Abstract

we will prove today and in the first our next week a Theorem:

$\forall \delta > 0$ define $E_\delta = \{(\alpha,\beta) \in [0,1]^2 \mid \inf_n n \cdot \|\alpha n\| \cdot \|\beta n\| \geq \delta\}$.

then the upper box dimension of E_δ is 0.

we will use entropy methods for the proof - today we will talk about entropy and we will prove some Theorems and claims. and we will use them and the theorem that Alon proved last lecture for the proof of the theorem.

why it's so interesting us that the upper box dimension is 0? if it's 0, so for every $\varepsilon > 0$, can cover E_δ by cubes of side length r such that $\sum vol(Q_i) \leq C_\varepsilon \cdot r^{2-\varepsilon}$ for $r < r_0$ small (as $\to 0^+$).

$C_\varepsilon \cdot r^{2-\varepsilon} \xrightarrow[r \to 0^+]{} 0$ ($\forall \varepsilon < 2$ fixed). but moreover, we know what is the rate of the decay.

In generally for bounded set with measure 0, if we cover by cubes of same side length r, It may even not tend to 0. for example if we take $\mathbb{Q} \cap [0,1]$ and cover them by intervals of side length r, by density it's easy to see that we need at least $\frac{1}{r}$ intervals, so the bound is at least 1.

Theorem

Let X' be a compact metric space, $\{a_t\} \cong \mathbb{R}$ be an $\mathbb{R}$-group that act cantinously on X'.

Let X'_0 be an $\{a^+_t\}$ invariant compact subspace (actually, in the article the Theorem formulated such that X'_0 invariant to all $\{a_t\}$. and in this case we don't need X', but it's enough that X'_0 invariant to $\{a^+_t\}$ and cantain in a space which invariant to all $\{a_t\}$.

assume that for every $x \in X'_0$ $h_{top}(Y_x, a^+_t) = 0$, then $h_{top}(X'_0, a^+_t) = 0$ (Y_x is the close of the orbit among the semi-group: $Y_x = \overline{\{a_t.x \mid t \in \mathbb{R}_+\}}$).

proof: first I want to understand for ~~Y~~ such that what is the entropy $h_{top}(Y, a_t)$ or $h_{top}(Y, a^+_t)$ (Y is $\{a_t\}$ invariant or Y is $\{a^+_t\}$ invariant and $Y \subset Y'$ which is $\{a_t\}$ invariant).

$\forall k \in \mathbb{N}$ $h_{top}(Y, T^k) = k\, h_{top}(Y, T)$. we saw this for $T: Y \to Y$ $x \mapsto a_1.x$

entropy with respect to a measure, for topological entropy the proof use equivalence defenition with cover entropy.

we can generelize this, and get that for every $s > 0$ ($s \neq 0$ in the case Y invariant to all the group) $\frac{1}{s} h_{top}(Y, T_s)$ not depend on s we will look on one map, for our comfort $T: Y \to Y$ $x \mapsto a_1.x$. and by variational principle $h_{top}(Y, a^+_t) = \frac{1}{1} h_{top}(Y, T) = \sup\limits_{\mu \text{ Borel regular measure such } T \text{ preserving } \mu} h_\mu(Y, T)$.

Claim

T preserving μ if and only if μ invariant to multiply by a_1. (i.e for every $U \subset Y$ Borel $\mu(a_1.U) = \mu(U)$)
*(samely with T_s and a_s)

proof: $\Rightarrow$ if T preserving μ, since we can multiply by the inverse on the full space (Y in the case of group, Y' in the case of semi-group), so the map is one to one. then $T^{-1}(a_1.U) = U$ so $\mu(a_1.U) = \mu(U)$.

$\Leftarrow$ if $\mu(a_1.U) = \mu(U)$. $\mu(T^{-1}(U)) = \mu(T(T^{-1}(U))) =$
$= \mu(U \cap \mathrm{Im}\,T)$. $\mu(\mathrm{Im}(T)) = \mu(Y) = 1 \Rightarrow \mu(a_1.U) = \mu(U)$.

