Setting: Random walk of $G=SL_2(\mathbb{R})$ w/ law μ

E.g. $\operatorname{supp}\mu = \left\{ \begin{bmatrix} 2 & \\ & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ & \frac{1}{2} \end{bmatrix} \right\}$

More general, assume that μ on $SL_D(\mathbb{R})$ is in "(M,N)-upper block form"

if (i) $\operatorname{supp}\mu \subset P = KAU$ where

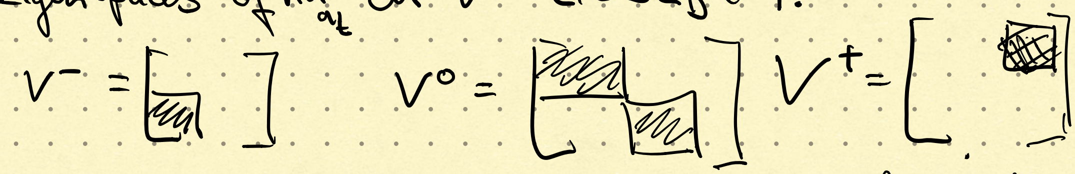
$$K = \begin{bmatrix} SO_M(\mathbb{R}) & \\ & SO_N(\mathbb{R}) \end{bmatrix} \quad A = \left\{ a_t = \begin{bmatrix} e^{t/M}\,\mathrm{Id}_M & \\ & e^{-t/N}\,\mathrm{Id}_N \end{bmatrix} \right\}_{t\in\mathbb{R}}, \quad U = \left\{ u_\alpha = \begin{bmatrix} \mathrm{Id}_M & \alpha \\ & \mathrm{Id}_N \end{bmatrix} \right\}_{\alpha \in \mathrm{Mat}_{M,N}(\mathbb{R})}$$

(ii) $c_1 := \mu(\Theta_1) > 0$

where $\Theta_1(g) := t$ if $g = k a_t u_\alpha$

(iii) $H := \overline{\langle \operatorname{supp}\mu \rangle}^{Z}$ contains $\operatorname{Lie} U =: V^+$

Eigenspaces of Ad_{a_t} on $V = \operatorname{Lie} SL_D(\mathbb{R})$:



for $d < \dim SL_D$ also consider decomp for $V^{\wedge d} = \bigwedge_{i=1}^{d} \mathfrak{g}$

for $\mathrm{ad}_{\log a_1} = \left[\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \cdot \right]$: $V^{\wedge d} = \bigoplus_{\psi} V_{\psi}^{\wedge d}$

E.g. $SL_2(\mathbb{R})$ $V^{\wedge 2} = V^- \wedge V^0 \oplus V^- \wedge V^+ \oplus V^0 \wedge V^+$

$= V_{-2}^{\wedge 2} \oplus V_0^{\wedge 2} \oplus V_2^{\wedge 2}$

$\simeq V^{\wedge 1}$

Define: $W^{\wedge d} := V_{>0}^{\wedge d} = \bigoplus_{\psi > 0} V_{\psi}^{\wedge d}$

Osedets: $G \to GL(V)$, $\beta = \mu^{\mathbb{N}}$ on $B = G^{\mathbb{N}}$

$\exists\, \lambda_1 > \cdots > \lambda_n$, and $\forall\, \beta$ a.e. b equivariant subspaces

$V = V_{\leq \lambda_1} > V_{\leq \lambda_2}^{(b)} > \cdots > V_{\leq \lambda_n}^{(b)} > 0$ st

$\forall v \in V_{\leq \lambda_i} \setminus V_{\leq \lambda_{i-1}}$ $\displaystyle \lim_{n\to\infty} \frac{1}{n} \log \frac{\| b_n \cdots b_1 . v\|}{\|v\|} = \lambda_i$

uniformly

$\displaystyle \sum \dim V_i / V_{i-1}\ \lambda_i = \int \log |\det g|\, d\mu(g)$

Apply to $V = V^{\wedge d}$. Define: $V_b^{<\max} = V^{\wedge 1}_{\leq \lambda_2}(b)$

$V_b^{\leq 0} = V^{\wedge d}_{\leq 0}(b)$

Thm: If μ is in (M,N)-uppr block form and for $d = 1 \ldots \dim S_D$ $W^{\wedge d}$ as above, then

(I) $\forall g \in \operatorname{supp}\mu$, $W^{\wedge d}$ is Ad_g invariant.

