Definition: The hyperplane absolute game on $\mathbb{R}^d$ is played by two players, say Alice and Bob, who take turns making their moves. Bob starts by choosing a parameter $0 < \beta < \frac{1}{3}$, which is fixed throughout the game, and a ball $B_0 \subseteq \mathbb{R}^d$ of radius $r_0 > 0$.

For $n = 0, 1, 2, \ldots$, first, Alice chooses a neighborhood $A_{n+1}$ of any hyperplane in $\mathbb{R}^d$ of radius $\varepsilon r_n$ for some $0 < \varepsilon \leq \beta$; and second, Bob chooses a ball $B_{n+1} \subseteq B_n \setminus A_{n+1}$ of radius $r_{n+1} \geq \beta r_n$.

where $r_n$ is the radius of $B_n$.
A set \( S \subseteq \mathbb{R}^d \) is called hyperplane absolute winning (HAW) if Alice has a strategy which ensures
\[
S \cap \left( \bigcap_{n \geq 0} b_n \right) \neq \emptyset.
\]

Definition: The restricted hyperplane absolute game on \( \mathbb{R}^d \) is played by two players, say Alice and Bob, who take turns making their moves. Bob starts by choosing a parameter \( 0 < \beta < 1 \), which is fixed throughout the game, and a ball \( B_0 \subseteq \mathbb{R}^d \) of radius \( r_0 > 0 \).

For \( n = 0, 1, 2, \ldots \), first, Alice chooses
a neighborhood $A_{n+1}$ of some hyperplane of $\mathbb{R}^d$ of radius $r_{n+1} = \beta^{n+1} r_0$; and second, Bob chooses a ball $B_{n+1} \subseteq B_n \setminus A_{n+1}$ of radius $r_{n+1} = \beta r_n = \beta r_0$, where $r_n$ is the radius of $B_n$. If there is no such ball the game stops and Alice wins by default. Otherwise, the outcome is the unique point $\cap B_n$.

$$n \geq 0 \quad \Rightarrow \quad r_2 = \beta^2 r_1 = \beta^2 r_0$$

$$B_0 \supseteq B_0 \setminus A_1 \supseteq B_1 \setminus A_2 \supseteq B_2 \supseteq \ldots$$

$$r_0 = \beta r_0 \quad r_1 = \beta r_0 \quad \beta r_1 = \beta^2 r_0$$
A set $S \subseteq \mathbb{R}^d$ is called **restricted hyperplane absolute winning** if Alice has a strategy which ensures that either she wins by default or the outcome lies in $S$.

Proposition 8 [BNY] Let $S \subseteq \mathbb{R}^d$. Then $S$ is restricted hyperplane abs. winning if and only if $S$ is hyperplane abs. winning.

Proof: Later if time permits.

Notation: For any $S \subseteq \mathbb{R}^d$, $r > 0$ denote $B(S, r) = \{x \in \mathbb{R}^d : d(x, S) \leq r\}$

Definition: A nonempty closed subset $K \subseteq \mathbb{R}^d$ is called **hyperplane diffuse**
if there exists $\beta > 0$, $r_0 > 0$ s.t. for every $\overline{x} \in K$, $0 < r \leq r_0$ and every hyperplane $H \subseteq \mathbb{R}^d$ we have

$$K \cap (B(\overline{x}, r) \setminus B(H, \beta r)) \neq \emptyset.$$ 

**Main Result**

**Proposition 13 [BNY]:** If $S \subseteq \mathbb{R}^d$ is HAW then $S \cap K \neq \emptyset$ for any hyperplane diffuse set $K \subseteq \mathbb{R}^d$. Moreover, if $S$ is Borel then converse is true.