So we take the sup about ~~random~~ Borel regular measures that invariant to multiply by a_1. (or, for every $s \geq 0$ look about T_s and take sup about the measures that invariant to a_s) by other ~~variational~~ version of the variational principle for groups and semi-groups, we can take the sup about the measures that invariant ~~except~~ to all the group/semi group, so we will work with these measures during the lecture.

Claim

Let X_0', a_t and μ as above, then for every $x \in X_0'$ generic point μ supported in Y_x. (x is generic if for every $f \in C_0(X_0')$, $\frac{1}{T}\int_0^T f(a_t.x)\,dt \to \int_{X_0'} f(y)\,d\mu(y)$.

(since X_0' compact, $C_0(X_0')$ is the set of all $f: X_0' \to \mathbb{R}$ continous)

~~proof: since X~~

proof: let assume by contradiction that μ does not supported on Y_x. then $\mu(Y_x^c) > 0$

$$Y_x^c = \bigcup_n \{y \mid \operatorname{dist}(y, Y_x) \geq \tfrac{1}{n}\}$$

$$\Rightarrow \exists n_0 : \mu\{y \mid \operatorname{dist}(y, Y_x) \geq \tfrac{1}{n_0}\} > 0$$

define
$$f = \begin{cases} 1 & y \in \underbrace{\{y \mid \operatorname{dist}(y, Y_x) \geq \frac{1}{n_0}\}}_{A} \\ 0 & y \in Y_x \end{cases}$$

so clearly then f cantinous

the map $y \mapsto \operatorname{dist}(y, Y_x)$ is continous

$\Rightarrow A$ is close.

$\Rightarrow Y_x \cup A$ is close.

by ~~Aiza~~ Tietze there is continousely expansion $0 \leq g \leq 1$ such that $g|_{A \cup Y_x} = f$.

$$\frac{1}{T}\int_0^T g(\underbrace{h_t \cdot x}_{\in Y_x})\, dt = 0 \longrightarrow \int_{X_0'} g(y)\, d\mu(y) \geq$$

$$\geq \int_A g(y)\, d\mu(y) = \mu(A) > 0.$$

contradiction. then μ supported on Y_x.

assume by contradiction $h_{top}(X_0', a_t) > 0$.

by variational principle there is a_t invariant ~~measure~~ probability measure μ ~~supported on~~ on X_0' such that $h_\mu(X_0', a_t) > 0$. ~~since~~

by the pointwise ergodic theorem μ a.e ~~there is a~~ point $x \in X_0'$ is generic. take such x.

then $\mu(Y_x^c) = 0$. and for every $n \in \mathbb{N}$ we have

$$\mu(a_1^{-n} Y_x^c) = \mu(a_1^{n} \cdot (a_1^{-n} Y_x^c)) = \mu(Y_x^c) = 0$$

so Y_x^c will not affect on the entropy calculation. so $h_\mu(Y_x, a_t^+) = h_\mu(X_0', a_t) \le h_{top}(Y_x, a_t^+) = 0$

contradiction.

so $h_{top}(X_0', a_t) = 0$

Theorem

let $(\alpha,\beta)\in\mathbb{R}^2$ such that $A^+\cdot x_{\alpha,\beta}$ is bounded.

for every $\sigma,\tau\geq 0$ we define $a_{\sigma,\tau}(t)=a(\sigma t,\tau\cdot t)$

$$\left(a(s,t)=\begin{pmatrix} e^{t+s} & 0 & 0\\ 0 & e^{-s} & 0\\ 0 & 0 & e^{-t}\end{pmatrix}\right).$$

Then the topological entropy of $a_{\sigma,\tau}$ that acts on $\overline{\{a_{\sigma,\tau}(t)\cdot x_{\alpha,\beta} \mid t\in\mathbb{R}_+\}}$ is 0. (clearly that this set is $a_{\sigma,\tau}$ invariant by continuous of the action).

* we define ~~from~~ the entropy of the action of semi group on the same way we define for a group

proof: let assume by contradiction that it's >0.

since it's compact metric we can use the variational principle, so there is an $a_{\sigma,\tau}$ invariant probability measure on $\overline{\{a_{\sigma,\tau}(t)\cdot x_{\alpha,\beta} \mid t\in\mathbb{R}_+\}}$ such that

$h_\mu(\overline{\{a_{\sigma,\tau}(t)\cdot x_{\alpha,\beta} \mid t\in\mathbb{R}_+\}}, a_{\sigma,\tau})>0$.

