First, before we will talk about our results for shmidet games we will talk about hausdroff dimension.

Let (X,ρ) be a metric space, $s\geq 0$, $\delta>0$. for $\delta>0$, $E\subseteq X$ define $\mathcal{H}_{s,\delta}(E):=\inf\{\sum_j(\operatorname{diam}(A_j))^s : E\subseteq\bigcup_j A_j,\ \operatorname{diam}(A_j)<\delta\}$

$\operatorname{diam}(A)=\sup\{\rho(x,y)\mid x,y\in A\}$

$\mathcal{H}_s(E)=\lim_{\delta\downarrow 0}\mathcal{H}_{s,\delta}(E)$.

בבירור $\mathcal{H}_s(E)$ מונוטונית יורדת באופן החלש

נגדיר כעת $\dim_{\mathcal{H}}(E)=\inf\{s\geq 0\mid \mathcal{H}_s(E)=0\}$

$=\sup\{s\geq 0\mid \mathcal{H}_s(E)=\infty\}$

$0\qquad\infty$

$\dim_{\mathcal{H}}(\mathbb{R}^n)=n$

$\forall E\subseteq\mathbb{R}^n$ $m(E)>0$ $\dim_{\mathcal{H}}(E)=n$

בבירור $E\subseteq\mathbb{R}^d$ לכל $s>d$ $H_s(E)=0$ (נוכל לכסות את $B(2^n)$

ע"י כדורים שסכום הנפח שלהם חסום ע"י $C\cdot n^d$ והרדיוס קטן מ-$\tilde{\delta}$. ואז נוכל לכסות את $B(2^n)$ ע"י כאלה רק שסכום

של $(\operatorname{diam}(A_j))^s$ חסום ע"י $C\cdot n^d\cdot\tilde{\delta}^{\tilde{c}}$ כאשר $\tilde{c}=s-d$.

ונבחר $\tilde{\delta}$ מספיק קטן כך ש $C\cdot n^d\,\tilde{\delta}^{\tilde{c}}<\frac{\varepsilon}{2^n}$

אם כן, לכל ε נוכל לכסות רק שסכומם יהיה קטן מ-ε.

שימו לב שאם $(E_n\supseteq E_{n+1})$ סדרה יורדת של קומפקטים אז $H_{s,\delta}(E)=\lim_{n\to\infty}H_{s,\delta}(E_n)$

לכל $\delta>0$.

Defenition

we say that a countable family $\mathcal{A}$ of compact subsets of $\mathbb{R}^n$ is tree-like (with respect to m) if every $\mathcal{A}_k$ is finite sub-collection and.

1) $\mathcal{A}_0 = \{A_0\}$

2) $m(A) > 0 \;\forall A \in \mathcal{A}$

3) $\forall k \in \mathbb{N} \;\forall A, B \in \mathcal{A}_k \quad A = B$ or $m(A \cap B) = 0$

4) $\forall k \in \mathbb{N}, B \in \mathcal{A}_k \;\exists A \in \mathcal{A}_{k-1} \quad B \subset A$

5) $\forall k \in \mathbb{N}, \forall A \in \mathcal{A}_{k-1} \;\exists B \in \mathcal{A}_k$ such that $B \subset A$.

define the (non-empty limit set of $\mathcal{A}$).

$A_\infty = \bigcap_{k \in \mathbb{N}} \bigcup \mathcal{A}_k$, and define the k-th stage diam: $d_k(\mathcal{A})$

by $d_k(\mathcal{A}) = \max_{A \in \mathcal{A}_k} \operatorname{diam}(A)$.

- we say that $\mathcal{A}$ is strongly tree-like if also $\lim_{k \to \infty} d_k(A) = 0$

finally, we define $\Delta_k(\mathcal{A}) = \min_{B \in \mathcal{A}_k} \dfrac{m(\bigcup \mathcal{A}_{k+1} \cap B)}{m(B)}$

Lemma

let $\mathcal{A}$ be a strongly tree-like collection of subsets of A_0. Then for any open U which intersect A_∞

$$\dim(A_\infty \cap U) \geq n - \limsup \frac{\sum_{i=0}^{k} \log \Delta_i(\mathcal{A})}{\log d_k(\mathcal{A})}$$

