Equidistribution on affine symmetric spaces

1 Sources

- Eskin-McMullen - Mixing, Counting, and Equidistribution in Lie Groups
- Schlichtkrull - Hyperfunctions and Harmonic Analysis on Symmetric Spaces
- Knapp - Representation Theory of semisimple groups: Beyond an introduction

2 Affine Symmetric Spaces

**Definition 2.1.** Let $G$ be a connected semi-simple Lie group with finite center. Let $\sigma : G \to G$ be an involution (i.e. a Lie group automorphism with $\sigma^2 = \text{id}$) and let $H < G$ the fixpoint set of $\sigma$. Then $G/H$ is called affine symmetric space and $H$ is called a symmetric group.

Recall that $G$ is semisimple if its Lie algebra $\mathfrak{g}$ is a direct sum of simple Lie algebras. The differential of $\sigma$ at the identity gives a Lie automorphism that is an involution, also denoted by $\sigma$. Any linear involution is diagonalizable - splitting into $\pm \sigma$-eigenspaces. This decomposition keeps holding in the group level, where however, only one eigenspace is a lie algebra. For a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$ we can then write $\sigma(g) = hb$.

**Example 2.2.** Let $G = \text{SL}_n(\mathbb{R})$ the group of $n \times n$-matrices of det 1 and $\sigma(g) = g^{-T}$ inverse transpose. $\text{stab}(\sigma) = \text{SO}_n(\mathbb{R})$. More generally, any classical Lie group that is closed under transposition. For an involution $\sigma$ with $H$ compact, $G/H$ defines a Riemannian symmetric space.

**Example 2.3.** $G \times G/G$ where $G$ is diagonally embedded comes from the convolution $\sigma(g, h) = \sigma(h, g)$. $\{ M \in \text{Mat}_{dd}(\mathbb{R}) | \text{det} M = 1 \} = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})/\Delta \text{SL}_2(\mathbb{R})$

**Example 2.4.** $\text{SL}_2(\mathbb{R})/A$ where $A$ the diagonal group coming from $\sigma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$.

**Example 2.5.** We let $I_{p,q} = (\text{id}_p, -\text{id}_q)$, $p + q = n$, and define $\sigma_{p,q}$ the involution on $\text{SL}_n(\mathbb{R})$ obtained by conjugation with $I_{p,q}$. The isotropy group is by definition $\text{SO}_{p,q}(\mathbb{R})$, the group of orientation preserving isometries of the indefinite form $\sum_{i=1}^q x_{i+p}^2 - \sum_{i=1}^p x_i^2$. Note that $\text{SO}_{1,1}(\mathbb{R})$ is the diagonal group in $\text{SL}_2(\mathbb{R}) \simeq \text{SO}_{1,2}(\mathbb{R})$. One can also take $G = \text{SO}_{p,q}(\mathbb{R})$, and $\sigma_{p',q'}$ giving rise to some $\text{SO}_{p',q'}(\mathbb{R}) < \text{SO}_{p,q}(\mathbb{R})$. Of particular importance is $\text{SO}_{p,q-1}(\mathbb{R}) < \text{SO}_{p,q}(\mathbb{R})$ from $I_{p+q-1,1}$ since $\text{SO}_{p,q}(\mathbb{R})/\text{SO}_{p,q-1}(\mathbb{R})$ is identified with the hyperboloid $\sum_{i=1}^q x_{i+p}^2 - \sum_{i=1}^p x_i^2 = 1$.

Any involution $\sigma$ on $G$ induces an involution on $\mathfrak{g}$, which we shall denote by the same letter. Then $\mathfrak{g}$ splits into $\sigma$-eigenspaces for the eigenvalues $\pm 1$

$$\mathfrak{g} = \mathfrak{h}_\sigma \oplus \mathfrak{q}_\sigma.$$ 

In particular $\mathfrak{h}$ is the Lie algebra of $H$. Note that we have

$$[\mathfrak{h}_\sigma, \mathfrak{h}_\sigma] \subset \mathfrak{h}_\sigma, \quad [\mathfrak{h}_\sigma, \mathfrak{q}_\sigma] \subset \mathfrak{q}_\sigma, \quad [\mathfrak{q}_\sigma, \mathfrak{q}_\sigma] \subset \mathfrak{h}_\sigma$$

and for any decomposition with such brackets relations there is an involution giving raise to this decomposition.

