Lecture

Def^n Let $(X, \mathcal{B})$ is a Borel space. A subcollection $\mathcal{N} \subseteq \mathcal{B}$ is called a σ-ideal if

1. $\mathcal{N}$ is closed under countable unions.
2. $B \in \mathcal{B}$ and $N \in \mathcal{N} \Rightarrow B \cap N \in \mathcal{N}$.

$A \equiv B \bmod (\mathcal{N})$ if $A \Delta B = (A-B) \cup (B-A) \in \mathcal{N}$.

Ex:- If m is a countable additive measure on $\mathcal{B}$, then the collection of sets in $\mathcal{B}$ of measure zero form a σ-ideal.

Def^n A one-one measurable map $T: (X, \mathcal{B}) \to (X, \mathcal{B})$ onto itself such that T^{-1} is also measurable map is called Borel automorphism.

Def^n Let $(X, \mathcal{B}_X)$ be a Borel set. A measurable set W is said to be wandering with respect to Borel automorphism T if

$T^n W : n \in \mathbb{Z}$ are pairwise disjoint.

The σ-ideal generated by wandering sets in $\mathcal{B}$ is denoted by $\mathcal{W}_T$

(W intersect orbit of any point at most 1).
never intersect periodic orbit).

Lemma Let T: be a Borel automorphism of a standard Borel space $(X, \mathcal{B})$. Then given any $A \in \mathcal{B}$ there exists $N \in \mathcal{W}_T$ such that each $x \in A - N$, the points $T^n x$ return to A infinitely many positive n.

Lemma $(X, \mathcal{B}, \mu, T)$ is p.p.T, and $A \in \mathcal{B}$ and ~~If $\mu(A) > 0$, then~~ then for almost all $x \in A$, the points $T^n x \in N$ for infinitely many n.

Let $(X, \mathcal{B}, \mu, T)$ be a invertible measure preserving system and let A be a measurable set with $\mu(A) > 0$. By Poincare recurrence, the first return time defined by

$$r_A(x) = \inf_{n \geq 1} \{ n \mid T^n x \in A \}$$

Exists almost every where.

<u>Defⁿ</u> The map $T_A : A \to A$ defined by

$$T_A(x) = T^{r_A(x)}(x)$$

(r_A and T_A are measurable maps)

For $n \geq 1$, $A_n = \{ x \in A \mid r_A(x) = n \}$

$$A_1 = A \cap T^{-1}(A)$$

$$A_2 = A \cap T^{-2}(A) \setminus A_1$$

$$\vdots$$

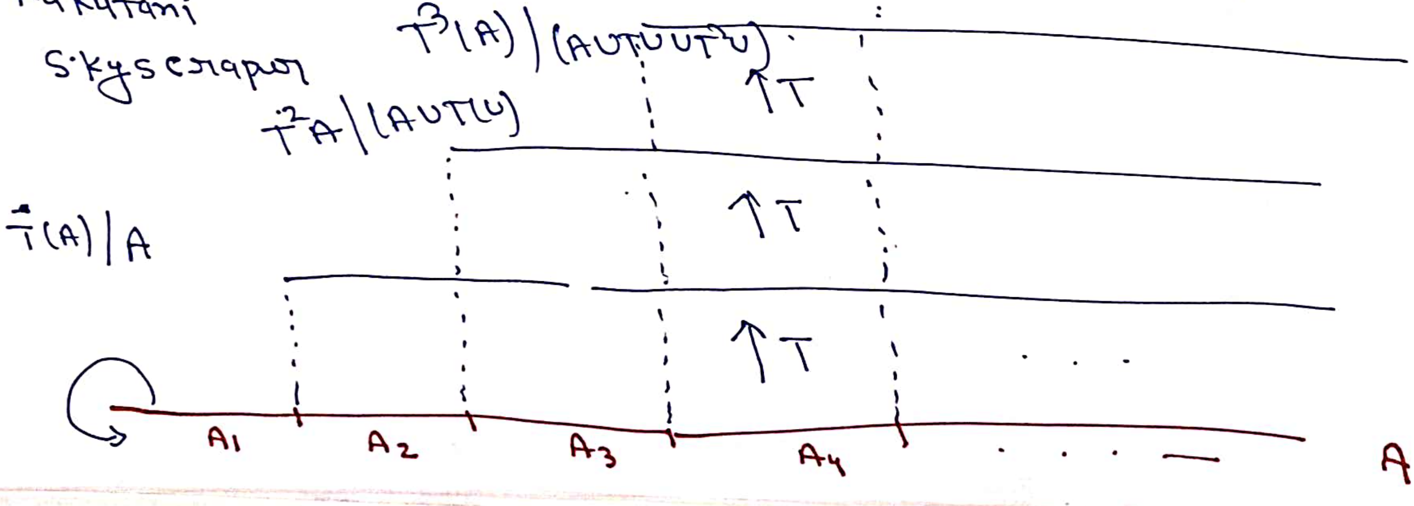
$$A_n = A \cap T^{-n}(A) \setminus \bigcup_{i<n} A_i$$