For β ae $b \in B$ $\quad W^{\wedge 1} \oplus V_b^{<\max} = V$

$d \geq 2 \quad W^{\wedge d} \cap V_b^{\leq 0} = 0$

(II) $\forall g \in \operatorname{supp}\mu$, $\mathrm{Ad}_g|_{W^{\wedge 1}}$ similarity and

$$\int_G \log \| \mathrm{Ad}_g|_{W^{\wedge 1}} \|\, d\mu > 0$$

(III) $\forall L \subset V^{\wedge d}$ st $\langle \operatorname{supp}\mu \rangle_+ . L = \{L_1, \ldots, L_r\}$ is a finite orbit $\Rightarrow$ $W^{\wedge d} \cap L \neq 0$

Thm 2.1: (I)-(III) $\Rightarrow$ $\begin{cases} \exists \text{ unique } \mu\text{-stat measure on } G/\Gamma \\ \beta \text{ a.e. } b, \ b_n \cdots b_1 \cdot x \text{ equi on } G/\Gamma \end{cases}$

Lemma: Sps $0 \to W \to V \to V/W \to 0$ st W invarient and Lyap's $\lambda_i^W, \lambda_i^V, \lambda_i^{V/W}$

st $\min_i \lambda_i^W > \max_i \lambda_i^{V/W}$ (21)

then $\{\lambda_i^V\} = \{\lambda_i^W\} \sqcup \{\lambda_i^{V/W}\}$ (22)

w/ multiplicity

Cor: a) β a.e. $b \in B$, $W \oplus V_{\leq \max \lambda_i^{V/W}}(b) = V$

b) Sps $\exists$ decomp $V = \bigoplus W_i$

st $\forall g \in \operatorname{supp} \mu$: $g = \begin{array}{c} W_1 \\ W_2 \\ \vdots \\ W_m \end{array} \begin{bmatrix} e^{\alpha_1(g)} O_1(g) & * & * \\ & \ddots & * \\ & & e^{\alpha_m(g)} O_m(g) \end{bmatrix}$

then

$$\lambda_i^V = \int \alpha_i(g) \, d\mu(g)$$

w/ multiplicity

Pf of b) Take $W = W_1$, then by last item of Oseledets

$$\dim W_1 \cdot \lambda_1^W = \int \log|\det e^{\alpha_1(g)} O_1(g)| \, d\mu$$
$$= \dim W_1 \cdot \int \alpha_1(g) \, d\mu$$

then Induction w/ $W_2 < \bigoplus_{i>1} W_i \simeq V/W_1$ ∎

Pf Lemma: Fix $\lambda = \lambda_i^{V/W}$ and $u \in V^{V/W}_{\leq \lambda_i^{V/W}}(b) \setminus V^{V/W}_{< \lambda_i^{V/W}}(b)$

Let $V_u = \mathbb{R}u + W < V$

Claim: $\lambda = \min\{ \lambda_j^V : V^V_{\leq \lambda_j}(b) \cap V_u \neq 0 \}$

Sps Claim true $\forall u$. Then $\exists$ Lift v of u st $v \in V^V_{\leq \lambda}$.

$\Rightarrow \{\text{Lyaps } V/W\} \subset \{\text{Lyaps } V\}$

Also, lift is unique (else $v - \tilde{v} \in W$ growths $\geq \min_i \lambda_i^W > \max_i \lambda_i^{V/W} \geq \lambda$ ↯)

$\Rightarrow \forall i \; \exists \text{map } (V/W)_{\leq \lambda_i} \longrightarrow V_{\leq \lambda_i}$

(start w/ smallest)

Pf of Claim: Take $\langle \cdot , \cdot \rangle$ on V and identify $W^\perp \simeq V/W$