**Example 2.6.** Let $\mathfrak{g} = \text{sl}_n(\mathbb{R})$ and $\sigma(X) = -X^T$ inverse transpose. Then the above decomposition is between symmetric and skew-symmetric traceless matrices.
Definition 2.7. We shall write \( \text{ad}_X \) the map \( Y \mapsto [X,Y] \). The killing form \( B(X,Y) = \text{Tr}(\text{ad}_X \circ \text{ad}_Y) \) is non-degenerate iff \( G \) is semi-simple and negative definite if \( G \) is compact. An involution \( \theta \) is called Cartan involution if \( B_\theta = -B(X,\theta(Y)) \) is symmetric and positive definite. Note that the adjoint of \( \text{ad}_X \) with respect to this inner product becomes \( -\text{ad}_X \), and thus selfadjoint on \( p_\theta \).

Example 2.8. For \( sl_n(\mathbb{R}) \), \( B(X,Y) = 2n \text{Tr}(XY) \), so that \( B_\theta(X,Y) = -2n \text{Tr}(X\theta(Y)) = 2n \text{Tr}(XY^T) \). But \( \text{Tr}(XY^T) \) is an inner product on the space of \( n \times n \) matrices making \( \theta \) a Cartan involution.

Proposition 2.9. \( B_\theta \) is symmetric and positive definite

- \( \mathfrak{t} \perp \mathfrak{p} \) with respect to both \( B \) and \( B_\theta \)

Definition 2.10. The decomposition \( \mathfrak{h}_\theta \oplus \mathfrak{q}_\theta \) for a Cartan involution \( \theta \) is called a Cartan pair.

Example 2.11. For \( sl_n(\mathbb{R}) = \mathfrak{t} \oplus \mathfrak{p} = so_n(\mathbb{R}) \). Since \( \mathfrak{p} \) consists of symmetric matrices, any \( Y \in \mathfrak{p} \) can be diagonalized, \( Y = kZk^{-1} = \text{Ad}_k Z \) for some \( Z \) diagonal (and traceless) and \( k \in K \). Let \( \mathfrak{a} \subset \mathfrak{g} \) be the diagonal traceless matrices then \( \mathfrak{p} = \text{Ad}_K \mathfrak{a} \).

Theorem 2.12. A Cartan involution is unique up to an inner automorphism, i.e. \( \theta = f \circ \theta' \circ f^{-1} \) and \( f = \text{Ad}_g \) for some \( g \in G \). For any involution \( \sigma \), there exists a Cartan convolution that commutes with \( \sigma \).

Theorem 2.13. For a Cartan pair \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \), \( K \) is a maximal compact subgroup of \( G \). Let \( \mathfrak{a} \) be a maximal abelian subspace of \( \mathfrak{p} \) and \( A = \exp \mathfrak{a} \). Then \( G = K \exp \mathfrak{p} \) (in fact \( (k,\mathfrak{t}) \to k \exp \mathfrak{t} \)) is a diffeo, \( \mathfrak{p} = \bigcup_{k \in K} \text{Ad}_k \mathfrak{a} \) (in fact for any maximal \( \mathfrak{a}, \mathfrak{a}' \in \mathfrak{p} \) are \( K \)-conjugates and \( G = KA'K \)).