all are measurable as is

$$T^n(A_n) = A \cap T^n(A) \setminus (TA \cup T^2A \cdots T^{n-1}A)$$

Since T is invertible.

Kakutani Skyscraper



Lemma The induced transformation T_A is measure preserving on the space $(A, \mathcal{B}|_A, \mu_A, T_A)$,

where

$$\mu_A = \frac{1}{\mu(A)} \cdot \mu|_A .$$

If T is ergodic with respect to μ, then T_A is ergodic with respect to μ_A.

Proof If $B \subseteq A$ is measurable, then $B = \bigsqcup_{n \geq 1} B \cap A_n$

$$\mu_A(B) = \sum_{n=1}^{\infty} \mu(B \cap A_n)$$

Now $T_A(B) = \bigsqcup_{n \geq 1} T(B \cap A_n) = \bigsqcup_{n \geq 1} T^n(B \cap A_n)$

$$\Rightarrow \mu_A(T_A(B)) = \sum_{n \geq 1} \mu_A(T^n(B \cap A_n))$$

$$= \frac{1}{\mu(A)} \cdot \sum_{n \geq 1} \mu(T^n(B \cap A_n))$$

$$= \frac{1}{\mu(A)} \cdot \sum_{n \geq} \mu(B \cap A_n) = \mu_A(B).$$

If T_A is not ergodic, then there is a T_A invariant measurable set $B \subseteq A$

$$0 < \mu(B) < \mu(A)$$

the set $S = \bigcup_{n \geq 1} \bigcup_{j=0}^{n-1} T^j(B \cap A_n)$ is nontrivial and $\nRightarrow$ μ is ergodic.

Theorem (Kac's Lemma). Let $(X, \mathcal{B}, \mu, T)$ is ergodic measure preserving system. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then

$$\frac{1}{\mu(A)} \int_A r_A(x)\, d\mu = \frac{1}{\mu(A)}.$$

($\Rightarrow r_A$ is integrable with respect to μ).

Very recently in Oct. 2024, Tom-Meyerovitch and Benjamin Weiss generalized Kac lemma for probability preserving action of arbitary countable group.

Theorem (W.B. Wright. 1960). If T is incompressible measure preserving transformation on a measure space $(X, \mathcal{B}, \mu)$

$$\left(m(E - T^{-1}E) = 0 \Rightarrow m(T^{-1}E - E) = 0 \right)$$

If $E \in \mathcal{B}$ has finite measure, then

$$\int_E r_E(x)\, d\mu = \mu\left(\bigcup_{j=1}^{\infty} T^{-j} E \right).$$

(If T is ergodic and $\mu(E) > 0$, then

$$\int_E r_E(x)\, d\mu = \mu(X) = 1.$$

Theorem Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system and $r : X \longrightarrow \mathbb{N}_0$ be a map in L^1_μ. The system $(X^r, \mathcal{B}^r, \mu^r, T^{(r)})$ defined as

- $X^{(r)} = \{ (x, n) \mid 0 \le n \le r(x) \}$
- $\mathcal{B}^{(r)}$ is the product σ-algebra on $\mathcal{B}$ and the Borel σ-algebra on $\mathbb{N}$
- $\mu^{(r)} = \mu^{(r)}(A \times N) = \dfrac{1}{\int r\, d\mu} \cdot \mu(A) \times |N|$

$A \in \mathcal{B},\ N \subseteq \mathbb{N}$.

