Stationary Measures On The Projective Space

Roie Salama

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Reminder from previous lectures

Let G be a locally compact semigroup equipped with a measure μ acting continuously on a compact metric space X .

Let $(B, \mathcal{B}, \beta)$ be the associated Bernouli space .

Let ν be a μ - stationary measure on X:

$$\mu * \nu = \int_G g_* \nu d\mu(g)$$

then for β - almost $b \in B$, $b = b_1, b_2, ...$ we have a limit measure

$$\nu_b = \lim_{n\to\infty} (b_1 \cdots b_n)_* \nu$$

We also have for every $m \in \mathbb{N}$ for $\beta \otimes \mu^{*m}$ - a.e $(b, g) \in B \times G$

$$\lim_{n\to\infty} (b_1 \cdots b_n g)_* \nu = \nu_b$$

Moreover, ν can be represented as an average of ν_b :

$$\nu = \int_B \nu_b d\beta(b)$$

Some required definitions

Let V be a finite dimensional real vector space , let $G = GL(V)$, and let $\Gamma \leq GL(V)$ be a subsemigroup. We have an action $\Gamma \curvearrowright V$ by matrx-by-vector multiplication .

Definition. (Irreducibility) We say that the action $\Gamma \curvearrowright V$ is irreducible if for every subspace $W \subset V$:

$\Gamma W = W \Rightarrow W = 0$ or $W = V$.

The action is strongly irreducible if for subspaces $\{V_k\}_{k=1}^{l}$:

$\bigcup V_k$ is $\Gamma - stable \Rightarrow \exists i. V_i = V$ or $V_1 = ... = V_l = \{0\}$

Definition. (Proximal dimension) : Let r_Γ be the smallest integer such that $\exists \pi \in End(V)$ of rank r_Γ such that

$$\pi = \lim_{n\to\infty} \lambda_n g_n$$

where $\lambda_n \in \mathbb{R}$, $g_n \in \Gamma$.

If $r_\Gamma = 1$, Γ is called proximal.

Fact. *If $\exists f \in \Gamma$ with a unique eigenvalue of maximal absolute value, then Γ is proximal. If Γ is also strongly irreducible then the converse is also true.*

Definition. The Grassmannian $G_r(V)$ is the space of all $r-dimensional$ subspaces of V . It has the structure of a compact metric space by

$$d(X,Y) = | Pr_X - Pr_Y |$$

where Pr_X is the orthogonal projection on X .

Note that $\mathbb{P}(V) = G_1(V)$

Stationary measures on the projective space and the Furstenberg boundary map

Let μ be a probability measure on $GL(V)$ and let Γ_μ be the smallest closed subsemigroup of G such that $\mu(\Gamma_\mu) = 1$.

The following result deals with the construction of The Furstenberg boundary map .

Theorem 1. *(Furstenberg boundary map) . Let μ be a probability measure on $GL(V)$. Assume Γ_μ is strongly irreducible. Let $r = r_\Gamma$ be the proximal dimension. Then*

a) There is a borel map $\xi : B \to G_r(V)$ such that for β almost any $b \in B$, for every nonzero limit point $f = \lim \lambda_n b_1 ... b_n$, $im(f) = \xi(b)$. In particular $rank(f) = r$.

b) Let ν be a $\mu-$stationary Borel probability measure on $\mathbb{P}(V)$. Then, for $\beta -$ almost any $b \in B$, $\xi(b)$ is the smallest vector subspace V_b of V such that $\nu_b(\mathbb{P}(V_b)) = 1$

Corollary. *(The proximal case) : Let μ be a Borel probability measure on $GL(V)$ such that Γ_μ is proximal and strongly irreducible. Then the μ stationary measure ν on $P(V)$ is unique. Let $\xi : B \to \mathbb{P}(V)$ be the Furstenberg map . then, forβ - a.e $b \in B$, $\nu_b = \delta_{\xi(b)}$ (the Dirac Mass) . For $\beta -$almost any $b \in B$, every nonzero limit point $f \in End(V)$ of a sequence $\lambda_n b_1 ... b_n$ with $\lambda_n \in R$ has rank one and admits the line $\xi(b)$ as its image.*

Proof. (of the corollary) : The Corollary follows by Theorem 1 in the proximal case $(r = 1)$. The uniqueness is due to the fact that

$$\nu = \int_B \delta_{\xi(b)} d\beta(b)$$

(which can also be written as $\nu = \xi_* \beta$) $\square$

Note that in this case we have an example of a dynamical system, in which the space is a compact metric space, where the invariant measure is unique. Which was exactly the setting for the lecture 2 weeks ago.

Before proving the theorem we shall prove a lemma about stationary measures on the grassmanian. Note that Γ_μ naturally acts on the Grassmanian.

Lemma 2. *Let μ be a Borel probability measure on $GL(V)$ and ν a μ - stationary probability measure on $G_{r_0}(V)$ for some $r_0 > 0$, and let $W \subset V$ a proper subspace of V .*

a) If Γ_μ is irreducible then $\nu(G_{r_0}(W)) \neq 1$

b) If Γ_μ is strongly irreducible then $\nu(G_{r_0}(W)) = 0$

Proof. a) Let $A = \{W \subset V \mid \nu(G_{r_0}(W)) = 1\}$. We wish to prove that $A = \{V\}$. Let

$$W_0 = \cap A$$

Note that W_0 can be represented as a finite intersection (exercise) . Therefore, $\nu(G_{r_0}(W_0)) = 1$. Now, since ν is a stationary measure :

$$1 = \nu(G_{r_0}(W_0)) = \int\limits_G \nu(G_{r_0}(g^{-1}W_0))d\mu(g)$$

and so for $\mu-$ almost any $g \in G$

$$\nu(G_{r_0}(g^{-1}W_0)) = 1$$

In other words $gW_0 = W_0$ for a.e g.