Proof. Assume \( G \) is a classical group, say \( G \subset GL(C,n) \) and \( \theta \) is Inverse conjugate transpose. Then there is a unique polar decomposition \( g = k \exp X \) with \( k \) unitary and \( X \) Hermitian (exp is surjective on the positive definite Hermitian matrices since it is on diagonal matrices). Now \( k^T = k^{-1} \), \( XT = X \theta(g) = k \exp -X \), \( \theta(g)^{-1} Y = \exp 2X \)

which implies \( \exp X \in G \) (using the fact that \( \exp X \) in an algebraic group then \( X \) is in the Lie algebra). Since \( g \in G \), also \( k \in G \cap U(n) = K \) is compact. We see that \( K \) must be maximal, since else it contains an element of \( \exp \mathfrak{p} \) but any non-trivial element gives an unbounded subgroup.

Given \( \mathfrak{a}, \mathfrak{a}' \) take \( Z, Z' \) such that no root \( \Sigma^a \) resp. \( \Sigma^a' \) vanishes. Consider the curve \( K \ni k \mapsto B(\text{Ad}_k Z, Z') \).

Let \( k \in K \) be the minimum (which exists by compactness of \( K \)). Its derivative,

\[
B(\text{ad}_H \text{Ad}_k Z, Z') = B([\text{Ad}_k Z, Z'], H) = 0
\]

for \( H \in \mathfrak{t} \) vanishes, but \( B(H, H) < 0 \) for any \( H \in \mathfrak{t} \), and thus \([\text{Ad}_k Z, Z'] = 0 \). Since \( Z' \) has non-trivial projection to any \( \mathfrak{g}_k \), \( \text{Ad}_k Z \in \mathfrak{g}_0 \). Since \( \mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \), \( \mathfrak{m} \perp \mathfrak{t} \) and as \( \text{Ad}_k Z \in \mathfrak{p} \), \( \text{Ad}_k Z \in \mathfrak{a}' \).

By symmetry of the argument, \( \text{Ad}_k Z \in \mathfrak{a}' \). Note also \( Z_\theta(Z) = Z_\theta \) generates the centralizer by construction. But \( \mathfrak{a}' \) commutes now with \( Z \) implying that \( \mathfrak{a}' \subset \text{Ad}_k \mathfrak{a} \), and by maximality they are equal.

\( KAK \) follows from the previous statements.

Theorem 2.14. Let \( \sigma \) be an involution of \( G \) with affine symmetric group \( H \) and giving rise to \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \). Let \( \theta \) be a commuting Cartan decomposition with symmetry group \( K \) giving rise to \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \). Let \( \mathfrak{a} \subset \mathfrak{p} \subset \mathfrak{q} \) be a maximal abelian subspace, \( A = \exp \mathfrak{a} \). Then \( G \subset HAK \).

Proof Sketch. Step 1: \( (X,Y,k) \mapsto \exp X \exp Y k \) from \( (\mathfrak{p} \cap \mathfrak{h}) \times (\mathfrak{p} \cap \mathfrak{q}) \times K \) to \( G \) is a local diffeo onto. Diffgeo by local dimensions argument. Assuming a decomposition \( g = \exp X \exp Y k \) for the moment. Since \( \theta(g^{-1}) = \theta(k^{-1} \exp -Y \exp -X) = k^{-1} \exp(Y) \exp(X) \), we have

\[
g\theta(g^{-1}) = \exp X \exp 2Y \exp X
\]

We already know \( G = \exp \mathfrak{p} K \) uniquely, and we want to assume \( g = \exp S \) for \( S \in \mathfrak{p} \), in particular \( \theta(g^{-1}) = g \) and the LHS is \( \exp 2S \) Apply \( \sigma \), to see \( (\sigma \text{ fixes } \mathfrak{h} \text{ thus } X) \)

\[
\exp 2\sigma(S) = \exp X \exp -2Y \exp X.
\]
Introduce the dual image is also open. Thus everything. Now also multiplication from A take a subsequence where the Combining both gives exp 2σ(S) = exp 2X exp −2S exp 2X and thus exp −S exp 2σ(S) exp −S = (exp −S exp 2X exp −S)^2 which we may rewrite as exp 2X = exp S exp T exp S with exp 2T = exp −S exp 2σ(S) exp −S These formulas show that X and Y are uniquely determined, and how to construct them given g.

