# Lecture



**Def$^n$** Let $(X, \mathcal{B})$ is a Borel space. A subcollection $\mathcal{N} \subseteq \mathcal{B}$ is called a $\sigma$-ideal if

1. $\mathcal{N}$ is closed under countable unions.
2. $B \in \mathcal{B}$ and $N \in \mathcal{N} \Rightarrow B \cap N \in \mathcal{N}$.

$A \equiv B \bmod (\mathcal{N})$ if $A \Delta B = (A-B) \cup (B-A) \in \mathcal{N}$.

Ex: If $m$ is a countable additive measure on $\mathcal{B}$, then the collection of sets in $\mathcal{B}$ of measure zero form a $\sigma$-ideal.

**Def$^n$** A one-one measurable map $T: (X, \mathcal{B}) \to (X, \mathcal{B})$ onto itself such that $T^{-1}$ is also measurable map is called Borel automorphism.

**Def$^n$** Let $(X, \mathcal{B}_X)$ be a Borel set. A measurable set $W$ is said to be wandering with respect to Borel automorphism $T$ if

$T^n W : n \in \mathbb{Z}$ are pairwise disjoint.

The $\sigma$-ideal generated by wandering sets in $\mathcal{B}$ is denoted by $\mathcal{W}_T$

($W$ intersect orbit of any point at most 1).
never intersect periodic orbit).

**Lemma** Let $T$: be a Borel automorphism of a standard Borel space $(X, \mathcal{B})$. Then given any $A \in \mathcal{B}$ there exists $N \in \mathcal{W}_T$ such that each $x \in A - N$, the points return to $A$ infinitely many positive $n$. $T^n x$.

**Lemma** $(X, \mathcal{B}, \mu, T)$ is p.p.T, and $A \in \mathcal{B}$ and ~~if $\mu(A) > 0$, then~~ then for almost all $x \in A$, the points $T^n x \in N$ for infinitly many $n$.

Let $(X, \mathcal{B}, \mu, T)$ be a invertible measure preserving system and let $A$ be a measurable set with $\mu(A) > 0$. By Poincare recurrence, the first return time defined by

$$r_A(x) = \inf_{n \geq 1} \{ n \mid T^n x \in A \}$$

Exists almost every where.

Def$^n$ The map $T_A : A \to A$ defined by

$$T_A(x) = T^{r_A(x)}(x)$$

($r_A$ and $T_A$ are measurable maps)

For $n \geq 1$, $A_n = \{ x \in A \mid r_A(x) = n \}$

$$A_1 = A \cap T^{-1}(A)$$

$$A_2 = A \cap T^{-2}(A) \setminus A_1$$

$$\vdots$$

$$A_n = A \cap T^{-n}(A) \setminus \bigcup_{i<n} A_i$$

all are measurable as is

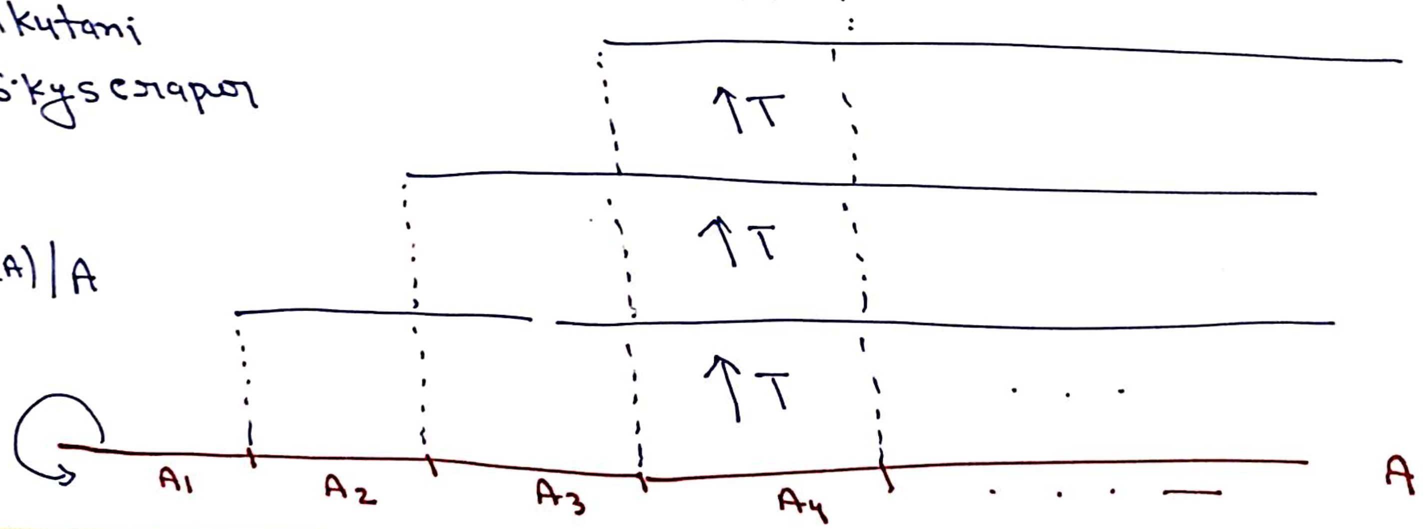
$$T^n(A_n) = A \cap T^n(A) \setminus (TA \cup T^2 A \ldots T^{n-1} A)$$

Since $T$ is invertible.