Theorem

For every S winning set in $\mathbb{R}^n$, and non-empty winning set U $\dim_H(S \cap U) = n$. i.e. $U \cap S$ have full hausdroff dimension

Idea of the proof of the lemma

we define sequence of measures by:

$$V_k(A) = \sum_{B \in A_{k-1}} \frac{\mu(\bigcup A_k(B) \cap A)}{\mu(\bigcup A_k(B))} V_{k-1}(B) \quad (\text{base}, \ V_0 = \mu|A_0)$$

put attention that for every k $V_k(\bigcup A_k) = \mu(A_0)$ and for every $A \in A_k$, $l \geq k$ $V_l(A) = V_k(A)$.

there exist $V_{n_k} \xrightarrow{weak} V$ and we can prove that it's cauchy sequence on the weak topology so the limit exist, $V_n \xrightarrow{weak} V$. now use Frostman Lemma and look at V.

Let $E=\mathbb{R}^n$, and define $\Psi: \mathbb{R}^n \times \mathbb{R}_+ \to C(\mathbb{R}^n)$ where $C(\mathbb{R}^n)$ is the set of nonempty compact subsets of E $\Psi(x,t)=\overline{B(x,e^{-t})}$. say $w \le w' (w,w' \in \mathbb{R}^n \times \mathbb{R})$ if $\Psi(w') \subset \Psi(w)$

we look on this (a,b) schmidt game - each time if Bob choose x, r on his k-th step, then $B_k = \Psi(x,r)$ (I mean, the r which he choose on the orriginal game is e^{-r}), and Alice choose $x' \in B_k$, $r' = r - \log \alpha$ such that $B(x', e^{-r+\log\alpha}) = B(x', \alpha \cdot e^{-r}) \subset B(x, e^{-r})$

Defenition

$\Omega_t = \{(x,t) \mid x \in \mathbb{R}^n\}$ so Ω is a disjoint union of slices Ω_t : $t \in \mathbb{R}$. for every $s>0$ and $w \in \Omega_t$ define $I_s(w) = \{w' \in \Omega_{t+s} : w' \le w\}$. In other words $I_a(w)$ and $I_b(w)$ are the sets of allowed moves of Bob and Alice.

by Defenition clearly that (take x to be the center of the ball):

MSG(0) Condition: $\forall s>0$ $I_s(w) \neq 0$ $\forall w \in \Omega$ and $s>0$.
Moreover, It's suttisfies MSG(1), and MSG(2) consitions and M(1), M(2) consitions

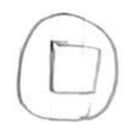
MSG(1): For any $\phi \neq U \subset E$ there is $\omega \in \Omega$ such that $\Psi(\omega) \subset U$. (clearly since U contain $B_\varepsilon(x)$ and take t such that $e^{-t} < \varepsilon$)

MSG(2): $\forall t \in \mathbb{R}, \omega \in \Omega_t$ $\operatorname{diam}(\Psi(\omega)) \le 2e^{-t}$ (traingle inequality)

from (2) we get that in step k of Bob $\operatorname{diam}(B_k) \xrightarrow[k\to\infty]{} 0$ (and same with alice) so the intersection contain at most 1 point (and contain a point because we can take convergence sub-sequence) and from (1) every (a,b)-winning set must be dense (if $U \cap S = \phi$ Bob choose ω such that $\Psi(\omega) \subset U$)

$\mu(1)$ $\mu(\Psi(\omega)) > 0$ $\forall \omega \in \Omega$ (clearly)

$\mu(2)$: For any $a > 0$ there exist $c > 0$ with the following property: $\forall \omega \in \Omega$ ~~...~~ and $\forall b > 0$ $\exists \theta_1, \dots, \theta_N \in I_b(\omega)$ such that $\Psi(\theta_i)$ $i = 1, \dots, N$ are essentially disjoint and that $\mu(\bigcup_i \Psi(\theta_i)) \ge c\,\mu(\Psi(\omega))$. (if $b < 10$ take a ball centered in the center of ω, else



$\Psi(\omega)$ contain a cube of vol $\ge$ ~~...~~ $c \cdot vol(\Psi(\omega))$)

and we can cover at least half of it by cubes of side length $2e^{-(t+b)}$ and each cube I can cover $\frac{M}{6}$ of her with ball of radius $e^{-(t+b)}$