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$$T^{(r)}(x,n) = \begin{cases} (x,n+1) & \text{if } n+1 < r(x) \\ (T(x),0) & \text{if } n+1 = r(x) \end{cases}$$

is a finite measure-preserving system.

Now to show T^{r} is μ^{r} ~~pre~~ measure preserving. Infact it suffices to check this on the generators. So let $B \in \mathcal{B}$ be T-measurable

- If ~~i~~ $i>0$, then $T^{(r)^{-1}}(B,i) = (B,i-1)$

$$\mu(T^{(r)^{-1}}(B,i)) = \mu(B,i-1) = \frac{\mu(B)}{\int r \, d\mu} = \mu(B,i)$$

- If $i=0$

$$T^{(r)^{-1}}(B,0) = \bigcup_{n=1}^{\infty} \left(\underbrace{\{x \in B \mid r(x)=n\}}_{A_n} \cap T^{-1}(B) \right)$$

$$\mu^{r}(T^{(r)^{-1}}(B,0)) = \sum_{n=1}^{\infty} \frac{\mu(A_n \cap T^{-1}(B))}{\int_B r \, d\mu} = \frac{\mu(B)}{\int_B r \, d\mu} = \mu(B,0).$$

Def^n Let $(X,\mathcal{B})$ be a standard Borel space. A group $a_t : t \in \mathbb{R}$ of Borel automorphisms on $(X,\mathcal{B})$ is called flow if

1. The map $(t,x) \longrightarrow a_t x$ from $\mathbb{R} \times X \longrightarrow X$ is measurable where $\mathbb{R} \times X$ have usual Borel product structure.
2. $a_0 x = x$.
3. $a_{t+s} \cdot x = a_t \circ a_s \cdot x \quad \forall\, t,s \in \mathbb{R},\ x \in X$.

Lecture

<u>Def</u>

Def^n Let $(X, \mathcal{B})$ be a standard Borel space. A group $a_t : t \in R$ of Borel automorphism on $(X, \mathcal{B})$ is called flow if

1. The map $(t, x) \longrightarrow a_t \cdot x$ from $R \times X \longrightarrow X$ is measurable where $R \times X$ have usual Borel product structure.

2. $a_0 \cdot x = x$

3. $a_{t+s} \cdot x = a_t \circ a_s \cdot x \quad \forall\ s, t \in R,\ x \in X.$

- The flow preserves μ, if

$$\mu(a_t(A)) = \mu(A) \quad \forall\ t \in R,\ A \in \mathcal{B}.$$

Examples

1. Let $X = \mathbb{R}$, and $a_t \cdot x = x + t$, $x, t \in R$, is a continous flow which preserve the Lebesgue measure.

2. Let $X = S^1$ and $a_t \cdot x = e^{it} x$

3. Let $X = S^1 \times S^1$, $a_t(x, y) = (e^{it} x, e^{it\alpha} y)$

Example Let $\theta \in [0, \frac{\pi}{2})$. Define

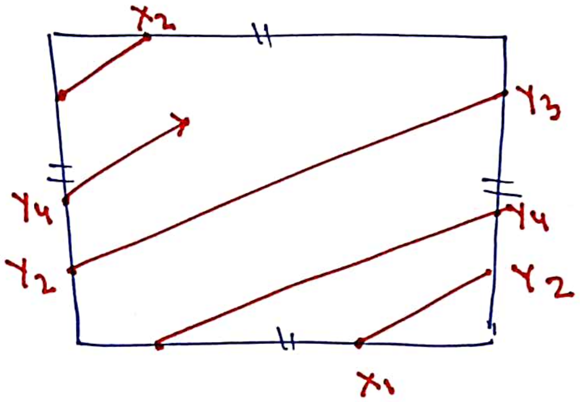
$$\phi_t^\theta(x, y) = (x(t), y(t)), \quad \text{where}$$