~~we can think about this measure as a measure on X_3 which supported on $\overline{\{a_{\sigma,\tau}(t)\cdot x_{\alpha,\beta} \mid t\in\mathbb{R}_+\}}$.~~ 0

~~proof: take the map $T: X_3\to X_3$~~

~~$x\mapsto a_{\sigma,\tau}(1)\cdot x$~~

~~for $E\subset X_3$ Borel set, define V_n~~

define measure V_n on X_3 $V_n(E)=\frac{1}{n}\sum_{i=0}^{n-1}\mu(E\cap\{a_{\sigma,\tau}\cdot x_{\alpha,\beta} \mid t\in\ldots$

* actually the theorem holds for every $x\in X_3$ such that $A^+\cdot x$ bounded

we want to think about this measure as measure on X_3, we can look about the natural induce measure, $\tilde{\mu}_\infty$ which define by $\tilde{\mu}_\infty(E) = \mu_\infty(E \cap \overline{\{a_{\sigma,\tau}(t).X_{\alpha,\beta} \mid t \in \mathbb{R}_+\}})$

we want that the measure will be invariant except to the semi group $\overline{\{a_{\sigma,\tau}(t) \mid t \in \mathbb{R}_+\}}$.

In $V = \overline{\{a_{\sigma,\tau}(t).X_{\alpha,\beta} \mid t \in \mathbb{R}_+\}}$ I know that the measure is invariant. if I take $U \subset V^c$, and act with an element of $\{a_{\sigma,\tau}(t) \mid t \in \mathbb{R}_+\}$ I can go inside V, so why it is still invariant? for every $t_0 \in \mathbb{R}_+$ we will define $T_{t_0}: X_3 \to X_3$

$x \mapsto a_{t_0}.x$, Then $\mu(T(V)) = \mu(V) = 1$ and since I can multiply by the inverse the map is one to one, so $\tilde{\mu}((T(V))^c) \le \mu((T(V))^c) = 0$
$\Rightarrow \tilde{\mu}(U) = 0$.

so $\tilde{\mu}_\infty$ invariant except to the semi group. we want to show that $\tilde{\mu}_\infty$ invariant to all the group. in general if $G \curvearrowright X$, μ measure on X then $stab_\mu$ is subgroup of G
$(stab_\mu = \{g \in G \mid \forall U \subset X_3 \;\; g.U = U\})$

so if it containes the semi-group it contain all the group.

finally (we will not use the original measure μ so we can call $\tilde{\mu}_\infty$ μ, and as we did before clearly that $h_{\tilde{\mu}_\infty}(X_3, a_{\sigma,\tau}) = h_\mu(X_3, a_{\sigma,\tau})$

so there exist an $a_{6,\tau}$ invariant probability measure (μ) supported on A, such that $h_\mu(X_3, a_{6,\tau}) > 0$.

define for every $s>0$ measure $\mu_s = \frac{1}{s^2}\int\int_0^s a(s,t)_* \mu \, ds\, dt$

$a(s,t)_* \mu(E) = \mu(T^{-1}(E))$ $\quad T: X_3 \to X_3 \quad x \mapsto a(s,t).x$

claim: for every μ' $a_{6,\tau}$ invariant measure on X_3, $\forall s,t \geq 0$ $h_{a(s,t)_*\mu'}(a_{6,\tau}) = h_{\mu'}(a_{6,\tau})$

proof: in the calculation of the entropy it's enough to take suprimum on finity partitions,

since $a_{6,\tau}$ and $a(s,t)$ מתחלפים

$h_{a(s,t)_*\mu'}(a_{6,\tau}, P) = h_{\mu'}(a_{6,\tau}, a(s,t)^{-1}_* P)$

$\Rightarrow h_{a(s,t)_*\mu'}(a_{6,\tau}) \leq h_{\mu'}(a_{6,\tau})$

the other side by the same way.

since μ is $\{a_{6,\tau}(t)\}$ invariant measure, there is an ergodic decomposition $\mu = \int_Y \mu_\xi \, d\nu(\xi)$.