Let $\tilde{\lambda}$ be RHS of claim

$\tilde{\lambda} \geq \lambda$: Let $v \in V_u$. If $[v] \neq 0$ (ie $\in V_u(W)$): $\| b_i^n \cdot v \| \geq \| b_i^n \cdot [v] \| \geq e^{(\lambda - \varepsilon) n} \| [v] \|$

else $v \in W$ and $\| b_i^n \cdot v \| > e^{(\lambda - \varepsilon) n} \| v \|$ by (2i) as $\lambda \in \{\text{Lyp of } V/W\}$

$\lambda \geq \tilde{\lambda}$: Let v_u lift of u st $b_i^n \cdot v_u \in W^\perp$ (W is invariant)

$$e^{(\lambda + \varepsilon) n} \geq \frac{\| b_i^n \cdot u \|}{\| u \|} = \frac{\| b_i^n \cdot v_u \|}{\| u \|} \geq \frac{\| b_i^n \cdot v_u \|}{\| v_u \|} \overset{\text{uniform}}{\geq} e^{(\tilde{\lambda} - \varepsilon) n}$$

∎

Proof of Theorem (μ (1,1) uppr triangular $\Rightarrow$ (I)–(III))

The main example $\mu = p_1 \delta_{a_{\log 2}} + p_2 \delta_{a_{\log 2} u_1}$ is (1,1) uppr triangular since

i) $\operatorname{supp}\mu \subset P = AU = \begin{bmatrix} * & \\ & * \end{bmatrix} \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix}$

ii) $c_1 = \int \Theta_1(g)\,d\mu$

By Cor b) $= \text{top Lyap} = \log 2 > 0$

iii) Lie alg of $\overline{\langle \operatorname{supp}\mu \rangle}^{Z} = P$ contains V^+

($\langle \operatorname{supp}\mu \rangle$ contains a lattice of both A and U)

$V^+ = \begin{bmatrix} & * \\ & \end{bmatrix}$ $V^0 = \begin{bmatrix} * & \\ & * \end{bmatrix}$ $V^- = \begin{bmatrix} & \\ * & \end{bmatrix}$

$U = \exp V^+$

One checks that $\forall u \in U$

(27) $\quad \operatorname{Ad}_u v - v \in \begin{cases} 0 & v \in V^+ \\ V^+ & v \in V^0 \\ V^+ + V^0 & v \in V^- \end{cases}$

(and analogous for $N = \exp V^-$)

$V^{\wedge d} = \bigoplus_{\chi} V_{\chi}^{\wedge d}$ decomp wrt $\operatorname{ad}_{\log a_1}$

we have:

(A) $V_{\geq \chi}^{\wedge d}$, $V_{>\chi}^{\wedge}$ invariant under P

(B) Ad_g on $V_{\geq\chi}^{\wedge d} / V_{>\chi}^{\wedge d}$ is similarity w/ dilation $e^{\chi\Theta_1(g)}$

(C) $P \curvearrowright V_{\geq\chi}^{\wedge d} / V_{>\chi}^{\wedge d}$ has only one Lyap $c_1 \chi$

$\ulcorner$ Follows for $\underline{d=1}$: since $\bigoplus_\chi V_\chi^{\wedge 1} = V^+ + V^0 + V^-$ and from (27), e.g. $[g.v] = [au.v] = [a.v] = e^{\chi\Theta_1(g)}[v]$ and ⓒ follows then from Cor b).

$\underline{d=2}$: ($\hat{=}$ induction step in general setting) consider $v \wedge w \in \hat{V}^1_{\chi_1} \wedge \hat{V}^1_{\chi_2}$

$\chi_1+\chi_2=\chi$ $\quad \subset V^{\wedge 2}_{\geq \chi} / V^{\wedge 2}_{>\chi}$

$$\begin{aligned}[g.v\wedge w] &= [au.v \wedge au.w] \\ &= [av \wedge aw] \quad (\text{since } u.v-v \text{ has higher weight}) \\ &= e^{(\chi_1+\chi_2)\Theta_1(g)}[v\wedge w]\end{aligned}$$