We reduce to show Step 2: exp p ∩ q ⊂ HAK.

Define g_0 = t_0 ⊕ p_0 = (t ∩ h) ⊕ (p ∩ q). By the bracket relations of involution, it is a sub lie algebra. Since σ and θ commute, θ preserves the eigenspace decomposition with respect to σ, and thus preserves g_0 but also the decomposition g_0 = t_0 ⊕ p_0 (σ acts by ±1, so any intersection of an eigenspace is preserved). The associated Lie group G_0 is by definition reductive, and again allows a K_0A_0K_0 decomposition where A_0 = A and K_0 = H ∩ K. We now conclude that exp p ∩ q ⊂ G_0 ⊂ HAK.

The maps ad_Z for Z ∈ a are commuting, and as remarked before, selfadjoint with respect to B_θ. Introduce the dual a^* and for λ ∈ a^*,
\[ g_λ = \{ X ∈ g : ad_Z(X) = λ(Z)X \text{ for all } Z ∈ a \} \]
Let Σ consists of all λ ≠ 0 with g_λ, the set of restricted roots. Having chosen a basis on a^*, one might introduce an ordering on Σ let Σ^+ be the positive restricted roots. A root in Σ^+ is called simple if it cannot be written as sum as any other two. Remark: Given a basis of a^* coming from elements of Σ, then these are simple with respect to some choice of Σ^+ if any other root in Σ^+ can be expressed in either all positive or all negative integer coefficients.

**Example 2.15.** Let E_{ij} be the elementary matrices in sl_n(ℝ) and Z = diag(h_1, ..., h_n) ∈ a then ad_Z(E_{ij}) = (h_i - h_j)E_{ij}. Let e_j ∈ a^* by e_j(H) = h_j, then e_i - e_j are precisely such λ for which g_λ ≠ 0 forming Σ. Taking the order induced from e_1, ..., e_n, a root is positive if the first coefficient is positive in that basis (so that e_1 - e_n is the largest positive root and e_{n-1} - e_n the smallest), and e_i - e_{i+1} form a base of simple positive roots.

**Theorem 2.16.**
- \[ g = g_0 ⊕ \sum_{λ ∈ Σ^+} g_λ \] (orthogonal sum)
- \[ [g_λ, g_μ] ⊂ g_{λ+μ} \]
- \[ θg_λ = g_{−λ} \text{ and hence } λ ∈ Σ \text{ implies } −λ ∈ Σ. \] Same for σ.
- \[ g_λ ⊥ g_μ \text{ with respect to } B_θ \]

We study now the Lie subalgebra of g,
\[ n = \sum_{λ ∈ Σ^+} g_λ \]

**Theorem 2.17.** Assume for the moment that σ = θ. Then the above theorem can be extended to say
\[ g_0 = a ⊕ m \]
and the Iwasawa decomposition:
\[ g = k ⊕ a ⊕ n \]
and K × A × N → G is a diffeo onto.

**Proof.** Any X ∈ I has non-zero projection to m or \[ \sum_{Σ^+} g_{−λ} \text{ together with } g = n + g_0 + n \] making k + a + n a direct sum. It is everything since
\[ a + m + (n + n) ⊃ Z + X_0 + \sum X_λ = (X_0 + \sum (X_{−λ} + θX_{−λ})) + Z + \sum (X_λ - θX_{−λ}) ∈ k + a + n \]
For the group level one uses that if \[ g = g ⊕ t \] of two subalgebras then the differential of the multiplication map vanishes nowhere. The image is closed since K is compact and AN are closed (for any subsequence, take a subsequence where the K part converges, then take limit in AN, still of product form). The image is also open. Thus everything. Now also multiplication from A × N to AN is smooth and onto. \[ \square \]
**Definition 2.18.** The hyperplanes in $\mathfrak{a} \simeq \mathfrak{a}^*$ defined by $\ker \lambda$ cut $\mathfrak{a}$ into finitely many open regions $\{C\}$ called Weyl chambers. For any set of simple roots $\Delta \subset \Sigma$ there is a unique $C_\Delta$ defined by the intersection of the half-spaces $\lambda > 0$ in $\mathfrak{a}$ where $\lambda \in \Delta$, and $\Sigma_\Delta$ denotes the positive roots with respect to $\Delta$, i.e. those $\lambda$ for which $\lambda(\mathcal{W}_\Delta) > 0$. Denote by $\mathfrak{n}_\Delta = \sum_{\lambda \in \Sigma_\Delta} \mathfrak{g}_\lambda$ and

