Theorem

for every $\delta > 0$, define $E_\delta = \{(\alpha,\beta) \in [0,1]^2 \mid \inf_n n \cdot \|n\alpha\| \cdot \|n\beta\| \geq \delta\}$

then E_δ have upper box dimension 0.

proof: ~~as~~ the upper box dimension defined to be

$\overline{\lim_{r \to 0^+}} \frac{\log N(r)}{\log r}$ when $N(r)$ is the minimal

number of cubes of side length r that we need to cover E_δ.

since $\overline{E_\delta} \subset \bigcup_{x \in E_\delta} C_r(x)$ and $\overline{E_\delta}$ it's compact

there exist finite sub-cover and $N(r)$ is finite

so the upper box dimension is well define.

for proving the lim is 0, we need to prove that $N(r) = O_\varepsilon(r^{-\varepsilon})$ $\forall \varepsilon > 0$ $r \to 0^+$

Claim

it's enough to prove that for every $\varepsilon > 0$, the maximal cardinality of r-seperated subset of E_δ is $O_\varepsilon(r^{-\varepsilon})$ when $r \to 0^+$. (a subset called r seperated if ~~a~~ for every $x \neq y$ $\|x-y\|_\infty \geq r$.)

proof: suppose this, and take r-seperated set, A, of max cardinality ~~such that~~, $|A| \leq O_\varepsilon(r^{-\varepsilon})$.

then $E_\delta \subset \bigcup_{x \in A} C_{2r}(x)$ because else take

$y \in E_\delta$ $y \notin \bigcup_{x \in A} C_{2r}(x)$, then we can add y to A

and it's still r-seperated. contradiction to the maximality.

let $r_0 < 1$ be a const, then we can cover the unit cube by finite cubes of side length $\frac{r_0}{2}$ which contain in $[0,1]^2$. so it's enough to show that the maximal cardinality of r-seperated subset of G_σ which contain in a cube of the form $C_{\frac{r}{2}}(x)$, is $O_\varepsilon(r^{-\varepsilon})$.

In the notation of the article we look about the Equivalence classes of the form $\Gamma \cdot g$, but we want to stick to our notation from previous lectures, so we will look about the Equivalence classes of the form $g \cdot \Gamma$.

In general for every topological group locally compact which suttisfies the second countability Axiom, there is a Right invariant Riemannian metric d, and in particular for $SL_3(\mathbb{R})$.

then d induce a metric on X_3, $\tilde{d}$, which define by $\inf_{g_1, g_2 \in \Gamma} d(xg_1, yg_2) = \inf_{g_1 \in \Gamma} d(xg_1, y)$

Right invariant

define $g_{a,b} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}$.

as alon said in the previous lecture, there is compact $G_\sigma \subset X_3$ such that for every $(\alpha, \beta) \in G_\sigma$ $A^+ x_\alpha$

Lemma

There exist C_0, r_0 such that for any $x \in G$, $|a|, |b| < r_0$ $\tilde{d}(x, g_{a,b} \cdot x) \geq C_0 \max(|a|, |b|)$.

An Idea to prove this, is that ~~the~~ locally, $X_3 \underset{Dif}{\cong} SL_3(\mathbb{R}) \underset{Dif}{\cong} \mathbb{R}^8$, i.e, there exist $\varphi: X_3 \to \mathbb{R}^8$ locally Difeomorphism, so $\tilde{d}$ and the ~~oclidead~~ ~~oclicad~~ ~~euclead~~ euclidean metric on $\mathbb{R}^8$ are ~~[illegible]~~ B-Lipshiz, (since they induce by Riemanian metrics and ~~the act $g_{a,b}(x) = \pi(g_{a,b} \cdot x)$ in $\mathbb{R}^8$ is~~ ~~[illegible] $g_{a,b} \cdot x$ is shifting~~ In $\mathbb{R}^8$ there are constants C_0^x, r_0^x which hold in a neighboor of x, so also in X_3 (in $\mathbb{R}^8$ $g_{a,b} \cdot x$ will translate to metrices multiply).

we can take a finite subcover, and get constants C_0, r_0.