$x(t) = x + t\cos\theta \mod 1$
$y(t) = y + t\sin\theta \mod 1,$

which are solution of differential equation

$$\frac{dx}{dt} = \cos\theta$$

$$\frac{dy}{dt} = \sin\theta.$$



$\phi^{\theta}_{t}(x,y)$ as $t \geq 0$ increases moves along the line through (x,y) in direction θ until $\phi^{\theta}_{t}(x,y)$ hits the boundary.

~~Def^n A cross section for a flow $\mathbb{R} \curvearrowright X$ is a Borel set $C \subseteq X$ which intersect every orbit in a non-empty countable set.~~

- When a flow a_t is free, any orbit can be "identified" with a copy of the real line. Concrete identification cannot be done in a Borel way through all orbit, unless the flow is smooth. But we may transfer invariant notions from $\mathbb{R}$ to each orbit.
- Distance between points within an orbit. $\text{dist}(x,y) = r \geq 0$ if $a_r x = y$
- Lebesgue measure on an orbit: for any Borel $A \subset \Omega$ we set
$$\lambda_x(A) = \lambda(\{r \in \mathbb{R} : a_r(x) \in A\})$$
where λ is the Lebesgue measure on $\mathbb{R}$. Note that $\lambda_x = \lambda_y$ whenever $x \sim y \Leftrightarrow or(x) = or(y)$.

- Linear order within orbit. $x < y$ if $a_r(x) = y$ for some $r > 0$.

Def^n Let $(X, \mathcal{B}_X, \{a_t\})$ be a Borel space with flow $\{a_t\}$. A Borel cross-section is a Borel subset S with the following properties:

1. For any $x \in X$, the set of visit times

$$Y_x = \{t \in \mathbb{R} : a_t x \in S\}$$

are all discrete and totally unbounded, that is for any $T > 0$,

$$Y_x \cap [T, \infty) \neq \phi, \quad Y_x \cap [-\infty, -T] \neq \phi$$

and $Y_x \cap [-T, T] < \infty$

2. The return time function

$$\tau_S : S \longrightarrow \mathbb{R}_{>0}$$

$$\tau_S(x) = \min\{Y_x \cap \mathbb{R}\}$$ is Borel.

If S is Borel cross-section we will denote by

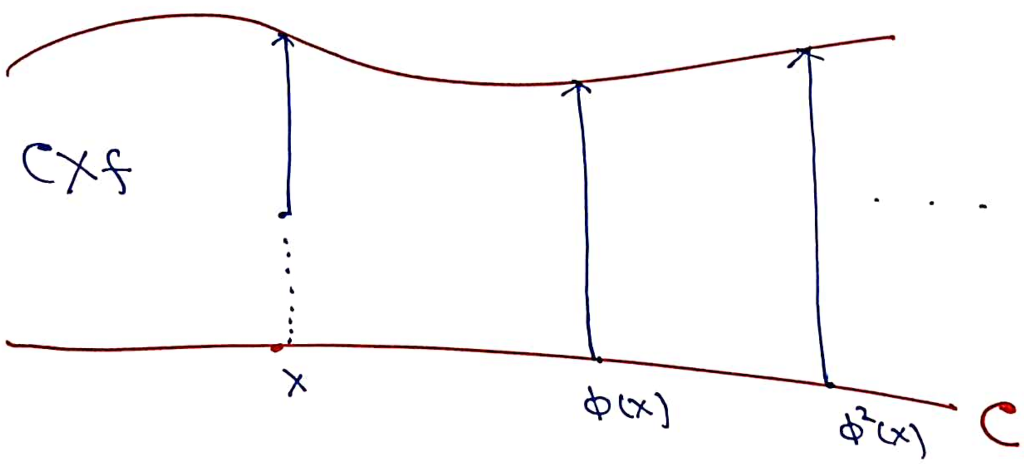
$T_S : S \longrightarrow S$ the first return map