$$N_\Delta = \langle \exp \mathfrak{n}_\Delta \rangle, \quad A_\Delta = \exp \mathcal{C}_\Delta$$

Any Weyl chamber contains exactly one root, the maximal element with respect to the ordering.

**Example 2.19.** Picture of triangulation of equilateral triangles coming from $A_2$. If $\alpha, \beta$ are two simple roots $\alpha + \beta$ is maximal and contained in the cone of the corresponding Weyl chamber. It is the highest weight of the adjoint representation.

**Proposition 2.20.** There exists a a shrinking family of open neighborhoods $N_e$ of $e \in N_\Delta$ invariant under conjugation by $A_\Delta$, i.e. for any open $e \in U$ there is $V_e \subset O$ with

$$e \in a^{-1}V_0a \subset V_e \subset U$$

for any $a \in A_\Delta$.

**Proof.** Let $X = \sum_{\lambda \in \Sigma^+} x_\lambda X_\lambda \in \mathfrak{n}$ where $X_\lambda$ spans the one-dimensional space $\mathfrak{g}_\lambda$. Let $c_a : N \to N$ the conjugation map $n \mapsto ana^{-1}$, its derivative acts on $\mathfrak{n}$ by $\text{Ad}(a) : \mathfrak{n} \to \mathfrak{n}$ which is related the previous adjoint action by $\text{Ad}(\exp Z) = \exp(\text{ad} Z)$, and so $\text{Ad}(a^{-1})X_\lambda = \exp(-\lambda(Z))$ for $a \in \exp Z \in A_\Delta$.

$$\text{Ad}(a^{-1})X = \sum_{\lambda \in \Sigma^+} x_\lambda \exp(-\lambda(Z))X_\lambda \in \mathfrak{n}$$

and we see that $a^{-1}$ contracts as $\lambda(Z) > 0$. Take $V_e$ to be a product neighbourhood. \qed

**Theorem 2.21.** Let $M = Z_K(A)$, then $H \times M \times A \times N \to G$ is open in a neighborhood of the identity in $G$.

**Proof.** It suffices to show $\mathfrak{h} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n} = \mathfrak{g}$. We have $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{g}_0 \oplus \mathfrak{h}$.

We decompose any $X$ with respect to that decomposition and thus assume $X \in \mathfrak{n} \oplus \mathfrak{g}_0$. For the $\mathfrak{n}$ part we observe that also $\sigma(\mathfrak{g}_0) = \mathfrak{g}_{-\lambda}$ since

$$[Z, \sigma(X)] = \sigma([\sigma(Z), X]) = -\sigma([Z, X]) = -\lambda(Z)\sigma(X)$$

for $X \in \mathfrak{g}_\lambda$ and $X + \sigma(X) \in \mathfrak{h}$.

Thus for any $X \in \mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_{-\lambda}$,

$$X = (X + \sigma(X)) - \sigma(X) \in \mathfrak{h} \oplus \mathfrak{n}.$$
3 Wavefront Lemma

Theorem 3.1. For any open neighbourhood $U$ of $e \in G$ there is $V \subset G$ open such that

$$HVg \subset HgU$$

for all $g \in AK$.

