forall$ hyperplane diffuse set $K \subseteq \mathbb{R}^d$, $S \cap K \neq \phi$.

Assume by contradiction S is <u>not HAW.</u>

Then by <u>Borel determinancy theorem</u> [FLC]

Bob has a winning strategy, which we fix for rest of the proof.

* let β and B_0 be first move by Bob.

Define $\mathcal{B}_0 = \{B_0\}$ and continue by induction to construct balls $\mathcal{B}_n$ as follows:

Given $\mathcal{B}_n$, $\forall B \in \mathcal{B}_n$, let $\mathcal{H}_B$ be the collection of hyperplanes arising from Lemma 15 [BNY]. Define $\mathcal{B}_{n+1}(B)$ to be collection of all of Bob's responses according to winning strategy while considering $\mathcal{H}_B$ as possible moves by Alice.

$$\mathcal{B}_0 = \{B_0\}$$

$$\mathcal{B}_1 = \{\text{Bob's response to } \mathcal{H}_{B_0}\}$$
$$= \{B_1', B_2', \ldots, -\}$$

$\mathcal{B}_2 = \{$ Bob's response to $\mathcal{H}_{B_1}, \mathcal{H}_{B_2} \ldots$ etc $\}$

$\vdots$

Define $K = \bigcap_{n \geq 0} \bigcup_{B \in \mathcal{B}_n} B$ — (4)

* If $x \in K$, then it is an outcome of some hyperplane absolute game played according to Bob's winning strategy.

$\Rightarrow x \notin S \Rightarrow K \cap S = \phi$. (contradiction)

To finish the proof we need to show that K defined above is Hyperplane diffuse (HD)

We will show K is HD for $\frac{\beta' \beta}{2}$ where β' is as in Lemma 15.

* Assume $x \in K$, $0 < r < r_0$, (where

r_0 is radius of B_0) and $H' \subseteq \mathbb{R}^d$ is a hyperplane.

Let n be unique positive integer s.t

$$2\beta^n r_0 \leq r < 2\beta^{n-1} r_0 \quad \text{—— (5)}$$

Such 'n' exists since $\beta \in (0,1)$.

* Since $x \in K$, by def of K (see (4))

$\Rightarrow \exists$ ball $\boxed{B = B(x_0, \beta^n r_0)} \in \mathcal{B}_n$ s.t $x \in B$.

(B is Bob's choice for nth step)

By (5) $\beta^n r_0 \leq \frac{r}{2} \Rightarrow B \subset B(x, r)$

* Apply Lemma 15 for x_0 & $\rho = \beta^n r_0$

i.e

$$B(x_0, \beta^n r_0) \cap B(H', \underbrace{\beta' \beta^n r_0}) \subset B(H, \beta^{n+1} r_0) \quad \text{—— (6)}$$

By 5 $r < 2\beta^{n-1} r_0$

$$\Rightarrow \frac{\beta\beta'}{2} r < \frac{\beta\beta'}{\cancel{2}} \cdot \cancel{2} \beta^{n-1} r_0 = \beta' \beta^n r_0$$

ie $\frac{\beta\beta' r}{2} < \underbrace{\beta'\beta^n r_0}$

By 6

$$\Rightarrow B(x_0, \beta^n r_0) \cap B\left(H', \frac{\beta'\beta' r}{2}\right) \subset B(H, \beta^{n+1} r_0) \quad (7)$$

* By definition of $\mathcal{B}_{n+1}(B) \to B \in \mathcal{B}_n$

$\exists$ a ball $B' \in \mathcal{B}_{n+1}(B)$ such that

$B' \cap \underbrace{B(H, \beta^{n+1} r_0)}_{\text{Alice's choice}} = \phi$.

By definition of K (see 4)

$$K \cap B' \neq \phi \Rightarrow$$

$$\phi \neq K \cap B' \subseteq K \cap B \subseteq K \cap B(x, r)$$

$\Rightarrow K \cap B(x, r) \neq \phi$ and since

$$K \cap \underbrace{B\left(H', \frac{\beta\beta'}{2}, r\right)}_{\text{deleted by Alice (see 7)}} = \phi$$

$$\Rightarrow K \cap \left(B(x, r) \setminus B\left(H', \frac{\beta\beta'}{2}, r\right) \right) \neq \phi$$

$\Rightarrow K$ is Hyperplane diffuse.

$\boxed{//}$