Kakutani

Skyscraper

$T^{-1}(A) \setminus A$



**Lemma** The induced transformation $T_A$ is measure preserving on the space $(A, \mathcal{B}|_A, \mu_A, T_A)$,

where
$$\mu_A = \frac{1}{\mu(A)} \cdot \mu|_A .$$

If $T$ is ergodic with respect to $\mu$, then $T_A$ is ergodic with respect to $\mu_A$.

**Proof** If $B \subseteq A$ is measurable, then $B = \bigsqcup_{n \geq} B \cap A_n$

$$\mu_A(B) = \sum_{n \geq 1}^{\infty} \mu(B \cap A_n)$$

Now $T_A(B) = \bigsqcup_{n \geq 1} T(B \cap A_n) = \bigsqcup_{n \geq 1} T^n(B \cap A_n)$

$$\Rightarrow \mu_A(T_A(B)) = \sum_{n \geq 1} \mu_A(T^n(B \cap A_n))$$
$$= \frac{1}{\mu(A)} \cdot \sum_{n \geq 1} \mu(T^n(B \cap A_n))$$
$$= \frac{1}{\mu(A)} \cdot \sum_{n \geq} \mu(B \cap A_n) = \mu_A(B).$$

If $T_A$ is not ergodic, then there is a $T_A$ invariant measurable set $B \subseteq A$

$$0 < \mu(B) < \mu(A)$$

the set $S = \bigcup_{n \geq 1} \bigcup_{j=0}^{n-1} T^j(B \cap A_n)$ is nontrivial and $\nRightarrow$ $\mu$ is ergodic.

**Theorem** (Kac's Lemma). Let $(X, \beta, \mu, T)$ is ergodic measure preserving system. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then

$$\frac{1}{\mu(A)} \int_A r_A(x)\, d\mu = \frac{1}{\mu(A)}.$$

($\Rightarrow r_A$ is integrable with respect to $\mu$).

very recently in Oct. 2024, Tom-Meyerovitch and Benjamin Weiss generalized Kac Lemma for probability preserving action of arbitary countable group.

**Theorem** (W.B. Wright. 1960). If $T$ is incompressible measure preserving transformation on a measure space $(X, \mathcal{B}, \mu)$

$$\left( m(E - T^{-1}E) = 0 \Rightarrow m(T^{-1}E - E) = 0 \right)$$

If $E \in \mathcal{B}$ has finit measure, then

$$\int_E r_E(x)\, d\mu = \mu\left( \bigcup_{j=1}^{\infty} T^{-j}E \right).$$

(If $T$ is ergodic and $\mu(E) > 0$, then

$$\int_E r_E(x)\, d\mu = \mu(X) = 1.$$

**Theorem** Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system and $r : X \longrightarrow \mathbb{N}_0$ be a map in $L^1_\mu$. The system $(X^{r}, \mathcal{B}^{r}, \mu^{r}, T^{(r)})$ defined by

- $X^{(r)} = \{ (x, n) \mid 0 \le n \le r(x) )$
- $\mathcal{B}^{(r)}$ is the product $\sigma$-algebra on $\mathcal{B}$ and the Borel $\sigma$-algebra on $\mathbb{N}$
- $\mu^{(r)} = \mu^{(r)}(A \times \mathbb{N}) = \dfrac{1}{\int r\, d\mu} \cdot \mu(A) \times |N|$

$A \in \mathcal{B}$, $N \subseteq \mathbb{N}$.

- $$T^{(r)}(x,n) = \begin{cases} (x, n+1) & \text{if } n+1 < r(x) \\ (T(x), 0) & \text{if } n+1 = r(x) \end{cases}$$

is a finite measure-preserving system.

Now to show $T^{r}$ is $\mu^{r}$ ~~pre~~ measure preserving. Infact it suffices to check this on the generators. So let $B \in \mathcal{B}$ be T-measurable

- If ~~\#~~$i > 0$, then $T^{(r)^{-1}}(B, i) = (B, i-1)$

$$\mu\left(T^{(r)^{-1}}(B,i)\right) = \mu(B, i-1) = \frac{\mu(B)}{\int r \cdot d\mu} = \mu(B, i)$$

- If $i = 0$

$$T^{(r)^{-1}}(B,0) = \bigcup_{n=1}^{\infty} \left( \underbrace{\{x \in B \mid r(x) = n\}}_{A_n} \cap T^{-1}(B) \right) \}$$

$$\mu^{r}\left(T^{(r)^{-1}}(B,0)\right) = \sum_{n=1}^{\infty} \frac{\mu\left(A_n \cap T^{-1}(B)\right)}{\int_B r\,d\mu} = \frac{\mu(B)}{\int_B r\,d\mu} = \mu(B,0)).$$

**Def$^n$** Let $(X, \mathcal{B})$ be a standard Borel space. A group $a_t : t \in \mathbb{R}$ of Borel automorphisms on $(X, \mathcal{B})$ is called flow if

1. The map $(t, x) \longrightarrow a_t x$ from $\mathbb{R} \times X \longrightarrow X$ is measurable where $\mathbb{R} \times X$ have usual Borel product structure.
2. $a_0 x = x$.
3. $a_{t+s} \cdot x = a_t \circ a_s \cdot x \quad \forall\, t, s \in \mathbb{R},\ x \in X$.