Proof. Assume first that $g \in A$. Then $g \in \exp(\mathcal{L})$ for some Weyl chamber. Let $N$ be the corresponding unipotent subgroup, with a contraction invariant neighborhoods $V_N$. We also let $V_M, V_A$, neighbourhoods in $M$ and $A$ and put $V = HV_MV_AV_N$ a neighbourhood of $G$ by $HMAN$ decomposition, by which we may also assume that $V_MV_AV_N \subset U$

$$HVg = HV_MV_AV_NV_N \subset H_MV_AV_NV_N \subset HgU$$

This $V = V_C$ depends on the Weyl chamber, and we take the intersection of all of them.

For general $g = ak$, we may choose that $U' \subset U$ which is $K$-conjugation invariant and take $V$ coming the above construction for $a$. Then

$$HVg = HVak \subset HaU'k = Hakk^{-1}U'k = Hgk^{-1}U'k \subset HgU$$

4 Equidistribution

Let $\Gamma < G$ be a lattice and let $X = \Gamma \backslash G$. We assume that $\Gamma$ projects densely onto $G/G'$ for any $G'$ normal noncompact Liegroup $G' \subset G$. This implies that $L^2(X)$ does not contain non-trivial $G_i$-invariant vectors for any $i$, and therefore, by Howe-Moore,

Theorem 4.1. The action of $G$ on $X$ is mixing, that is for any $\alpha, \beta \in L^2(X)$,

$$\int_X \alpha(xg)\beta(x)dx \to \frac{1}{m(X)} \int_X \alpha \int_X \beta$$

Assume that $H$ is such that $\Gamma \cap H$ intersects $H$ in a lattice. Then $\Gamma H$ is a closed orbit of finite volume, naturally identified with $\Gamma \backslash H \backslash H$ of measure $m(Y)$ induced by a fixed Haar measure on $H$. We may push these measures to measures on $\Gamma \backslash Hg$. Theorem 4.2. The translates $Yg, Y = \Gamma H$ become equidistributed in $X$ as $Hg \to \infty$ in $H/G$:

$$\frac{1}{m(Y)} \int_{Yg} \alpha(y)dy \to \frac{1}{m(X)} \int_X \alpha(x)dx.$$

for any $\alpha \in C_c(X)$.

Proof. Let $Hg_n \to \infty$ in $H \backslash G$, $g_n \in AK$. Let $(U, \epsilon)$ such that $\alpha(gu)$ is $\epsilon$-close to $\alpha(g)$ fpr all $u \in U$. By the wave front lemma, there is $HVg \subset HgU$ for all $g$ in $AK$ and by mixing,

$$\frac{1}{m(YV)} \int_{YVg_n} \alpha(g)dg = \frac{1}{m(YV)} \int_{\Gamma \backslash G} \chi_{YV}(g)\alpha(gg_n)dg \to \frac{1}{m(X)} \int_X \alpha(g)dg.$$ 

The LHS is a convex combination of the integrals

$$\frac{1}{m(Y)} \int_{Yg_nu} \alpha(h)dh$$

which are $\epsilon$-close to $\frac{1}{m(Y)} \int_{Yg_n} \alpha(h)dh$. 

\( \square \)
5 Counting

Theorem 5.1. \[ \left| \{ M \in \text{Mat}_{dd}(\mathbb{Z}) \mid \det M = a, \| M \| \leq R \} \right| 
\asymp c_a R^{d(d-1)} \]

\[ V = \{ M \in \text{Mat}_{dd}(\mathbb{R}) \mid \det M = a \} = \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) / \text{SL}_2(\mathbb{R}). \] Claim: \( V(\mathbb{Z}) \) finite union of \( \Gamma = \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \)-orbits. Action of \( G \times G \) on \( V \) by \( gMh^{-1} \). \( H = \Delta G \). The maximal abelian space \( a \) is \( A' = \{ (a, a^{-1}) \} \in A \times A \), and \( G \times G = (K \times K) A'H \)

Theorem 5.2. \( V_a \) level set of the standard quadratic surface of signature \((m, n)\), \( a \in \mathbb{Z} \) and assume \( V(\mathbb{Z}) \) not empty then \[ |V(\mathbb{Z}) \cap B^Y_R| \asymp c_m R^{m+n-2} \]