Theorem: Let $\phi \neq S$ be a winning set, $a = -\log\alpha$, $b = -\log\beta$, $0<\alpha, \beta<1$. Then for any $\phi \neq U \subset E$ $\dim_{\text{hausdroff}}(S \cap U) \geq d + \frac{\log c}{a+b}$

(c from $\mu(2)$).

proof

we need to find a strongly tree-like collection $\mathcal{A}$ of sets, whose limit set is subset of $S \cap \mathcal{U}$.
Bob can start the game and choose $t_1 > 0, \omega_1 \in \Omega_{t_1}$ such that $\psi(\omega_1) \subset \mathcal{U}$. then, since S is winning, Alice can choose $\omega_1' \in I_a(\omega)$ such that $A_0 \stackrel{def}{=} \psi(\omega_1')$ has non-empty intersection with S.
Now let $\theta_1, \ldots\ldots, \theta_N \in I_b(\omega_1')$ be as in (μ_2). for $\omega = \omega_1'$. Each of these could be chosen by bob at the next step. Since S is (a,b) winning for each above choices θ_i Alice can pick $\theta_i' \in I_a(\omega_i)$ such that every sequence of possible further moves of Bob can be countred by Alice to win.
The collection of Images $\psi(\theta_i')$ of these choices of Alice, essentialy disjoint in view of (μ_2) will comprise the first level A_1 of the tree.
Repeating this for $A_2, A_3, \ldots$ clearly that it tree like collection.
clearly that $\operatorname{diam} A_k \leq C \cdot e^{-(a+b)k} \xrightarrow[k\to\infty]{} 0 \Rightarrow d_k \xrightarrow[k\to\infty]{} 0$
since θ_i satisfy (μ_2), $\mu_k(A) \geq C > 0$

$$\underset{\mathcal{H}}{\dim}(A_\infty \cap \mathcal{U}) \geq n - \limsup \frac{(k+1)\log \tilde{C}(a)}{\log C - (a+b)k} = n + \frac{\log \tilde{C}(a)}{a+b} \xrightarrow[\beta\to 0]{} n$$

$\downarrow$

$\forall \beta$

Theorem

There exist a complete metric space, which have haustoff dimension >0, and contain a winning set of hausdroff dimension 0.

proof

define $X = \{0,1,2\}^{\mathbb{N}}$ (the space of seauences of $0,1,2$) with the metric $d(x,y) = 3^{-k}$ $k = \min\{i \mid x_i \neq y_i\}$ ($d(x,y) = 0$ for $x = y$). obviously it's compact and complete space. define E to be the set of sequences in which the digit 0 can only be followed by 0.

Then E is close subset of X, so it's a complete metric space with the enduce topology.

Let S be the set of sequences in E for which the digit 0 appears. then S is countable subset of E. since for every ε I can cover a point by open set of diam $< \varepsilon$. Clearly that the hausdroff dimension is 0. its not hard to check that $\dim_H(E) = \frac{\log 2}{\log 3} > 0$. we will prove that S is $\alpha = \frac{1}{27}$ winning set. since on step $k > k_0$ bob choose $r < 1$, then we assume WLOG on step 1 he choose $\omega = (x,r)$ $x = (x_1, \ldots, x_n)$ take ℓ such that $3^{-(\ell+1)} < r \leq 3^{-\ell}$

Note that $B(\omega)$ contains all sequences (y_n) with $y_i = x_i$ $i \leq \ell$

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then Alice choose $(x_1, \dots, x_\ell, x_{\ell+1}, 0, 0, \dots) \in B_r(x)$

then for every next choice $B(x', r') \subset B_{\frac{r}{27}}(x_1, \dots, x_\ell, x_{\ell+1}, 0, \dots)$

so $d(y, x) < \frac{r}{27} \leq 3^{-(\ell+3)}$. then for $1 \leq i \leq \ell+3$

$y_i = (x_1, \dots, x_{\ell+1}, 0, 0)_i \Rightarrow y_{\ell+2} = 0$. ~~and for every~~

~~$z \in B(y, r')$ $r' \leq \frac{r}{27}$ we have $d(y,z) < 3^{-(\ell+3)}$~~

~~so $z_{\ell+2} = y$~~ so S is α-winning.