So we get that the set

$$G_{W_0} = \{g \mid gW_0 = W_0\}$$

is a closed subsemigroup of full measure and so $\Gamma \subset G_{W_0}$. By irreducibility , $W_0 = V$ as desired.

b) let $r \geq r_0$ be the smallest integer such that there exists $W \subset V$ with $\nu(G_{r_0}(W)) \neq 0$. We wish to show that $r = dimV$. By minimality, for $W_1 \neq W_2$ in $G_r(V)$ we have that $\nu(G_{r_0}(W_1 \cap W_2)) = 0$. And so for countable family of subspaces

$\{W_j\}_{j \in \mathbb{N}}$

$$\sum_j \nu(G_{r_0}(W_j)) = \nu(\bigcup_j G_{r_0}(W_j)) \leq 1$$

Hence, for any $m > 0$ there are finitely many $W \subset V$ such that $\nu(G_{r_0}(W)) \geq m$. Let

$$m = \sup_{W \in G_r} \nu(G_{r_0}(W))$$

Now define

$$M = \{W \in G_{r_0}(V) \mid \nu(G_{r_0}(W)) = m\}$$

so M is finite and non empty.

Like before, we have

$$\nu(G_{r_0}(W_0)) = \int_G \nu(G_{r_0}(g^{-1}W_0))d\mu(g)$$

for any $W \in M$. And so for μ - almost $g \in G$, $g^{-1}W_0 \in M$. So the finite union $\cup M$ is Γ_μ- stable , and so by strong irreducibility , $V \in M$. In particular, $r = dimV$. $\square$

We shall now prove Theorem 1 . Note that every $f \in End(V)$ induces a continuous map $\mathbb{P}(V)\backslash\mathbb{P}(kerf) \to \mathbb{P}(V)$.

Proof. Let ν be a μ stationary measure on $\mathbb{P}(V)$. Then for μ almost $g \in G$ and β almost $b \in B$

$$(b_1 \cdot ... \cdot b_n g)_* \nu \to \nu_b$$

Now define $\xi(b)$ to be the smallest vector subspace of V such that

$$\nu_b(\mathbb{P}(\xi(b))) = 1$$

Let $f \in End(b)$, $f = \lim \lambda_n b_1 \cdots b_n$ a nonzero limit point with $\lambda_n \in \mathbb{R}$. For any $g \in G$, $ker(fg) \subsetneq V$ and so $\nu(\mathbb{P}(ker(fg))) = 0$ by Lemma 1. So fg induces a well defined continuous map $\mathbb{P}(V) \to \mathbb{P}(V)$.

We claim that for a.e $b \in B$, and a.e $g \in G$:

1)

$$(fg)_*\nu = \nu_b$$

indeed, take some test function $\varphi \in C(\mathbb{P}(V))$, then

$$\int_{P(V)} \varphi(x)d((fg)_*\nu)(x) = \int_{P(V)} \varphi(fgx)d\nu(x)$$

$$= \int_{P(V)} \varphi(\lim \lambda_n b_1 \cdots b_n g(x))d\nu(x) \underbrace{=}_{continuity} \int_{P(V)} \lim \varphi(\lambda_n b_1 \cdots b_n gx)d\nu(x)$$

$$= \int_{P(V)} \lim \varphi(b_1 \cdots b_n gx)d\nu(x) \underbrace{=}_{dominated-convergence} \lim \int_{P(V)} \varphi(b_1 \cdots b_n gx)d\nu(x)$$

$$= \lim \int_{P(V)} \varphi(x)d((b_1 \cdot ... \cdot b_n g)_*\nu(x)) = \int_{P(V)} \varphi(x)d\nu_b(x)$$

Now, the elements $g \in G$ where the equality in 1) occurs form a closed subsemigroup of measure 1 (the proof that it's closed is similar to the above argument) . And so we have (1) for every $g \in \Gamma_\mu$. Now, since we also have for a.e b

$$\nu_b = \lim_{n\to\infty} (b_1 \cdots b_n)_* \nu$$

we could also deduce (in the same manner with $g = id$) :

$$f_* \nu = \nu_b$$

From this (and the way we defined $\xi(b)$) , we deduce that $\nu(f^{-1}\xi(b)) = 1$ and so by lemma 1 , $f^{-1}(\xi(b)) = V$.

In other words , $\xi(b) = Imf$.

So we proved that every limit point has the same image $\xi(b)$, and so $\xi(b)$ is independent of the choice of stationary measure ν and limit point f .

We are left to prove that $dim(\xi(b)) = r$. By the definition of r there exists $\pi \in End(V)$ with rank r such that

$$\pi = \lim_{n\to\infty} \lambda_n g_n$$

where $\lambda_n \in \mathbb{R}$ and $g_n \in V$.

Note that since Γ_μ is irreducible, we may choose π such that

$$f\pi \neq 0$$

($kerf \neq V$ and so we can always find g such that $g\pi \notin Kerf$) .

By Lemma 1 , $\nu(kerf\pi) = 0$. By equation (1) applied to $g = g_n$ and taking limit

$$(f\pi)_* \nu = \nu_b$$

As before we get $\xi(b) = im(f\pi)$ and so $dim(\xi(b)) \leq r$ (remember r is the rank of π) .

Therefore, by minimality (definition of r) :

$$dim(\xi(b)) = r$$

as desired. □