Before we prove Prop 13 we need
following Lemmas

Lemma 15 [BNY]: For any $\beta > 0$ there exists $0 < \beta' < \beta$ and $N$ such that for every ball $\mathcal{B} = B(\bar{x}, \bar{r}) \subseteq \mathbb{R}^d$, there is a collection of at most $N$ hyperplanes $\mathcal{H}_\mathcal{B}$ such that for any hyperplane $H'$ there exists $H \in \mathcal{H}_\mathcal{B}$ for which

$$B(\bar{x}, \bar{r}) \cap B(H', \beta' r) \subseteq B(H, \beta r)$$

Proof: W.l.o.g. we will prove (1) for $B(\bar{0}, 1)$ instead of $B(\bar{x}, \bar{r})$. We will show that

$$B(\bar{0}, 1) \cap B(H', \beta') \subseteq B(H, \beta)$$
Denote by $\mathcal{H}$ the set of all hyperplanes in $\mathbb{R}^d$. Recall that every $d \in \mathcal{H}$ can be expressed as

$$v_1 \vec{x}_1 + v_2 \vec{x}_2 + \ldots + v_d \vec{x}_d = t$$

Where at least one of $v_i$ is non-zero and $t$ is constant.

W.l.o.g. by scaling $t$, we can assume $(v_1, \ldots, v_d) \in S^{d-1}$ i.e. $\sqrt{v_1^2 + \ldots + v_d^2} = 1$

Thus there is a surjective map

$$\Pi : S^{d-1} \times \mathbb{R} \to \mathcal{H} \quad \text{s.t.}$$

$$\Pi(\vec{v}, t) = \{ \vec{x} \in \mathbb{R}^d : \vec{v} \cdot \vec{x} = t \}$$

Let $F$ be a maximal $\beta'$-separated subset of $S^{d-1} \times [-1 - \beta', 1 + \beta']$.
i.e. \( \forall (\vec{v}_1, t_1), (\vec{v}_2, t_2) \in F \) then \( \|\vec{v}_1 - \vec{v}_2\| \geq \beta' \) and \( |t_1 - t_2| \geq \beta' \)

Denote by \( F = \Pi (F) \), we claim \( F \) satisfies (2), i.e. \( F = H_B \).

Since \( |F| < \infty \) its image under \( \Pi \) is also finite, i.e. \( |F| < \infty \).

Fix \( H' = \Pi (\vec{v}_i, \vec{e}_i) \in H \)

If \( |t_1| > 1 + \beta' \) then

\( B(\vec{0}, 1) \cap B(H', \beta') = \emptyset \)

To see this in \( \mathbb{R}^2 \):

\( (u_1, u_2) \in S^1 \)

\[ u_1 x_1 + u_2 x_2 = t \]
\[ d(\vec{0}, H) = \frac{|u_0 u_2 - t_1|}{\sqrt{u_1^2 + u_2^2}} = |t_1| \]

Hence \((2)\) holds trivially.

Now assume \(|t_1| \leq 1 + \beta'\), we need to find \(H \in \mathcal{F}\) s.t.

\[ B(\vec{0}, 1) \cap B(H', \beta') \subseteq B(H, \beta) \tag{3} \]

Pick \(H = \tau_1 (\vec{v}_2, t_2) \in \mathcal{F}\) such that

\[ |\vec{v}_1 - \vec{v}_2| < \beta' \text{ and } |t_1 - t_2| < \beta' \]

For \(\bar{x} \in B(\vec{0}, 1) \cap B(H', \beta')\), we have

\[ |\vec{v}_2 \cdot \bar{x} - t_2| \leq |(\vec{v}_2 - \vec{v}_1 + \vec{v}_1) \cdot \bar{x} - t_2| \]

\[ = |(\vec{v}_2 - \vec{v}_1) \cdot \bar{x} + \vec{v}_1 \cdot \bar{x} - t_2 - t_1 + t_1| \]

\[ \leq |(\vec{v}_2 - \vec{v}_1) \cdot \bar{x}| + |\vec{v}_1 \cdot \bar{x} - t_1| + |t_1 - t_2| \]
\[ \leq \| (\overrightarrow{v_2} - \overrightarrow{v_1}) \| \| \overrightarrow{x} \| + |\overrightarrow{v_1} \cdot \overrightarrow{x} - t_1| + |t_1 - t_2| \]

\[ \leq \beta' \cdot 1 + \beta' + B' \]

\[ \overrightarrow{x} \in B(0,1) \quad \text{sine } \overrightarrow{x} \in B(H', \beta') \]

Put \( \beta' = \beta/3 \)

\[ = \beta \quad \text{. Hence } (3) \text{ holds} \]