$\lrcorner$

$$W^{\wedge d} = V^{\wedge d}_{>0} = \bigoplus_{\chi>0} V^{\wedge d}_\chi$$

$\underline{(I)}$:
- $W^{\wedge d}$ invariant under supp μ by Ⓐ
- Lyaps of $W^{\wedge d}$ are >0:

Action is in block diagonal form by Ⓐ

$\chi_1 > \dots > \chi_m > 0$

$$\begin{array}{c} V_{\chi_1} \\ \vdots \\ V_{\chi_m} \end{array} \begin{bmatrix} \square & * & * \\ & \square & * \\ & & \square \end{bmatrix} \quad \text{w/ Lyaps } c_i\chi_i > 0 \text{ by ⓒ}$$

similarly, Lypas on $V^{\wedge d}/W^{\wedge d}$ are ≤ 0

$\Rightarrow$ Assumption of Lemma holds

and by cor a) $\quad W^{\wedge d} \oplus V^{\wedge d}_{\leq 0}(b) = V$

and note $V^{\wedge 1}_{\leq 0}(b) = V_b^{<\max}$. $\blacksquare$

(II) $\mathrm{Ad}_g|_{W^{\wedge 1}}$ similarity since $au.w \overset{(27)}{=} a.w$
$= e^{2\theta_1(g)} w$

and $\int \log \|\mathrm{Ad}_g|_{W^{\wedge 1}}\| \, d\mu = 2\int \theta_1(g) \, d\mu$
$= 2c_1 > 0$

(III) Sps $\mathrm{supp}\,\mu$ permutes $\{L_1, \dots, L_r\}$ subspaces of $V^{\wedge d}$

(Need: $L_i \cap W^{\wedge d} \neq 0$)

$\Rightarrow H = \overline{\langle \mathrm{supp}\,\mu \rangle}^Z$ permutes $\{L_i\}$ $\left[\begin{bmatrix} 0 & \ast \\ \ast & 0 \end{bmatrix} \right]$

$\Rightarrow H^0$ fixes each L_i [h invertible and const action so $h.v \neq 0$ only intersection point of L_i]

Already noted that $P \subset H^0$

Sps $L \in \{L_i\}$ has $L \cap W^{\wedge d} = 0$

$A \subset P$ inv $L = \bigoplus L_\chi \Rightarrow \chi < 0$
$\Rightarrow L \subset V^{\wedge d}_{\leq 0}(b)$

$V_{\leq 0}(b)$ is $Q = \begin{bmatrix} 1 & \\ \ast & 1 \end{bmatrix} A$ invariant

$\overline{QP}^Z = G$

$L' = G.L < V_{\leq 0}(b)$ is G-invariant

G contains Weyl element $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$
mapping L'_χ to $L'_{-\chi}$

$\Rightarrow L' = L'_0 \Rightarrow \mathrm{Ad}$ action has kernel $\supset A$

G simple $\Rightarrow$ trivial action $\Rightarrow L' = L = 0$ ↯

Backwards dynamical system $T^X : B\times X \to B\times X$

$$T^X(b,x) = (\tau b, b_1^{-1}.x)$$

$$\mathcal{B}_n = \sigma(\{b \mapsto b_i\}\ i=1..n) = \sigma([b_1,..,b_n])$$

so that for $\varphi : B \to \mathbb{R}$

$$\mathbb{E}(\varphi \mid \mathcal{B}_n)(b) = \int \varphi(b_1,..,b_n, b')\, d\beta(b')$$

ν μ-stat measure on X

<u>Def</u>: $\nu_{b,n} = b_1 \cdots b_n * \nu \in \mathcal{P}(X)$

<u>Def</u> β^X <u>on $B\times X$:</u> (this is not $\beta\times\nu$!)