$$T_S(x) = a_{\tau(x)} \cdot (x)$$

Flow under a function

Given a standard Borel space C, a free Borel automorphism $\phi : C \longrightarrow C$ and a bounded away from zero Borel function $f : C \longrightarrow R_{>0}$. Define

$$C \times f = \left\{ (x,s) \in C \times R_{>0} \;\middle|\; 0 \le s < f(x) \right\}$$



and a flow $\{F_t\}$ on $C \times f$ by letting points flow upward until they reach the graph of f and then jump to the next fiber as determined by ϕ

In symbol, $r \geq 0$

$$F_r(x+s) = \left(\phi^n(x),\ s + r - \sum_{i=1}^{n-1} f(\phi^i(x)) \right)$$

for the unique $n \in \mathbb{N}$, such that

$$0 \le s + r - \sum_{i=0}^{n-1} \cancel{\phi} f(\phi^i(x)) < f(\phi^n(x))$$

For $r<0$, (5)

$$F_r(x+s) = \left(\phi^{-n}(x),\ r+s+\sum_{i=1}^{n} f(\phi^{-i}(x)) \right)$$

for the unique $n \in \mathbb{N}$.

$$0 \le s+r+\sum_{i=1}^{n} f(\phi^{-i}(x)) < f(\phi^{-n}(x)).$$

Let $a_t, t \in \mathbb{R}$ be a free flow on $(X, \mathcal{B}_X)$ and $S \subseteq \mathcal{B}_X$ be a cross section, then the flow under the function τ_S and the induced automorphism $T_S : S \longrightarrow S$ is naturally isomorphic to $\{a_t\}$.

<u>Def</u>

Def^n The flow $\{F_t\}$ defined on $C \times f$ is called the flow built under the function f with base automorphism ϕ and base space C.

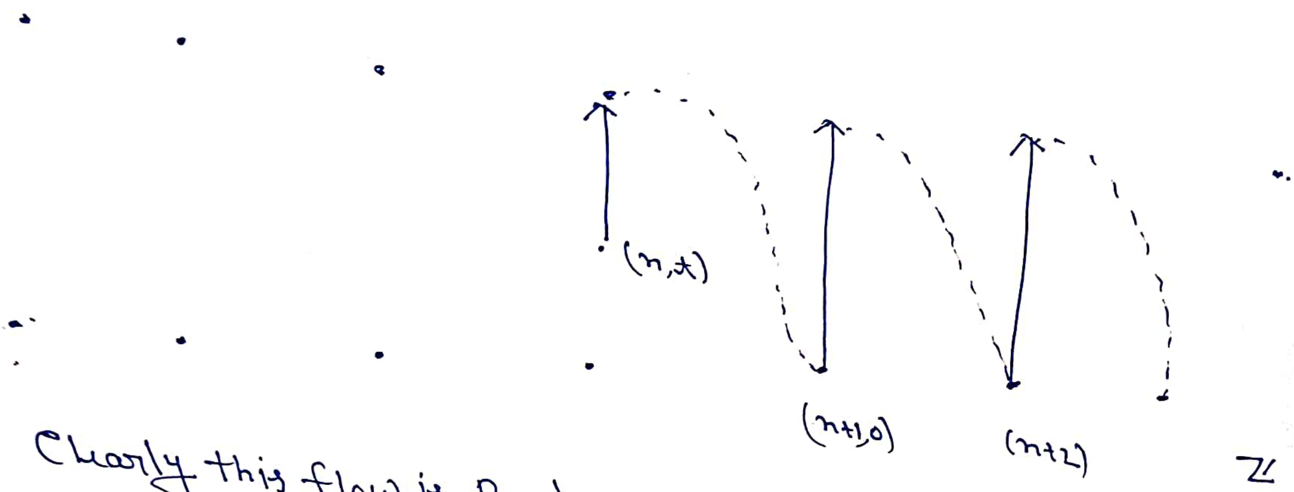
Example Let $q_t(x) = x+t$ be the flow on $\mathbb{R}$. Define