**Lemma 16**: Let \( K \subset \mathbb{R}^d \) be hyperplane diffuse. Then there exist \( 0 < \beta_0 < \frac{1}{3} \) and \( \rho_0 > 0 \) such that for any \( 0 < r \leq \rho_0 \) any \( \overrightarrow{x} \in K \) and any hyperplane \( H \) there exists \( \overrightarrow{x}' \in K \) such that

\[ B(\overrightarrow{x}', \beta_0 r) \subset B(\overrightarrow{x}, r) \setminus B(H, \beta_0 r) \]
Proof: Since $K$ is hyperplane diffuse, there exist $\hat{\mathbf{x}} \in K$ s.t.

$$\mathbf{x} \in B(\hat{\mathbf{x}}, (1-\beta) \mathbf{r}) \setminus B(H, \beta (1-\beta) \mathbf{r})$$

(we assume w.l.o.g. $0 < \beta < 1$, thus $(1-\beta) \mathbf{r} \leq \mathbf{r}_0$)

Put $\beta (1-\beta) = 3\beta_0$, then $\beta_0 = \frac{\beta (1-\beta)}{3} < \frac{\beta}{3}$

and

$$\mathbf{x}' \in B(\hat{\mathbf{x}}, (1-\beta) \mathbf{r}) \setminus B(H, 3\beta_0 \mathbf{r})$$

$$\Rightarrow B(\hat{\mathbf{x}}, \beta_0 \mathbf{r}) \subset B(\hat{\mathbf{x}}, \mathbf{r}) \setminus B(H, \beta_0 \mathbf{r})$$

Now we prove the main result (Prop 13). Recall
Proposition 13 [BNY]: If $S \subseteq \mathbb{R}^d$ is HAW then $S \cap K \neq \emptyset$ for any hyperplane diffuse set $K \subseteq \mathbb{R}^d$. Moreover, if $S$ is Borel then converse is true.

Proof "⇒" Assume that $S$ is HAW and $K$ is hyperplane diffuse. Let $\beta_0$ and $r_0$ be as in Lemma 16 and suppose Alice and Bob play restricted HAW. Suppose that on the first move Bob chooses $\beta = \beta_0$ and a ball $B_0$ of radius $r_0$ centered in $K$. Note that this is a valid assumption since $\beta_0 \in (0, 1)$, $K$ is not empty, and $B_0$ can be arbitrary.

Let $n \geq 0$ and suppose that $B_0, \ldots, B_n$
are the balls arising from the restricted hyperplane absolute game that Alice and Bob play with Alice using winning strategy.

* We can assume that all $B_i$'s have centers in $K$. To see this note that for $n=0$, $B_0$ has center in $K$ by choice.

* Let $A_{n+1}$ be the nbhd of any hyperplane of radius $\beta^{n+1} r_0$ that Alice chose.

\[ B_0 = B_0 \setminus A_1 \supset B_1 \supset B_1 \setminus A_2 \supset B_2 \supset \cdots \]

\[ r_0 = \beta r_0, \quad r_1 = \beta^2 r_0, \quad \beta r_1 = \beta^2 r_0 \]

\[ r_2 = \beta r_1 = \beta^2 r_0 \]

Recall
Then by Lemma 16, Bob can choose a ball \( B_{n+1} \) of radius \( \beta^{n+1} r_0 \), which is contained in \( B_n \setminus A_{n+1} \) and centered at \( K \).

* To see this, define \( B_{n+1} \) to be \( B(x', \beta_0 r) \) from Lemma 16 with \( B(x, r) = B_n \) and \( B(H, \beta_0 r) = A_{n+1} \).

* Thus for every \( n \) Bob has a legal move, which means Alice cannot win by default, and the sequence \( (B_n)_{m \geq 0} \) can be made infinite.

* Also we have shown that Bob can force the centers of \( B_n \) to be in \( K \), the unique point \( \cap B_n \) lies in \( K \).
On the other hand Alice wins.
\[ \Rightarrow \bigwedge_{m \in S} \Rightarrow K \cap S \neq \emptyset. \]

To prove the other direction we would need Borel determinacy Theorem.