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Lemma

There exist c_0, r_0 such that for any $x \in C_\sigma$, $|a|, |b| < r_0$

$d(x, g_{a,b} \cdot x) \geq c_0 \max(|a|, |b|)$.

~~the~~ an Idea to prove ~~this~~ is that locally $X_3 \cong \mathbb{R}^k$ and all the norms on $\mathbb{R}^k$ are equality (here $k = 8$). So for every $x \in C_\sigma$ there is constants r_0^x, c_0^x for a neighborhood of x. ~~and we~~ can ~~take~~ a finite subcover. we will not get into it.

For any $\alpha, \alpha', \beta, \beta'$ $x_{\alpha,\beta} = g_{\alpha'-\alpha', \beta'-\beta'} \cdot x_{\alpha',\beta'}$

(the act is ~~$x \cdot g = x \cdot g$~~ $x \cdot \pi_\Gamma(g) = \pi_\Gamma(x \cdot g) = (x \cdot g) \cdot \Gamma$

and in general, for any n $a_{1,1}^n \cdot x_{\alpha,\beta} = g_{e^{3n}(\alpha'-\alpha'), e^{3n}(\beta'-\beta')} \cdot (a_{1,1}^n(1) \cdot x_{\alpha',\beta'})$.

let $S \subset C_\sigma$ r_n seperated, $r_n = e^{-3n} \cdot r_0$. assume (wLoG) $S \subset C_{\frac{r_0}{2}}(x)$. define $S' = \{x_{\alpha,\beta} : (\alpha,\beta) \in S\}$.

Claim

S' is T $(n, \frac{c_0 r_0}{e^3})$ seperated, for $T: X_3 \to X_3$, $x \mapsto a_{1,1}(1) \cdot x$

~~proof: $e^{-3n} \cdot r_0 \leq$~~ proof: for $x_{\alpha,\beta} \neq x_{\alpha',\beta'} \in S'$

$r_0 > \max\{|\alpha - \alpha'|, |\beta - \beta'|\} \geq r_0 \cdot e^{-3n}$. then there exist $0 \leq m \leq n$ such that $r_0 > \max\{|\alpha-\alpha'|, |\beta - \beta'|\} \cdot e^{3m} \geq r_0 \cdot e^{-3}$

then $d(T^m(x_{\alpha,\beta}), T^m(x_{\alpha',\beta'})) \geq \frac{c_0 r_0}{e^3}$

$$(a^n_{1,1}(1)\cdot x_{\alpha',\beta'})$$

$$(d(T^m(x_{\alpha,\beta}), T^m(x_{\alpha',\beta'})) = d(g_{e^{3m}(\alpha-\alpha'), e^{3m}(\beta-\beta')}\cdot \cancel{(x_{\alpha',\beta'})}$$

$$a^n_{1,1}(1)\cdot x_{\alpha',\beta'}$$

$$\geq 6 e^{3m}\max\{|\alpha'-\alpha|, |\beta'-\beta|\} \geq \frac{C_0 r_0}{e^3}$$

remember that $h_{top}(X_{C\sigma}, T) = 0$

then $\overline{\lim}\ \frac{\log(N(X_{C\sigma}, T, n, \varepsilon))}{n} = H(X, T, \varepsilon)$

$$\lim_{\varepsilon\to 0^+} H(X_{C\sigma}, T, \varepsilon) = 0$$

$H(X_{C\sigma}, T, \varepsilon)$ is decreasing and non-negetive

then $H(X_{C\sigma}, T, \frac{C_0 r_0}{e^3}) = 0$. Let $\varepsilon > 0$

$$\Rightarrow \forall n > n_0 \quad N(X_{C\sigma}, T, n, \varepsilon) \leq e^{\varepsilon\cdot n_0}$$

$$|S| = |S'| \leq O_\varepsilon(r_n^{-\frac{\varepsilon}{3}}) = O_\varepsilon(r_n^{-\varepsilon})$$

and since for every $r < r_0$ there exist n such that $r\cdot e^{-3} \leq r_n \leq r$ we got this result, which prove the theorem.