$$\underset{T_S}{T} : \underset{S}{\mathbb{Z}} \longrightarrow \mathbb{Z}, \quad T(n) = n+1$$

and

$$\tau_S = f \equiv 1.$$

Then the flow under f is defined as



Clearly this flow is Borel isomorphic to q_t, $t \in \mathbb{R}$.

Theorem (Vivek M. Wagh – 1988) Any free Borel flow is isomorphic to a flow under a (bounded away from zero) function, which one may moreover assume to be bounded from above.

- Every measurable flow (free) on standard Borel space is a flow of homeomorphisms on a complete separable metric space.

- Every free flow $a_t; t \in \mathbb{R}$ on $(X, \mathcal{B}_X)$, $0 \le \alpha \le 1$

  there exists a measurable set B such that for every X the orbit of x spend α, proportion of time in B

$$\frac{1}{N} \lambda\left(\{t : a_t \cdot x \in B : 0 \le t \le N\}\right) \longrightarrow \alpha \text{ as } N \to \infty.$$

<u>Ques</u> How simple the function in Wagh's Theorem.

<u>Proposition</u> (Ambrose, 1941) A Borel flow ~~on~~ $a_t, t \in \mathbb{R}$ on $(X, \mathcal{B})$ can be written as a flow under a constant function

$$X = \mathcal{C} \times \{\lambda\}, \quad \lambda > 0$$

If and only if there is a Borel function

$$h : X \longrightarrow \mathbb{C} \setminus \{0\} \text{ such that}$$

$$h(w + r) = e^{\frac{2\pi i r}{\lambda}} h(w) \quad \forall\, w \in X, r \in \mathbb{R}.$$

<u>Ergodic Setting</u>

<u>Theorem</u> (Ambrose 1941) Every measurable ergodic flow is isomorphic to a flow built under a function.

- Ambrose also give some criteria for a flow to admit a cross section with constant gaps.

- Bernoulli flow Let $(X, \mathcal{B}, \mu)$ be a probability space. A flow $\{T^t\}$ on X is called Bernoulli flow if T^1 is Bernoulli automorphism.
If $\{T^t\}$ is Bernoulliflow, then T^t $t \neq 0$ are all Bernoulli automorph.

In 1973, Ornstein. ~~he~~ prove that a Bernoulli flow is isomorphic to flow under a ~~fn~~ two valued function $\{\alpha, \beta\}$, with $\frac{\alpha}{\beta} \notin \mathbb{Q}$.

In 1975 Rudolph proved for general ergodic flow.

U. Krengel, 1976, strengthened Rudolph's result, and proved that, given three real number α, β, γ with $\frac{\alpha}{\beta} \notin \mathbb{Q}$, we can choose function f

$$\mu(\{x \in X \mid f(x) = \alpha\}) = \gamma \mu(\{x \in X \mid f(x) = \beta\}).$$

- M. G. Nadkarni, asked can be have two-valued function f is Borel setting? (1998)

Theorem (Slutsky, 2015). Let $a_t, t \in \mathbb{R}$ be free Borel flow on $(X, \mathcal{B})$. Then a_t is Borel isomorphic to flow built under a function (two valued) $\{\alpha, \beta\}$ such that $\frac{\alpha}{\beta} \notin \mathbb{Q}$.

Invariant measures :- An important invariant of a flow is its set of invariant measures. Given a flow $\{a_t\}$ its set of ergodic invariant probability measures on (X, B_X) is denoted by

$$\mathcal{E}(\{a_t\})$$

Let $S \subset B_X$ be a cross section of $\{a_t\}$. Ambrose showed that for any finite $\{a_t\}$ invariant measure μ on (B_X, X) there exists a T_S invariant measure ν_μ on S such that μ is the product of the Lebesgue measure d on $\mathbb{R}$.