Theorem [FLC]: let \((X, d)\) be a complete metric space. Fix \(0 < \beta < \frac{1}{3}\) and \(S \subseteq X\). If \(S\) is Borel then \((\beta, S)\)-absolute winning game are determined.

Assume \(S\) is Borel s.t. \(A\) hyperplane diffuse set \(K \subseteq \mathbb{R}^d\), \(S \cap K \neq \emptyset\). Assume by contradiction \(S\) is not HAW. Then by Borel determinacy Theorem [FLC]
Bob has a winning strategy, which we fix for rest of the proof.

Let $B$ and $B_0$ be first move by Bob. Define $B_0 = \{B_0\}$ and continue by induction to construct balls $B_n$ as follows:

Given $B_n$, $\forall B \in B_n$, let $H_B$ be the collection of hyperplanes arising from Lemma 15 [BNY]. Define $B_{n+1}(B)$ to be collection of all of Bob’s responses according to winning strategy while considering $H_B$ as possible moves by Alice.

$$B_0 = \{B_0\}$$

$$B_1 = \{Bob's\ response\ to\ H_{B_0}\}$$

$$= \{B'_1, B'_2, \ldots, \}$$
\[ B_2 = \{ \text{Bob's response to } K^{(B_1)}, K^{(B_2)}, \ldots \} \]

Define \[ K = \bigcap_{m > 0 \ B \in B_m} U B \]

* If \( x \in K \), then it is an outcome of some hyperplane absolute game played according to Bob's winning strategy.

\[ \Rightarrow x \notin S \Rightarrow K \cap S = \emptyset. \text{ (contradiction)} \]

To finish the proof we need to show that \( K \) defined above is hyperplane diffuse (HD).

We will show \( K \) is HD for \( \beta^2 \beta \) where \( \beta \)' is as in Lemma 15.

* Assume \( x \in K, \ 0 < r < r_0 \) (where...
let $n$ be unique positive integer s.t.

$$2 \beta^m r_0 \leq r < 2 \beta^{m-1} r_0 \quad (5)$$

such $n$ exist since $\beta \in (0,1)$.

* Since $x \in K$, by def of $K$ (see (4))

$$\Rightarrow \exists \text{ ball } B = B(x_0, \beta^m r_0) \subset B_n$$

s.t. $x \in B$.

(B is Bob's choice for $n$th step)

By (5) $\beta^m r_0 \leq \frac{r}{2} \Rightarrow B \subset B(x, r)$

* Apply Lemma 15 for $x_0$ & $s = \beta^m r_0$

i.e.

$$B(x_0, \beta^m r_0) \cap B(H, \beta^m r_0) \subset B(H, \beta^{m+1} r_0) \quad (6)$$
By (5) \( r < 2\beta^{m-1} r_0 \)

\[ \Rightarrow \frac{\beta \beta' r}{2} < \frac{\beta \beta' \cdot 2 \beta^{m-1} r_0}{2} = \beta' \beta^m r_0 \]

i.e. \( \frac{\beta \beta' r}{2} < \beta' \beta^m r_0 \)

By (6)

\[ \Rightarrow B(x_0, \beta^m r_0) \cap B(H', \frac{\beta' \beta' r}{2}) \subset B(H, \beta^{m+1} r_0) \]

By definition of \( B_{m+1} (B) \rightarrow B \in B_m \)

If a ball \( B' \in B_{m+1} (B) \) such that \( B' \cap B(H, \beta^{m+1} r_0) = \emptyset \).

\( \Rightarrow \) Alice's choice

By definition of \( K \) (see (4))
\[ K \cap B' \neq \emptyset \Rightarrow \]
\[ \emptyset \neq K \cap B' \subseteq K \cap B \subseteq K \cap B(x, r) \]
\[ \Rightarrow K \cap B(x, r) \neq \emptyset \text{ and since } \]
\[ K \cap B(H', \beta' r) = \emptyset \]
\[ \Rightarrow K \cap \left( B(x, r) \setminus B(H', \beta' r) \right) \neq \emptyset \]
\[ \Rightarrow K \text{ is Hyperplane diffuse.} \]