$$\beta_n^X = \int \delta_b \otimes \nu_{b,n}\, d\beta \in \mathcal{P}(B\times X)$$

on $\mathcal{B}_n \times \mathcal{B}(X)$ $\beta_k^X = \beta_m^X$ $k,m \geq n$

φ $\mathcal{B}_n \times \mathcal{B}(X)$ meas

$$\beta_{n+1}^X(\varphi) = \int \varphi((b_1,..,b_n), x)\, d\, b_1 \cdots b_n b_{n+1} * \nu(x)\, d\beta(b)$$

$$= \int \varphi(b_1 \cdots b_n, x)\, d\, b_1 \cdots b_n \underbrace{(b_{n+1} * \nu))\, d\mu^{\otimes n}(b)\, d\mu(b_{n+1})}_{=\nu \quad \mu\text{stat}}$$

$$= \beta_n^X(\varphi)$$

$\Rightarrow \exists$ extension β^X on $\mathcal{B}(B) \otimes \mathcal{B}(X)$

(Parthasarathy Thm 4.1)

Check: T^X-invariant, $\pi_* \beta^X = \beta$

Now can disintegrate wrt $B \times X \xrightarrow{\pi} B$
to get measure
$\delta_b \otimes \nu_b$ on $\{b\} \times X$ st $\beta^X = \int \delta_b \otimes \nu_b \, d\beta(b)$

Lemma 3.2 BQI

a) $f : X \to \mathbb{R}$ bnd, β ae B $\nu_b(f) = \lim_n \nu_{b,n}(f)$

b) $\nu_b = b_1 * \nu_{Tb}$

c) $\nu = \int_B \nu_b \, d\beta(b)$

d) ν_b are unique measures w/ properties b) & c)

e) sps ν_b has b) and define ν by c) then ν is μ-stat

Need:

Martingale Convergence Thm

sps $f \in L^1(X, \mathcal{A}, \mu)$ $\mathcal{A}_n \nearrow \mathcal{A}$ or $\mathcal{A}_n \searrow \mathcal{A}$

then $\mathbb{E}_\mu[f \mid \mathcal{A}_n] \to \mathbb{E}_\mu[f \mid \mathcal{A}]$

ae and in L^1

Proof of Lemma: Let $f : X \to \mathbb{R}$ bnd
extend to $B \times X$ (trivially)

$\pi^{-1}\mathcal{B}_n = \mathcal{B}_n \times \mathcal{B}(X)$

$$\mathbb{E}[f \mid \pi^{-1}\mathcal{B}_n] = \nu_{b,n}(f) \qquad (*)$$

Sps ψ $\pi^{-1}\mathcal{B}_n$ meas, then

$$\int \psi\, \mathbb{E}[f \mid \pi^{-1}\mathcal{B}_n]\, d\beta^X \overset{\text{Cond Expect propty}}{=} \int \psi f \, d\beta^X$$

$$\underset{\uparrow}{=} \int \psi f \, d\beta_n^X(b,x) \underset{\substack{\text{cond expect} \\ \leadsto \\ \text{propty}}}{=} \int \psi(b)\, \nu_{b,n}(f)\, d\beta \qquad (*)$$

ψf $\mathcal{B}_n \times \mathcal{B}(X)$ meas
& Def β^X

$$\overset{\text{MCT}}{\Rightarrow} \quad \nu_{b,n}(f) \longrightarrow \mathbb{E}[f \mid \pi^{-1}\mathcal{B}] \underset{\uparrow}{=} \nu_b(f)$$

cond meas property

b) $\nu_{b,n} = b_1 * \nu_{Tb,n-1}$

$\Rightarrow \quad \nu_{b,n}(f) \to b_1 * \nu_{Tb}(f)$

Take $\{f = \mathbb{1}_O\}$ for $\{O\}$ countable basis of top

$\Rightarrow \quad b_1 * \nu_{Tb} = \nu_b$

c) $$\underset{\substack{\uparrow \\ f\ \mathcal{B}_0 \times X \text{ meas}}}{\beta^X(f)} = \beta_0^X(f) = \int f(x)\, d\nu(x)\, d\beta(b) = \nu(f)$$

II-Disintegration of β^X

$\int f(x)\, d\nu_b(x)\, d\beta(b)$

d) $\tilde{\nu}_b$ w/ b) & c Define $\tilde{\beta}^X = \int \delta_b \otimes \tilde{\nu}_b$

Check $\tilde{\beta}^X = \beta^X$ $\rightsquigarrow$ uniqueness of Disintegration

e) e for easy

$\blacksquare$