More formally, when X is viewed as subset of $S \times \mathbb{R}$ via the identification

$$\{(x, r) \in S \times \mathbb{R} : 0 \leq r < \tau_S(x)\}$$

then

$$\mu = \nu_\mu \times d \big|_X$$

- If $c \in \mathbb{R}_{>0}$ is such that $\tau_S(x) \geq c > 0$ for all $x \in S$ then for any Borel $A \subseteq S$

$$\nu_\mu(A) = \frac{\mu(A \times [0, c])}{c}$$

The definition is independent of c

- When S admits an upper bound on its T_S, we also have a map in other direction.

- For any T_S invariant finite measure ν we define an $\{a_t\}$ invariant μ_ν on $(X, \mathcal{B}_X)$ by setting

$$\mu_\nu(A) = \int_S \tilde{\lambda}_x(A)\, d\nu(x) \quad , \quad A \in \mathcal{B}_X$$

where

$$\tilde{\lambda}_x(A) = \lambda\left(\{t \in \mathbb{R} \mid 0 \le t < T_S(x) \text{ and } a_t(x) \in A\}\right)$$

The boundedness of gap from above is needed to ensure that the integral is finite.

- The maps $\mu \longrightarrow \nu_\mu$ and $\nu \longrightarrow \mu_\nu$ are inverse of each other and provide a bijection between finite $\{a_t\}$ invariant measure on $(X, \mathcal{B}_X)$ and finite T_S-invariant measure on a cross section S with bounded ~~gr~~ T_S.

- These maps preserve ergodicity, but do not in general, preserve normalization : $\mu(X)$ is generally not equal to $\nu_\mu(S)$

- $\mu \longrightarrow \dfrac{\nu_\mu}{\nu_\mu(S)}$ is a bijection between $\mathcal{E}(\{a_n\})$ and $\mathcal{E}(T_S)$.

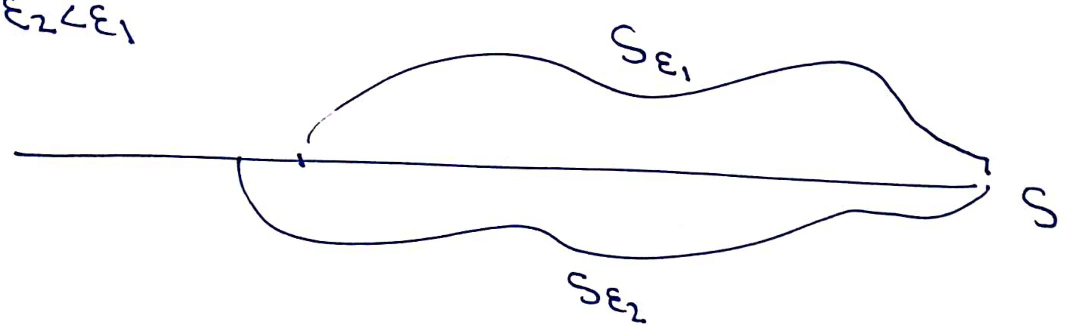
Theorem Let $\{a_t\}$ be a free Borel flow on (X, B_X) and $S \subset B_X$ be a cross section with bounded gaps. then

$$\#\xi(\{a_t\}) = \#\xi(T_S).$$

- For $\varepsilon > 0$ we let

$$S_{\geq\varepsilon} = \{x \in S : T_S(x) \geq \varepsilon\} \quad \text{and} \quad S_{<\varepsilon} = S / S_{\geq\varepsilon}$$

If $\varepsilon_2 < \varepsilon_1$



The sets $S_{\geq\varepsilon}$ are an increasing collection of Borel sets union in S

Given $E \in B_X$ and $I \subset \mathbb{R}$ we let

$$E^I \stackrel{def}{=} \{a_t x : x \in E, t \in I\}$$