so μ_s has ergodic decomposition

$$\mu_s = \frac{1}{s^2}\int\int_0^s \int_Y a(s,t)_* \mu_\xi \, d\nu(\xi)\, ds\, dt$$

$a(s,t)_* \mu_\xi$ is ergodic since if for every t

$a_{6,\tau}(t)\cdot U = U \Rightarrow a(s,t)^{-1}\cdot a_{6,\tau}(t) U = a_{6,\tau}(t)\cdot (a(s,t))^{-1}\cdot U = a(s,t)^{-1}\cdot U$

$\Rightarrow \mu_\xi(a(s,t)^{-1}\cdot U) \in \{0,1\}$

$= a(s,t)_*\mu_\xi(U)$

so $h_{\mu_S}(a_{\sigma,\tau}) = \frac{1}{S^2} \int_0^S \int_0^S \int_Y h_{a(s,t)_*\mu_\xi}(a_{\sigma,\tau})\, d\nu(\xi)\, ds\, dt$

$= \int_Y h_{\mu_\xi}(a_{\sigma,\tau})\, d\nu(\xi) = h_\mu(a_{\sigma,\tau})$

since $\forall s,t \geq 0 \quad a(s,t).(\overline{A^+.X_{\alpha,\beta}}) \subseteq \overline{A^+.X_{\alpha,\beta}}$

$\Rightarrow \forall s,t \geq 0 \quad a(s,t)_*\mu$ support on $\overline{A^+.X_{\alpha,\beta}}$,

so also μ_S. then $\forall S > 0$ μ_S supported on a uniform compact.

since $\forall S > 0$ μ_S supported on a uniform compact, if we look on $\{\mu_n\}$ there exist a subsequence $\mu_{n_k} \to \mu_\infty$ in the weak topology.

by the semi-continous of entropy,

$h_{\mu_\infty}(a_{\sigma,\tau}) \geq \overline{\lim}\, \underbrace{h_{\mu_{n_k}}(a_{\sigma,\tau})}_{=c>0} > 0$,

and since for every $s,t \in \mathbb{R}$ $|\mu_n(a(s,t).E) - \mu_n(E)| \leq \frac{C_{s,t}}{n} \xrightarrow[n\to\infty]{} 0$, so μ_∞ invariant to all the semi-group A.

<u>corollary (Alon presented)</u>

if μ in A-invariant measure on X_3, and $\exists a_t \subset A$ one parameter subgroup such ~~th~~ $h_\mu(X_3, a_t) > 0$, then μ is not compact supported.

(Alon presented this for μ ergodic, but we can generalize this:

let $\tilde{C} \subset X_\xi$ compact. look on the ergodic decomposition,
$\mu = \int_Y \mu_\xi$, $\forall \xi$ $\mu_\xi((\tilde{C})^c) > 0$ so $\mu((\tilde{C})^c) > 0$

(in general, if $f > 0$ so there exist n_0 such that $\mu(\{f \geq \frac{1}{n_0}\}) > 0$, then $\int f \, d\mu > 0$).

Theorem

Let $C \subset X_3$ compact set, define $X_C = \{x \in X_3 \mid A^+ . x \subset C\}$, then for every $\sigma, \tau \geq 0$ the topological entropy of the semi-group $\{a_{\sigma,\tau}(t) \mid t \geq 0\}$ that act on X_C is 0.

(X_C is close, $X_C \subset C$ so it's compact).

X_C close because if $x_n \to x_0$ $A^+ x_n \subset C$ $\forall s,t \geq 0$ $a(s,t).x_n \in C$ $a(s,t).x_n \to a(s,t).x$ (by continously of the action). and C close so close to limits so $a(s,t).x \in C$. so X_C close.

and clearly that X_C invariant to the semi-group (by continously of the action), so it's well define.

proof: as we said, it's enough to prove that for every $x \in X_C$, $Y_x = \{a_{\sigma,\tau}(t) \mid t \in \mathbb{R}_+\}$ $h_{top}(Y_x, a^+_{\sigma,\tau}) = 0$, and the previous Theorem holds for every $x \in X_3$ such that $A^+ X_{\alpha,\beta}$ is Bound. so we get our result.